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Heterogeneity and Clustering of Defaults

(Authors' names blinded for peer review)

This paper studies an economy where privately informed hedge funds trade a risky asset in order to exploit potential mispricings. Hedge funds are allowed to have access to credit, by using their risky assets as collateral. We analyse the role of the degree of heterogeneity among hedge funds in the emergence of clustering of defaults. We find that fire-sales caused by margin calls is a necessary, yet not a sufficient condition for defaults to be clustered. We show that the degree of heterogeneity plays a pivotal role; defaults are clustered only if the degree of heterogeneity exceeds a threshold value. Our results indicate that financial stability is inherently linked with the degree of heterogeneity, which suggests that the latter should be taken into consideration when designing the optimal regulation policy for the industry.

Key words: Financial crises; hedge funds; survival statistics; bankruptcy.

1. Introduction

The hedge fund (HF) industry has experienced an explosive growth in recent years. The total size of the assets managed by HFs in 2015 was estimated at US$2.74 trillion (BarclayHedge 2016). Due to the increasing weight of HFs in the financial market, failures of HFs can pose a major threat to the stability of the global financial system. The default of a number of high profile HFs, such as LTCM and HFs owned by Bear Stearns (Haghani 2014), testifies to this.

At the same time, poor performance of HFs—the prelude to the failure of a HF—is empirically found to be strongly correlated across HFs (Boyson et al. 2010), a phenomenon known as "contagion". Moreover, Boyson et al. (2010) point out that the correlation between HFs' worst returns—falling in the bottom 10% of a HF style's monthly returns—remains high, even after taking into account that HF returns are autocorrelated and the effect of the exposure of HFs to commonly known risk factors. The findings of Boyson et al. (2010) support the theoretical predictions of Brunnermeier and Pedersen (2009), who provide a mechanism revealing how liquidity shocks can lead to downward liquidity spirals and thus to contagion[1]. The mechanism that leads

[1] Other works which study the the causes of contagion in financial markets include Kyle and Xiong (2001), and Kodres and Pritsker (2002).

to contagion is closely related to the theory of the "leverage cycle", i.e. the pro-cyclical increase and decrease of leverage, due to the interplay between equity volatility and leverage, put forward by Geanakoplos (1997)[2].

The dominant role of HFs in the financial system in conjuction with the possibility of transmission of the risk, not only to other financial organisations, but also to the real economy, has placed the operation of HFs under close scrutiny and has highlighted the significance of regulation of the industry. Regulating the HF industry is not an easy task; designing the appropriate regulation requires a good understanding of many aspects, such as the mechanism which generates defaults at the individual level, the mechanism behind contagion, and finally the parameters which determine the persistency of the effect of a default of an individual HF on the industry. Although Brunnermeier and Pedersen (2009) propose a mechanism behind contagion, they failed to analyse the effect of the persistency of the impact of a default of an individual HF. Our paper aims to properly analyse the existence of such memory. In particular, we characterise the conditions under which the correlation between HF's defaults is persistent, i.e. defaults are clustered.

We find that the feedback between market volatility, and margin requirements (downward liquidity spiral), is a necessary, yet not a sufficient condition for clustering of defaults to occur, as has been suggested by Boyson et al. (2010). In this work we show that heterogeneity plays a pivotal role in the emergence of clustered defaults: only if the degree of heterogeneity exceeds a threshold value does the default time-sequence show infinite memory, and hence defaults are clustered.

We study the correlation of default events in time, by developing a simple dynamic model with a representative mean-reverting noise trader, and a finite number of HF managers who trade a risky asset. Each manager, before submitting his trading orders, acquires information through his analysts regarding the fundamental value, and the price volatility of the risky asset. We allow for a setup where the analysts of all HFs agree on the fundamental value, but they might disagree on the price volatility. The rationale behind this assumption is that the fundamental value of the asset is exogenously determined, whereas price volatility is determined endogenously. Price volatility depends on the HFs' trading strategy, which in turn, depends on the analysts' private information. Hence, a direct consequence of the lack of access to the other HFs analysts' private information is that the consensus about the market volatility becomes a zero probability event. In addition, we allow for the HFs to have access to credit, and we endogenise the probability of default by assuming that a HF would choose to default when its portfolio value falls below a threshold.

[2] In fact the theory of leverage cycle, in contrast to other models that endogenise leverage (Brunnermeier and Pedersen 2009, Brunnermeier and Sannikov 2014, Vayanos and Wang 2012), has the additional merit of making the endogenous determination of collateral possible.

In this environment we show that the degree of heterogeneity is the determinant of the memory depth of the underlying stochastic process determining the time-sequence of defaults, and in this sense, clustering of defaults. Moreover, we show that this "transition" is also reflected in the emergence of a fat-tail in the aggregate probability density function (PDF) of waiting-times between defaults. Furthermore, we establish a quantitative connection between the non-trivial aggregate statistics, and the presence of infinite memory in the underlying stochastic process governing the defaults of the HFs. The comparison between the theoretical prediction of the asymptotic behaviour of the autocorrelation function (ACF)[3] of defaults and the numerical findings, reveals that our theoretical predictions are valid even in a market with a finite number of HFs and the clustering of defaults is confirmed.

The structure of the rest of the paper is as follows. Section 2 discusses the relevant literature. Section 3 presents the economic framework that we use. In section 4.1 discusses the numerical findings. In Section 4.2, we provide analytical results linking the heavy-tailed aggregate density to the observed statistical character of defaults on a microscopic level, and the power-law decay of the ACF of the default time-series of defaults, identifying that defaults are clustered. Finally, section 5 provides a short summary with concluding remarks.

2. Relevant Literature

Our paper is related to the literature which studies the effects of leverage on financial stability. Probably the largest part of this literature is concerned with what is known as the leverage cycle theory. The leverage cycle models can be traced back to the collateral equilibrium models of Geanakoplos (1997), Geanakoplos and Zame (1997), Geanakoplos (2003), Fostel and Geanakoplos (2015) who provided a general equilibrium model of collateral. The key concept underlying these models is that lenders require a collateral from the borrowers in order to lend them funds This borrowing and lending is agreed through a contract repay the loan. The investor who sells the contract is borrowing money –using a collateral to back the promise– from the agent who buys the contract. Each contract is chosen from a menu of contracts with different loan to value (LTV) ratio. In Geanakoplos (1997) it was shown that a scarcity of collateral leads to a reduction in the number of contracts being traded, which has the effect of rendering leverage (LTV) endogenous. Finally, the investors default when the the value of the collateral is less than the value of the contract that the borrowers and lenders have agreed. Geanakoplos (2003) considered a continuum of risk neutral agents with different priors in a binomial economy with a small number of possible states of the

[3] Using the definition of Andersen and Bollerslev (1997) clustering is determined by the divergence of the sum (or integral in continuous time) of the autocorrelation function (ACF) of the default time sequence, and therefore, the presence of infinite memory in the underlying stochastic process describing the occurrence of defaults.

world. Changes in volatility were shown to lead to changes in equilibrium leverage, which in turn have a larger effect on asset prices than the direct effect of the news. Geanakoplos (2003, 1997) showed that in some cases all agents will choose the same contract from the contract menu. This result has been recently extended by Fostel and Geanakoplos (2015) who studied in more detail the relationship between leverage and default, and prove that in all binomial economies with financial assets, exactly one contract is chosen.

Fostel and Geanakoplos (2008) extended the economy of Geanakoplos (2003) to an economy with multiple assets, and two risk averse agents, instead of a continuum of risk neutral ones, and developed an asset pricing theory, which links collateral and liquidity to asset prices. Geanakoplos (2010) combined the advances made in Geanakoplos (1997), where the collateral was based on non-financial assets and Geanakoplos (2003), where the collateral was based on financial assets, to show that the introduction of CDS contracts reduced the asset prices. Doing this he put forward a model of a *double leverage cycle*, in housing and securities, which provides an explanation for the 2007-08 crisis. Fostel and Geanakoplos (2012) provided a further analysis of CDS contracts and showed: (i) why trenching and leverage initially raised asset prices, and (ii) why CDSs lowered them later. Simsek (2013a) considered a continuum of states, and two types of agents' beliefs (optimistic and pessimistic), regarding the effects of downside or upside states on prices. He showed that the type of disagreement between agents has more important effects on asset prices compared to the effects related to the degree of disagreement between optimists and pessimists. To our knowledge, this is the only paper in this literature that considers the effect of different degrees of heterogeneity[4]. Thurner et al. (2012) and Poledna et al. (2014) study the effects of the leverage cycle in an Agent Based Model with heterogeneous interacting HFs[5].

The effects of leverage have also been studied by Gromb and Vayanos (2002), Acharya and Viswanathan (2011), Brunnermeier and Pedersen (2009), Brunnermeier and Sannikov (2014), and Adrian and Shin (2010), among others. These approaches differ from the models previously outlined in two key aspects: The models of Acharya and Viswanathan (2011), Adrian and Shin (2010), Brunnermeier and Sannikov (2014), and Gromb and Vayanos (2002) focused on the ratio of an agent's total asset value to his total wealth (investor based leverage), whereas the leverage cycle models of Geanakoplos and coauthors[6] focus on LTV. The second aspect has to do with the fact that in the models of Brunnermeier and Pedersen (2009), and Gromb and Vayanos (2002) the

[4] In different context Simsek (2013b) shows that the level of belief disagreement affects the average consumption risks of individuals in a model which studies the effect of financial innovation on portfolio risks.

[5] Other works in the leverage cycle literature include Geanakoplos and Zame (2014), Geanakoplos (2014), and Fostel and Geanakoplos (2016). For a recent review of this literature see Fostel and Geanakoplos (2014).

[6] Also the models of Brunnermeier and Pedersen (2009), and Simsek (2013a) use the same ratio.

leverage ratio is exogenously given, whilst in the former is given by a *VaR* rule, and in the latter it is given by a maximin rule used to prevent defaults. In the cases of Brunnermeier and Sannikov (2014), Acharya and Viswanathan (2011), and Adrian and Shin (2010) leverage is endogenous but is not determined by collateral capacities. In Acharya and Viswanathan (2011), and Adrian and Shin (2010), leverage is determined by asymmetric information between borrowers and lenders; while in Brunnermeier and Sannikov (2014) it is determined by agents' risk aversion.

3. Model

3.1. Environment

We study an economy with two assets, one riskless (cash C) and one risky, two types of traders and a bank. The supply of the risky asset, which can be viewed as a stock, is fixed and equal to N, whereas there is an infinite supply of the riskless asset. The price of the riskless asset is normalised to 1, whereas the price of the risky asset at time t, p_t, is determined endogenously. The riskless and the risky asset are traded by a representative, mean-reverting noise trader and K types of hedge funds (HFs), whose objective is to exploit potential mispricings of the risky asset. The role of the bank, which is infinitely liquid, is to provide credit to HFs, by using the HF's assets as collateral.

Representative Hedge Fund: Each HF is run by a myopic portfolio manager, whose objective is to maximise her next period's CRRA utility function over his wealth, W_t:

$$U\left(W_t^j\right) = {W_t^j}^{1-\alpha} / (1-\alpha), \tag{1}$$

where $\alpha > 0$ is the measure of relative risk aversion, and $j \in \{1, \ldots, K\}$.

The manager's strategy is a mapping from her information set to trading orders for the risky and the riskless asset, where D_t^j (C_t^j) denotes the units of the risky (riskless) asset the jth HF is willing to trade. Thus, the available information plays a crucial role in determining orders. The manager's information set is formed as follows. The manager, before undertaking an investment decision, hires an analyst whose role is to collect information regarding the fundamental value of the risky asset V and the price volatility $\mathrm{Var}[\log p_t]$.

In order for the analyst to first acquire costly information, and second, truthfully transmit this information to the manager, he is offered a compensation contract, which aligns his incentives with the incentives of the manager. The alignment in the incentives requires the compensation of the analyst to be contingent to the HF's wealth. Specifically, the analyst's compensation, which is denoted as M_{t+1}^j, is a percentage of the wealth (standard management fee) and the realised profits (performance fee) of the previous period[7], i.e.,

$$M_{t+1}^j = \gamma \left[W_t^j + 10 \max\left(W_t^j - W_{t-1}^j, 0\right)\right], \tag{2}$$

[7] In practice, HFs' management fees are paid quarterly or monthly, with a typical structure 2% of the net asset value and 20% of the net profits per annum, for the standard management fee and for the performance fee, respectively (Goetzmann et al. 2003). Given that a time-period in the context of the model corresponds to a fraction of a month or a qurter, it follows that $\gamma \ll 1$.

where $\gamma \ll 1$. The analyst's compensation is going to be one of the three factors which affect the transition from period t to period $t+1$. In particular, the wealth of a HF evolves according to:

$$W_{t+1}^j = W_t^j + (p_{t+1} - p_t)D_t^j - M_{t+1}^j, \tag{3}$$

where the first term of the RHS captures the value of riskless assets, the second term captures the change in the value of the risky assets, whereas the third term captures the compensation of the analysts.

It is worth highlighting that the amount of cash required to complement the trading order for a risky asset, i.e., $D_t^j p_t$, may exceed the cash which is available at the beginning of each trading period. This can be the case because we allow for access to credit. However, this access to credit is not unbounded, and is assumed to be subject to regulation. Here the HF cannot become more leveraged than λ_{max}, a maximum ratio of the market value of the risky asset held as collateral by the bank to the net wealth of the risky asset. Thus, the maximum leverage constraint translates into:

$$D_t^j p_t / W_t^j \leq \lambda_{max}$$

Consequently, the maximum demand for the risky asset is given by:

$$D_{t,\max} = \lambda_{max} W_t^j / p_t, \ \forall j \in \{1, \ldots, K\}. \tag{4}$$

Default: We define as default any event in which the wealth of a HF falls below $W_{\min} \ll W_0$, where W_0 denotes the initial endowment of each HF upon entrance in the market. This enables us to endogenise the probability of default of each HF. The main objective of this paper is to study both the individual (HF) and collective (systemic) default probabilities over time. After $T_r \sim \mathrm{U}[b,c], \ c > b$, time-steps the bankrupt HF is replaced by a HF with identical characteristics. This allows us to maintain the character of the market (at a statistical level).

Noise traders: The second type of traders is noise-traders, who are supposed to trade for liquidity reasons. Following the related literature, we assume that the demand d^{nt} of the representative noise-trader for the risky asset, in terms of cash value, is assumed to follow a first-order autoregressive [AR(1)] process (Xiong 2001, Thurner et al. 2012, Poledna et al. 2014).

$$\log d_t^{\mathrm{nt}} = \rho \log d_{t-1}^{nt} + (1-\rho)\log(VN) + \chi_t, \tag{5}$$

where $\rho \in (0,1)$ is a parameter controlling the rate of reversion to the mean. Given that the expected value of χ_t and the auto-covariance function are time-independent, the stochastic process

is wide-sense stationary, $\chi_t \sim \mathcal{N}(0, \sigma^2_{\text{nt}})$, and V is the fundamental value of the risky asset[8].

Trading orders and Equilibrium prices: Finally, the price of the risky asset is determined endogenously by the market clearing condition [together with Eqs. (5), and (3)—(8)]

$$D^{\text{nt}}_{t+1}(p_{t+1}) + \sum_{j=1}^{K} D^{j}_{t+1}(p_{t+1}) = N, \tag{6}$$

where $D^{\text{nt}}_{t+1}(p_{t+1}) = d^{\text{nt}}_{t+1}/p_{t+1}$ stands for the demand of the noise traders whereas $D^{j}_{t+1}(p_{t+1})$ stands for the demand of the jth HF. Both values are in number of shares.

Source of Heterogeneity: A critical component, which lies in the heart of our analysis, is the heterogeneity across HFs. In particular we allow for a setup where the analysts of all HFs agree on the fundamental value, but they might disagree on the price volatility. The analysts of each HF believe that the behaviour of the noise-traders causes the price of the risky asset to fluctuate about its fundamental value according to a log-normal distribution. The analysts can correctly predict the fundamental value V of the risky asset, and hence the expected value of the logarithmic price at the next period $\log p_{t+1}$, but have different expectations about the volatility of the market. Their prior beliefs about the volatility are modelled by a Gamma probability density function (PDF), i.e. $\text{Var}[\log p_{t+1}] \sim \Gamma(\alpha_j, \beta_j)$, where α_j, β_j are the scale and rate parameters of the PDF[9].

The rationale behind the assumption that the analysts agree on the fundamental value of the asset, but disagree on price volatility, relies on the fact that the fundamental value, as opposed to price volatility, is not affected by the behaviour of HFs. In other words, the fundamental value of the asset is exogenously determined, whereas the volatility of the market is endogenously determined, with its value depending on the HFs' trading strategy, which depends on their analysts' private information. Hence, it is not feasible for the analyst to reach an agreement on the market volatility, because they have access to different information sets, and the market volatility is affected by the information each analyst has access to.

Timing: Each period t consists of 5 sub-periods

1. The analysts inform their managers about the fundamental value of the asset and the volatility of the market.

[8] The demand of the noise traders in terms of the number of shares of the risky asset D^{nt} and the price of the risky asset p_t at period t is $d^{nt}_t = D^{nt}_t p_t$. Hence, In the absence of the HFs, from Eq. (5), and Eq. (6) we have $\mathbb{E}[\log p_{t+1}] = \log V$.

[9] The particular choice for the parametrisation of the prior beliefs of the HFs by a Gamma distribution is inconsequential, since we do not allow HFs to update their beliefs. However, the specific choice has the advantage of yielding a closed-form formula if updating is considered.

2. The managers set their demand orders for the risky asset.
3. The price of the risky asset is determined, and the return of each portfolio is realised.
4. The analysts receive their compensation.
5. The next-period's wealth is determined.

3.2. Optimal Demand

The manager of the jth HF maximises his expected utility, given his beliefs about the asset's fundamental value and the volatility of the market, and subject to the constraint that the demand cannot exceed $D^j_{t,max}$. This is expressed as

$$D^j_t = \underset{D^j_t \in [0, D_{t,\max}]}{\arg\max} \left\{ \mathbb{E}\left[U(W^j_{t+1}) | \alpha_j, \beta_j \right] \right\}. \tag{7}$$

Solving the optimisation problem we obtain[10]

$$D^j_t = \min \left\{ \frac{1}{a} \left(s_j \log\left(V/p_t\right) + \frac{1}{2} \right), \lambda_{\max} \right\} \frac{W^j_t}{p_t}, \tag{8}$$

where $s_j = 1/\mathrm{Var}\left[\log p_{t+1} | \alpha_j, \beta_j\right]$. Therefore, the demand of the HFs is proportional to the expected logarithmic return and their wealth, and inversely proportional to the conditional variance of the logarithm of the price, given their beliefs.

The clustering of HFs' defaults is determined by the decay rate of the of the default time-sequence autocorrelation function (ACF) $C(t')$, with t' being the time-lag variable. If defaults are clustered, then $C(t')$ decays in such a way that the sum of the ACF over the lag variable diverges (Baillie 1996, Samorodnitsky 2006, 2007).

Definition 1. Let $C(t')$ denote the autocorrelation of the time series of defaults, with t' being the lag variable. Defaults are clustered if and only if

$$\sum_{t'=0}^{\infty} C(t') \to \infty. \tag{9}$$

Given that the ACF is bounded in [-1,1], it follows that the convergence of the infinite sum is in turn determined by the asymptotic behaviour $t' \gg 1$ of the ACF. In this limit, the sum can be approximated by an integral.

In the following we assume that the ACF of the default time sequence can be approximated by a continuous function for $t' \gg 1$. Then it follows that,

Remark 1. Defaults are clustered if the ACF asymptotically approaches zero not faster than $C(t') \sim 1/t'$. In this case defaults are interrelated (statistically dependent) for all times.

[10] For details see Appendix A.

Remark 2. If the decay of the ACF is faster than algebraic, then defaults are not clustered. The effect of the shock on the market caused by the default of a HF is only transient, and the defaults are asymptotically statistically independent.

Our main goal is to study the relationship between the degree of heterogeneity κ, identified with the difference between extreme values of s_j, and clustering of defaults. The question arises as to whether the leverage cycle is a sufficient condition for the defaults to be clustered, or rather whether there exists a critical value for the degree of heterogeneity above which the mechanism of the leverage cycle leads to clustering of defaults.

In the next section, we present the results of the model. The first subsection presents the numerical results obtained by iterating the model defined above. We present the ACFs for various values of κ and interpret these in light of Remarks 1 and 2. Section 4.2 provides an analytical insight into the numerical results. Our findings show that the degree of the heterogeneity is indeed the key determinant of the persistence of the effect of a HF's default on the probability of default of the other HFs, as measured by the ACF.

4. Results

Choice of Parameters

In all simulations we consider a market with $K = 10$ HFs. In the following we assume homogeneous preferences towards risk across HFs, and set $a_j = 3.2\ \forall j \in \{1, \ldots, 10\}$, this being a typical value for HFs (Gregoriou et al. 2007, p. 417). From Eq. (5) we have $\tilde{\sigma}^2_{\text{nt}} = \sigma^2_{\text{nt}}/(1-\rho^2)$, where ρ is the mean reversion parameter. The inverse of the expected volatility given the HF's prior beliefs, i.e. $s_j = 1/\text{Var}\left[\log p_{t+1} | \alpha_j, \beta_j\right]$ determines the responsiveness of the HFs to the observed mispricing. In our numerical simulations s_j is sampled from a uniform distribution in $[1, \delta]$, and $\delta \in [1.2, 10]$.

Moreover, the maximum allowed leverage $\lambda_{\max}$ is set to 5. This particular value is representative of the mean leverage across HFs employing different strategies (Ang et al. 2011). The remaining parameters are chosen as follows: $\sigma^2_{\text{nt}} = 0.035$, $V = 1$, $N = 10^9$, $W_0 = 2 \times 10^6$, $W_{\min} = W_0/10$, $\rho = 0.99$ (Poledna et al. 2014), and $\gamma = 5 \times 10^{-4}$. Bankrupt HFs are reintroduced after T_r periods, randomly chosen according to a uniform distribution in $[10, 200]$.

4.1. Numerical results

To assess the effect of the degree of heterogeneity on the persistence of the correlation between defaults, in Figure 1(a) we compare the numerically computed ACF of the default time-sequence[11] as observed on the aggregate level for 11 different degrees of heterogeneity κ, determined by the support of the distribution of s_j. The results were obtained by iterating the model described in

[11] The time-sequence considered is constructed by mapping defaults to 1s, irrespective of which HF defaulted, and to 0 otherwise.

Section 3 for up to 3×10^8 periods, and averaging over 40 realisations of the responsiveness s_j; namely, $s_j \sim \mathrm{U}[1, \delta]$, with $\delta = \{1.2, 1.4, 1.7, 2, 3, 5, 6, \ldots, 10\}$. Clearly, when the degree of heterogeneity $\kappa = \delta - 1 \leq 1$, the ACF decays far more rapidly in comparison with larger values of heterogeneity. In fact, as it can be observed in the figure, the ACF for $\kappa \leq 1$ decays faster than a power-law with exponent equal to -1 (black dashed line), which is the largest exponent (in absolute terms) leading to a non-integrable ACF [see Remark 1]. On the other hand, the converse is true for large degrees of heterogeneity ($\kappa > 2$), in which case the ACF decays asymptotically—$t' \gg 1$—as a power-law with exponent less than 1 in absolute value. Consequently,

Result 1 *For $\kappa \leq 1$, the mechanism of the leverage cycle, does not result into an infinite memory of the underlying stochastic process governing the defaults of the HFs, and the defaults are thus not clustered.*

Fig 1(a) also shows that for increasing heterogeneity the ACF converges to a limiting form as the heterogeneity is increased, which is reflected in the coalescence of the ACFs corresponding to $\kappa \geq 5$. The latter is more clearly demonstrated in Fig 1(b), where a blow-up of the area within the rectangle shown in panel (a) is presented. Therefore,

Result 2 *For sufficiently large values of the degree of heterogeneity κ, namely for $\kappa \geq 5$, the ACF converges to a limiting form exhibiting a power-law trend with an exponent less than 1 (in absolute value).*

To gain some insight into the qualitative difference with respect to the depth of the memory of the underlying stochastic process governing defaults as a function of the degree of heterogeneity κ, let us turn our attention to the default statistics. In Fig. 2 we present the aggregate PDF of waiting times between defaults[12] using a logarithmic scale on both axes for 6 different values of κ. We observe that for small degrees of heterogeneity $\kappa = \{0.2, 0.4, 0.7\}$ the density function asymptotically decays approximately exponentially. This is better demonstrated in the inset were we use semi-logarithmic axes[13]. On the contrary, for sufficiently large heterogeneity—such that the corresponding ACFs have converged to the limiting form—the PDFs exhibit a constant decay rate in the doubly logarithmic plot (power-law tail). Fitting the aggregate density for $\kappa = 9$[14], corresponding to the highest degree of heterogeneity considered, with the model $\tilde{P}(\tau) \sim \tau^{-\zeta}$ we obtain $\zeta = 2.84 \pm 0.03$ (red dashed line).

[12] The PDF of waiting-times between default is also known as the failure function in survival analysis theory.

[13] The use of a logarithmic scale for the vertical axis transforms an exponential function to a linear one.

[14] To increase the accuracy of the fit, we increase the number of realisations of s_j to 10^3.

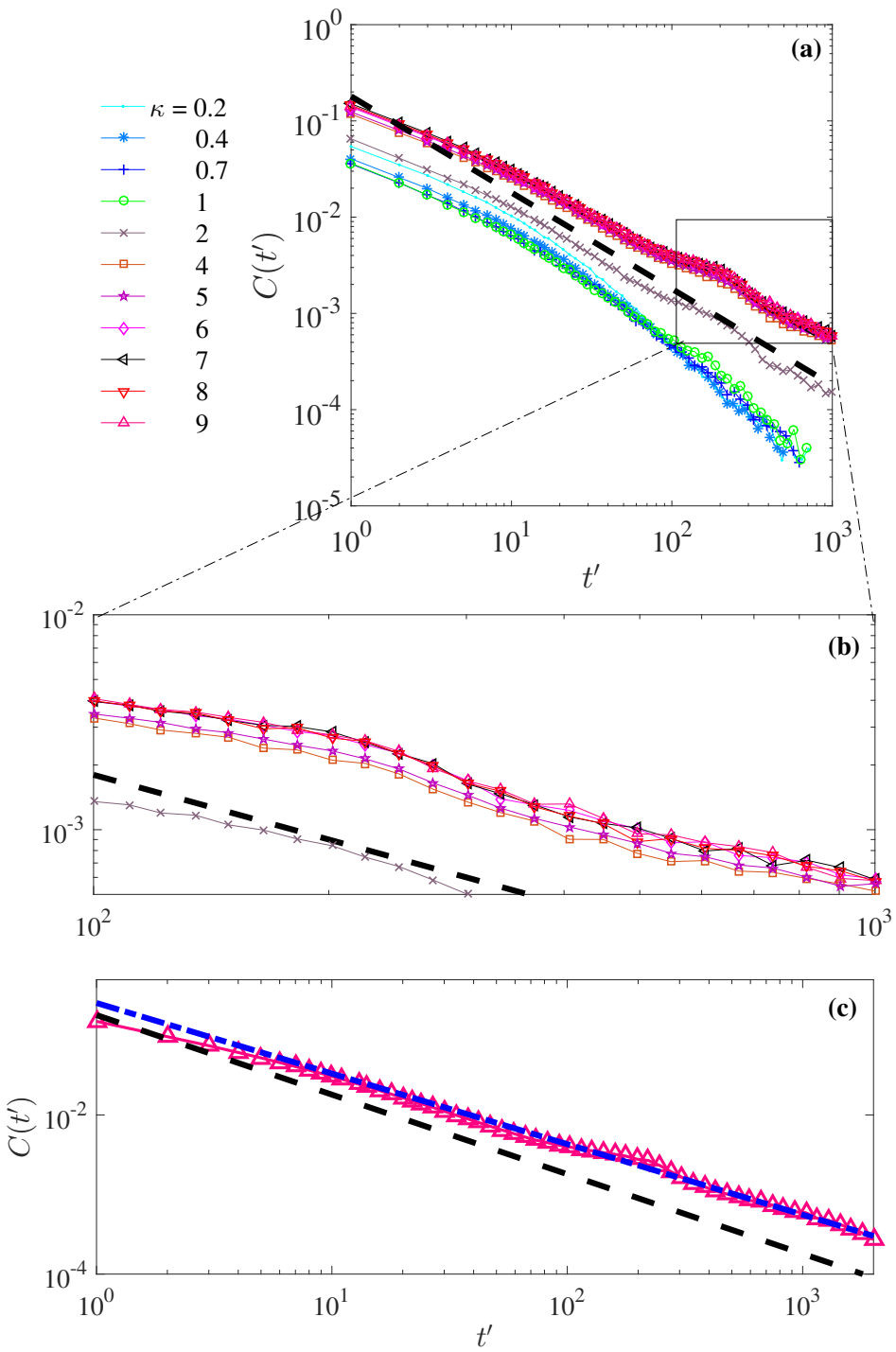

Figure 1 **(a) The ACF of the binary sequence of defaults corresponding to 11 different values of κ. The dashed black line corresponds to a power-law with exponent -1, which is the largest exponent that leads to clustering [see Remark 1].**

Let us now turn our attention to the statistical properties of HFs on a microscopic scale, i.e. study each HF default statistics individually. In Fig. 3 we show as an example the density function $P_j(\tau)$, of waiting times τ between defaults, for a number of HFs corresponding to high heterogeneity,

Figure 1 **(b) A blow-up of the rectangular area shown in panel (a) illustrating the coalescence of $C(t')$ for large values of the degree of heterogeneity, $\kappa = \{6,7,8,9\}$. (c) The ACF corresponding to $\kappa = 9$, averaged over 5×10^2 different realisations of s_j (red upright triangles). The blue dot-dashed line is the result of fitting $C(t')$ with a power-law model $C(t') \propto t'^{-\eta}$, $\eta = 0.887 \pm 0.003$ ($R^2 = 0.9927$). The power-law with exponent -1 is also shown for the sake of comparison (black dashed line).**

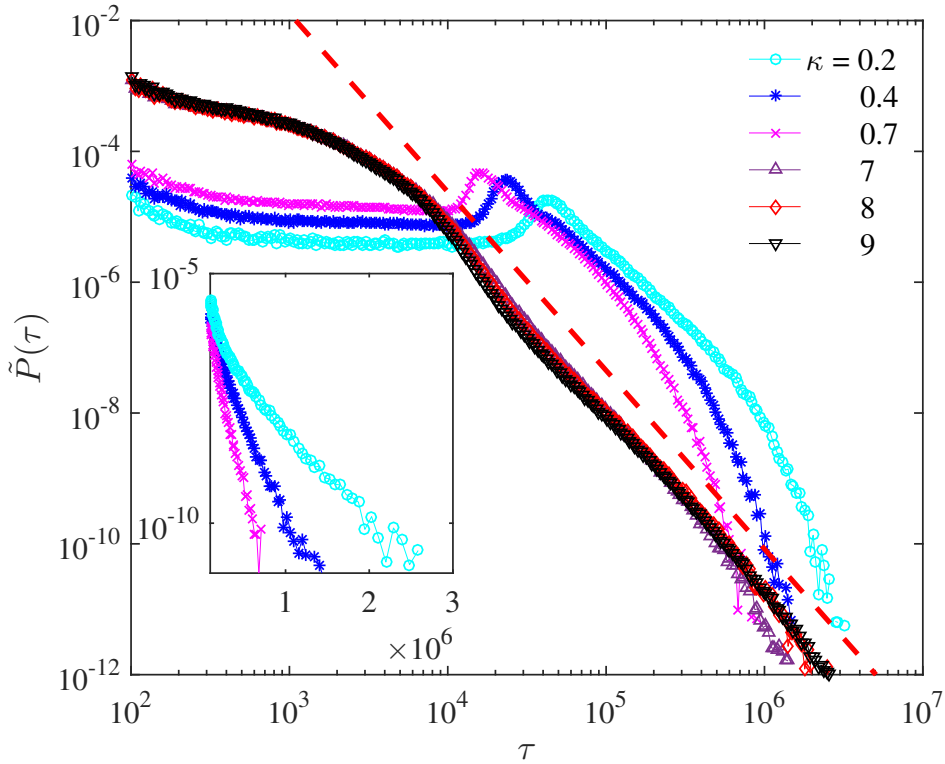

Figure 2 **The aggregate PDF of waiting times between defaults for 6 different degrees of heterogeneity using double logarithmic scale. For large heterogeneity $\kappa = \{7,8,9\}$, we observe that the PDF is decaying approximately linearly, corresponding to a power-law decay. Performing a fit with the model $\tilde{P}(\tau) \sim \tau^{-\zeta}$ we obtain $\zeta = 2.84 \pm 0.03$ ($R^2 = 0.9947$). To illustrate the approximate exponential asymptotic decay of the aggregate PDF for $\kappa = \{0.2, 0.4, 0.7\}$ we also show the corresponding aggregate densities using a logarithmic scale on the vertical axis (inset).**

$\kappa = 9$, with $s_j = \{2,4,6,8,10\}$ on a log-linear scale. The results were obtained by iterating the model for 3×10^8 periods and averaging over 100 different initial conditions[15], holding s_j fixed at $\{1,2,\ldots 10\}$. We observe that $P_j(\tau)$ for $\tau \gg 1$ decays linearly, and thus it can be well described by an exponential function.

Consequently, all HFs on a microscopic level—individually—are characterised by exponential PDFs of waiting-times, and therefore the default events approximately follow a Poisson process. The stability of each HF, quantified by the probability of default per time-step μ_j, is different

[15] We are averaging using different seeds for the pseudo-random number generator used in Eq. (5).

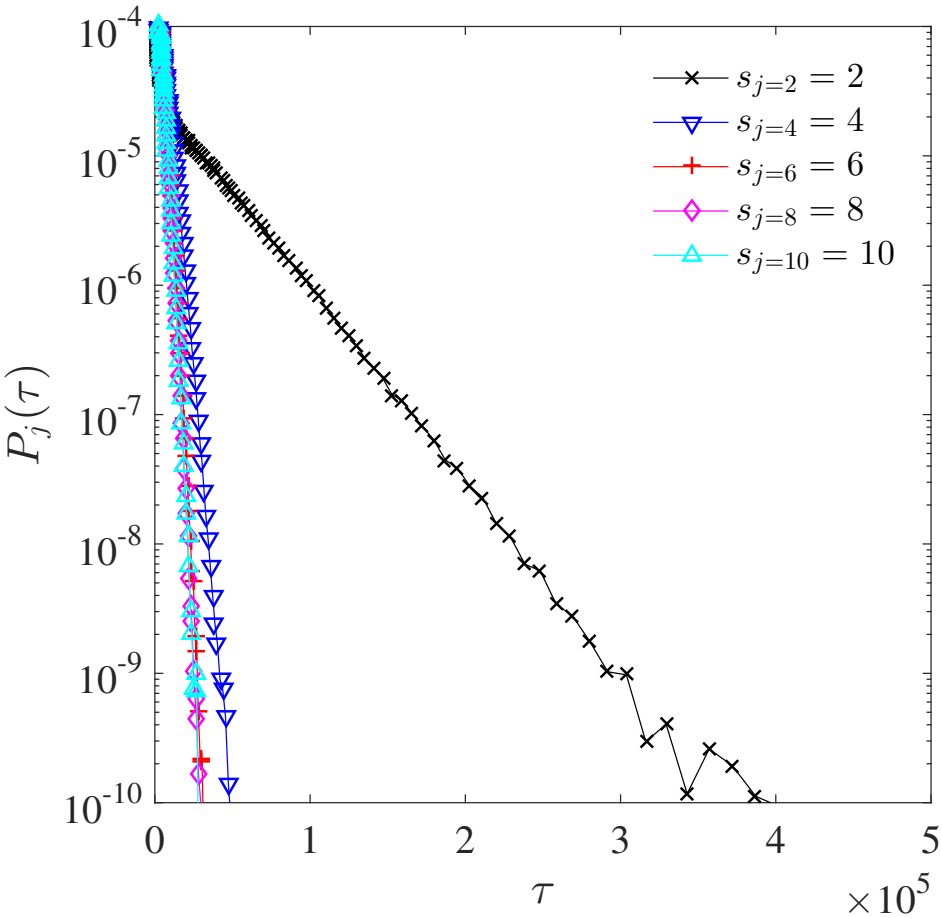

Figure 3 The PDF of waiting times between defaults τ for specific HFs, having different responsiveness $s_j = \{2, 4, 6, 8, 10\}$ (black diagonal crosses, downright triangles, red upright crosses, magenta diamonds and cyan upright triangles, respectively). Note the log-linear scale.

for each HF, and depends on its responsiveness s_j. This is reflected by the different slopes of the approximately straight lines shown in Fig 3 for the different values of s_j.

Thus, the default statistics on an aggregate level are qualitatively different for large values of κ compared to the corresponding ones observed when each HF is studied individually. Moreover we have already established that for such high values of the degree of heterogeneity the defaults are clustered. In the following we will investigate how the emergence of a fat-tail in the aggregate statistics is connected with the observed clustering of defaults.

4.2. Analytical Results

From the numerical results, we observe that $P_j(\tau)$, for $\tau \gg 1$ decays linearly (in log- linear scale) and thus it can be well described by an exponential function. Therefore we can assume that:

$$P_j(\tau; \tau \gg 1) \sim \mu_j \exp(-\mu_j \tau) \ \forall j \ \in \{1, \ldots, 10\} \tag{10}$$

When the above is true, we know that for sufficiently long waiting times between defaults; default events of individual HFs have the following statistical properties: (i) they are approximately independent and (ii) occur with a well defined mean probability per unit time step. From this we get that the probability $P_j(T=\tau)$, $\tau \in \mathbb{N}_+$, is given by a geometric probability mass function (PMF)

$$P_j(\tau) = p_j(1-p_j)^{\tau-1}, \tag{11}$$

where p_j denotes the probability of default of the jth HF.

Given that our focus is in the asymptotic properties of the PDFs, T can be treated as a continuous variable. In this limit, the renewal process given by equation (11), becomes a Poisson process; and the geometric PMF tends to an exponential PDF[16]. Thus equation (10) can be approximated by (11). Consequently, our intuition with respect to the statistical properties of default events when each HF is considered individually is aligned with our numerical findings presented in the previous Section [see Fig 3].

The question then arises as to how the aggregation of these very simple stochastic processes can lead to the non-trivial fat-tailed statistics we observed in Fig 2 for a sufficiently high degree of heterogeneity. Evidently, the aggregate PDF $\tilde{P}(\tau)$ we seek to obtain is a result of the mixing of the Poisson processes governing each of the HFs. In the limit of a continuum of HFs the aggregate distribution is

$$\tilde{P}(\tau) = \int_0^\infty \mu \exp(-\mu\tau)\rho(\mu)d\mu, \tag{12}$$

where $\rho(\mu)$ stands for the PDF of μ given the responsiveness s_j[17].

Assumption 1. *$\rho(\mu)$ in a neighbourhood of 0 can be expanded in a power series of the form $\rho(\mu) = \mu^\nu \sum_{k=0}^{n} c_k \mu^k + R_{n+1}(\mu)$, with $\nu > -1$* [18].

This assumption is quite general, and only excludes functions that behave pathologically in a neighbourhood around 0. Then from equation (10) and Assumption 1 we can show that the aggregation of the exponential densities determining the default statistic for each HF individually leads to a qualitatively different heavy-tailed PDF.

Proposition 1. *Consider the exponential density function $P(\tau;\mu)$ describing the individual default statistics of a HF with a constant mean default rate $\mu \in \mathbb{R}_+$, and statistical weight given by the PDF $\rho(\mu)$, which is determined by the interactions between the agents in the market and the distribution of the responsiveness s. Then from Assumption 1, the aggregate PDF $\tilde{P}(\tau)$ exhibits a power-law tail.*

[16] This limit is valid for $\tau \gg 1$ and $p_j \ll 1$ such that $\tau p_j = \mu_j$, where μ_j is the parameter of the exponential PDF [see equation (10)] (Nelson 1995).

[17] The distribution function of the random parameter μ is also known as the *structure* or *mixing* distribution (Beichelt 2010).

[18] Since $\rho(\mu)$ is a PDF it must be normalisable and thus, a singularity at $\mu = 0$ must be integrable.

T he aggregate density can be viewed as the Laplace transform $\mathcal{L}[.]$ of the function $\phi(\mu) \equiv \mu\rho(\mu)$, with respect to μ. Hence,

$$\tilde{P}(T=\tau) \equiv \mathcal{L}[\phi(\mu)](\tau) = \int_0^\infty \phi(\mu)\exp(-\mu\tau)d\mu. \tag{13}$$

To complete the proof we apply Watson's Lemma (Debnath and Bhatta 2007, p. 171) to the function $\phi(\mu)$, according to which the asymptotic expansion of the Laplace transform of a function $f(\mu)$ that admits a power-series expansion in a neighbourhood of 0 [see Assumption 1] of the form $f(\mu) = \mu^\nu \sum_{k=0}^{n} b_k\mu^k + R_{n+1}(\mu)$, with $\nu > -1$ is

$$\mathcal{L}_\mu[f(\mu)](\tau) \sim \sum_{k=0}^{n} b_k \frac{\Gamma(\nu+k+1)}{\tau^{\nu+k+1}} + O\left(\frac{1}{\tau^{\nu+n+2}}\right). \tag{14}$$

Given that $\phi(\mu)$ for $\mu \to 0_+$ is

$$\phi(\mu) = \mu^{\nu+1}\sum_{k=0}^{n} c_k\mu^k + R_{n+1}(\mu), \tag{15}$$

we conclude that

$$\tilde{P}(\tau) \propto \frac{1}{\tau^{k+\nu+2}} + O\left(\frac{1}{\tau^{k+\nu+3}}\right). \tag{16}$$

Corollary 1. *If $0 < k+\nu \leq 1$, then the variance of the aggregate density diverges (shows a fat tail). However, the expected value of τ remains finite.*

An important aspect of the emergent heavy-tailed statistics stemming from the heterogeneous behaviour of the HFs, is the absence of a characteristic time-scale for the occurrence of defaults (scale-free asymptotic behaviour[19]). Thus, even if each HF defaults according to a Poisson process with intensity $\mu(s)$—which has the intrinsic characteristic time-scale $1/\mu(s)$—after aggregation this property is lost due to the mixing of all the individual time-scales. Therefore, on a macroscopic level, there is no characteristic time-scale, and all time-scales, short and long, become relevant.

This characteristic becomes even more prominent if the density function $\rho(\mu)$ is such that the resulting aggregate density becomes fat-tailed, i.e. the variance of the aggregate distribution diverges. In this case extreme values of waiting times between defaults will be occasionally observed, deviating far from the mean. This will leave a particular "geometrical" imprint on the sequence of default times. Defaults occurring close together in time (short waiting times τ) will be clearly separated due to the non-negligible probability assigned to long waiting times. Consequently, defaults,

[19] If a function $f(x)$ is a power-law, i.e. $f(x) = cx^a$, then a rescaling of the independent variable of the form $x \to bx$ leaves the functional form invariant ($f(x)$ remains a power-law). In fact, a power-law functional form is a necessary and sufficient condition for scale invariance (Farmer and Geanakoplos 2008). This scale-free behaviour of power-laws is intimately linked with concepts such as self-similarity and fractals (Mandelbrot 1983).

macroscopically, will have a "bursty" or intermittent, character, with long quiescent periods of time without the occurrence of defaults and "violent" periods during which many defaults are observed close together in time. Hence, infinite variance of the aggregate density will result in the clustering of defaults.

In order to show this analytically, we construct a binary sequence by mapping time-steps when no default events occur to 0 and 1 otherwise. As we show below, if the variance of the aggregate distribution is infinite, then the autocorrelation function of the binary sequence generated in this manner, exhibits a power-law asymptotic behaviour with an exponent $\beta < 1$. Therefore, the ACF is non-summable and consequently, according to Definition 1 defaults are clustered.

Let T_i, $i \in \mathbb{N}_+$, be a sequence of times when one or more HFs default and assume that the PDF of waiting times between defaults $\tilde{P}(\tau)$, for $\tau \to \infty$, behaves (to leading order) as $\tilde{P}(\tau) \propto \tau^{-a}$. Consider now the renewal process $S_m = \sum_{i=0}^{m} T_i$. Let $Y(t) = 1_{[0,t]}(S_m)$, where $1_A : \mathbb{R} \to \{0,1\}$ denotes the indicator function, satisfying

$$1_A = \begin{cases} 1 : x \in A \\ 0 : x \notin A \end{cases}$$

THEOREM 1. *If the variance of the density function $\tilde{P}(\tau)$ diverges, i.e. $2 < a \leq 3$, then the ACF of $Y(t)$,*

$$C(t') = \frac{\mathbb{E}\left[Y_{t_0} Y_{t_0+t'}\right] - \mathbb{E}\left[Y_{t_0}\right] \mathbb{E}\left[Y_{t_0+t'}\right]}{\sigma_Y^2},$$

where t_0, $t' \in \mathbb{R}$ and σ_Y^2 is the variance of $Y(t)$, for $t \to \infty$ decays as

$$C(t') \propto t'^{2-\alpha} \tag{17}$$

A ssuming that the process defined by $Y(t)$ is ergodic we can express the autocorrelation as,

$$C(t') \propto \lim_{K \to \infty} \frac{1}{K} \sum_{t=0}^{K} Y_t Y_{t+t'}. \tag{18}$$

Obviously, in equation (18) for $Y_t Y_{t+t'}$ to be non-zero, a default must have occurred at both time t and t'[20]. The PDF $\tilde{P}(\tau)$ can be viewed as the conditional probability of observing a default at period t given that a default has occurred $t - \tau$ periods earlier. If we further define $C(0) = 1$ and $\tilde{P}(0) = 0$, the correlation function can then be expressed in terms of the aggregate density as follows:

$$C(t') = \sum_{\tau=0}^{t'} C(t' - \tau) \tilde{P}(\tau) + \delta_{t',0}, \tag{19}$$

where $\delta_{t',0}$ is the Kronecker delta. Since we are interested in the long time limit of the ACF we can treat time as a continuous variable and solve equation (19) by applying the Laplace transform

[20] A detailed exposition of the proof is given in Appendix B.

$\mathcal{L}\{f(\tau)\}(s) = \int_0^\infty f(\tau)\exp(-s\tau)d\tau$, utilising also the convolution theorem (Procaccia and Schuster 1983). Taking these steps we obtain

$$C(s) = \frac{1}{1-\tilde{P}(s)}, \tag{20}$$

where $\tilde{P}(s) = \int_1^\infty \tilde{P}(\tau)\exp(-s\tau)d\tau$, since $\tilde{P}(0) = 0$. After the substitution of the Laplace transform of the aggregate density in equation (20), one can easily derive the correlation function in the Fourier space $\mathcal{F}\{C(t')\}$ by the use of the identity (Jeffrey and Zwillinger 2007, p. 1129),

$$\mathcal{F}\{C(t')\} \propto C(s \to 2\pi i f) + C(s \to -2\pi i f). \tag{21}$$

to obtain ,

$$\mathcal{F}\{C(t')\} \overset{f \ll 1}{\propto} \begin{cases} f^{a-3}, & 2 < a < 3 \\ |\log(f)|, & a = 3 \\ \text{const.}, & a > 3 \end{cases}. \tag{22}$$

Therefore, for $a > 3$ this power spectral density function is a constant and Y_t behaves as white noise. Consequently, if the variance of $\tilde{P}(\tau)$ is finite, then Y_t is uncorrelated for large values of t'.

Finally, inverting the Fourier transform when $2 < a \leq 3$ we find that the autocorrelation function asymptotically ($t' \gg 1$) behaves as

$$C(t') \propto t'^{2-a},\ 2 < a \leq 3. \tag{23}$$

Turning back to the numerical results shown in Fig 2, the aggregate PDF as already discussed converges to a limiting form, characterised by a fat-tail with an exponent equal -2.84 ± 0.03. Therefore, from Eq. (23) we deduce that the ACF should show a power-law trend with exponent -0.84 ± 0.03. The result of the regression of the ACF for $\kappa = 9$ was -0.887 ± 0.003 [blue dashed-dotted line in Fig. 1(c)], in good agreement with the analytical result.

In this Section we have shown that when the default statistics of HFs are individually described by (different) Poisson processes (due to the heterogeneity in the prior beliefs among the HFs) we obtain a qualitatively different result after aggregation: the aggregate PDF of the waiting-times between defaults exhibits a power-law tail for long waiting-times. As shown in Theorem 1, if the relative proportion of very stable HFs approaches 0 sufficiently slowly (at most linearly with respect to the individual default rate μ, as $\mu \to 0$), then long waiting-times between defaults become probable, and as a result, defaults which occur closely in time will be separated by long quiescent time periods and defaults will form clusters. The latter is quantified by the non-integrability of the ACF of the sequence of default times, signifying infinite memory of the underlying stochastic process describing defaults on the aggregate level. It is worth commenting on the fact that the most stable (in terms of defaults) HFs are responsible for the appearance of a fat-tail in the aggregate PDF.

5. Conclusions

This paper studied the role of the heterogeneity in available information among different HFs in the emergence of clustering of defaults. The economic mechanism leading to the clustering of defaults is related to the leverage cycle put forward by Geanakoplos and coauthors. In these models the presence of leverage in a market leads to the overpricing of the collateral used to back-up loans during a boom, whereas, during a recession, collateral becomes depreciated due to a synchronous de-leveraging compelled by the creditors. In the present work we have shown that this feedback effect between market volatility and margin requirements is a necessary, yet not a sufficient condition for the clustering of defaults and, in this sense, the emergence of systemic risk.

We have shown that a large difference in the expectations of the HFs is an essential ingredient for defaults to be clustered. The heterogeneity among HFs in our model, realised as different prior beliefs across HFs about the volatility of the market, leads to the co-existence of many time-scales characterising the occurrence of defaults. This manifests itself in the emergence of scale-free (heavy tailed) statistics on the aggregate level. We show, that this scale-free character of the aggregate survival statistics, when combined with large fluctuations of the observed waiting-times between defaults, i.e. infinite variance of the corresponding aggregate PDF, leads to the presence of infinite memory in the default time sequence. Consequently, the probability of observing a default of a HF in the future is much higher if one (or more) is observed in the recent past, and as such, defaults are clustered. Therefore, our work shows that individual stability can lead to market-wide risk

This work raises several interesting questions, which we aim to address in the future. In this paper we have assumed that the difference in beliefs is due to disagreement about the long-run volatility of the risky asset, and remains constant over time, i.e. the agents do not update their beliefs given their observations. This assumption is crucial in order to be able to analyse the effects of different degrees of heterogeneity. Regarding this issue, future work can take two different directions: On the one hand, this assumption can be relaxed, allowing agents to update their beliefs on market volatility. However, given that market volatility is endogenous, it is not guaranteed that agents' beliefs will convergence. On the other hand, we can study the effects of heterogeneity stemming from different aversion to risk among the HFs, while retaining the common prior assumption. Furthermore, these two approaches can be combined by assuming both different aversion to risk, and different beliefs about price volatility. Finally, our work can also be extended in two further directions. The first being to give a more active role to the bank which provides loans, while the second is to study the effects of different regulations on credit supply.

Appendix A: Optimal demand

We seek to determine the optimal demand for each of the HFs given their beliefs about price volatility $q_j \sim \Gamma[a_j, \beta_j]$. This translates into the optimisation problem, assuming log-normal returns on the risky asset

$$\underset{D_t^j \in [0, D_{t,\max}]}{\arg\max} \left\{ \mathbb{E}\left[U(W_{t+1}^j) | q_j \right] \right\}, \tag{1}$$

where $U\left(W_{t+1}^j\right) = {W_{t+1}^j}^{1-a}/(1-a) \sim {W_{t+1}^j}^{1-a}$, and W_{t+1}^j is the wealth of the jth HF at the next period. To simplify the notation, in the following we will assume that the expected value is always conditioned on HF's prior beliefs, and moreover, we will drop the superscript j. Eq. (1) is equivalent to the maximisation of the logarithm of the expected utility. Furthermore, given that returns are log-normally distributed, it follows that (Campbell and Viceira 2002, pp. 17-21)

$$\log \mathbb{E}\left[{W_{t+1}}^{1-a} \right] = \mathbb{E}\left[\log {W_{t+1}}^{1-a} \right] + \frac{\mathrm{Var}\left[\log {W_{t+1}}^{1-a} \right]}{2} \tag{2}$$

Consequently, the problem becomes

$$\underset{D_t \in [0, D_{t,\max}]}{\arg\max} \left\{ (1-a)\, \mathbb{E}\left[\log W_{t+1} \right] + (1-a)^2 \frac{\mathrm{Var}\left[\log W_{t+1} \right]}{2} \right\}. \tag{3}$$

The wealth of the jth HF at the next period is

$$W_{t+1} = (1 + x_t R) W_t, \tag{4}$$

where x is the fraction of its wealth invested into the risky asset, and R the (arithmetic) return of the portfolio. Re-expressing Eq. (4) in terms of the logarithmic returns r we get

$$\log\left(W_{t+1}\right) = \log W_t + \log\left[1 + x_t \left(\exp(r_{t+1}) - 1\right)\right], \tag{5}$$

albeit a transcendental equation with respect to r. An approximative solution can be obtained by performing a Taylor expansion of Eq (5) with respect to r to obtain

$$\log W_{t+1} = \log W_t + x_t r \left(1 + \frac{r_t}{2}\right) - \frac{{x_t}^2}{2} r_t^2 + \mathcal{O}\left(r^3\right). \tag{6}$$

Substituting Eq. (6) into Eq. (3), and furthermore approximating $\mathbb{E}(r^2)$ with $\mathrm{Var}(r)$ we obtain

$$\underset{D_t \in [0, \lambda_{\max}]}{\arg\max} \left\{ \log W_t + x_t \mathbb{E}(r_t) + \frac{x_t}{2}(1 - x_t)\mathrm{Var}(r_t) \right\}. \tag{7}$$

Finally the first-order condition yields

$$x_t = \min\left[\frac{\mathbb{E}(r_t) + \frac{1}{2} a \mathrm{Var}(r_t)}{a \mathrm{Var}(r_t)}, \lambda_{\max} \right] \tag{8}$$

Consequently, the optimal demand for HF j in terms of the number of shares of the risky asset given the price at the current period is

$$D_t = \min\left\{ \frac{\log(V/p_t) + \frac{1}{2} a \mathrm{Var}\left[\log p_{t+1} | \alpha_j, \beta_j\right]}{a \mathrm{Var}\left[\log p_{t+1} | \alpha_j, \beta_j\right]}, \lambda_{\max} \right\} \frac{W_t}{p_t}. \tag{9}$$

As a last step we need to calculate the conditional variance of the logarithmic price. The prior predictive density is

$$\rho(p_t | \alpha_j, \beta_j) = \int_0^\infty L(p_t | q_j) \Gamma(q_j | \alpha_j, \beta_j) dq_j, \tag{10}$$

where

$$\Gamma_{\alpha_j,\beta_j}(q_j) = \frac{{\beta_j}^{\alpha_j}}{\Gamma(\alpha_j)} q_j^{\alpha_j - 1} \exp\left(-\beta_j q_j\right), \tag{11a}$$

and

$$L(p_t|q_j) = \sqrt{\frac{q_j}{2\pi}} \exp\left[-q_j \frac{(\log p_t - \log V)^2}{2}\right] / p_t. \tag{11b}$$

Thus,

$$\rho(p_t|\alpha_j,\beta_j) = \frac{2^a \beta^a \Gamma\left(a+\frac{1}{2}\right)\left[2\beta + (\log p_t - \log V)^2\right]^{-a-\frac{1}{2}}}{\sqrt{\pi} p_t \Gamma(a)}. \tag{12}$$

Assuming without loss of generality that $\alpha_j > 1$, it follows that

$$\mathbb{E}\left[\log p_{t+1}|\alpha_j,\beta_j\right] = \log V, \tag{13a}$$

as expected, and

$$\mathrm{Var}\left[\log p_{t+1}|\alpha_j,\beta_j\right] = \frac{\beta_j}{\alpha_j - 1}. \tag{13b}$$

Appendix B: Proof of theorem 1

As already stated in Section 4.2, Theorem 1, assuming that the process defined by $Y(t) = 1_{[0,t]}(S_m)$ is ergodic, the auto-correlation function can be expressed as a time-average

$$C(t') \propto \lim_{K\to\infty} \frac{1}{K} \sum_{t=0}^{K} Y_t Y_{t+t'}. \tag{1}$$

Given that $Y(t)$ is by definition a binary variable, the only non-zero terms contributing to the sum appearing on the right hand side (RHS) of equation (1) correspond to default events (mapped to 1) that occur with a time difference equal to t'. Therefore, the RHS of equation (1) is proportional to the conditional probability of observing a default at time t', given that a default has occurred at time $t = 0$. Therefore, we can express $C(t')$ in terms of the aggregate probability $\tilde{P}(\tau = t')$, i.e. the probability of a default event being observed after t' time-steps, given that one has just been observed. Moreover, we must take into account all possible combinations of defaults happening at times $t < t'$. For example, let us assume that we want to calculate $C(t' = 2)$. In this case there are exactly 2 possible set of events that would give a non-zero contribution. Either a default happening exactly 2 time-steps after the last one (at $t = 0$), or two subsequent defaults happening at $t = 1$, and $t = 2$. In this fashion, we can express the correlation function in terms of the probability the waiting-times between defaults as (Procaccia and Schuster 1983),

$$C(1) = \tilde{P}(1), \tag{2}$$

$$\begin{aligned} C(2) &= \tilde{P}(2) + \tilde{P}(1)\tilde{P}(1) \\ &= \tilde{P}(2) + \tilde{P}(1)C(1), \end{aligned} \tag{3}$$

$$\vdots$$

$$C(t') = \tilde{P}(t') + \tilde{P}(t'-1)C(1) + \ldots \tilde{P}(1)C(t'-1). \tag{4}$$

If we further define $C(0) = 1$ and $\tilde{P}(0) = 0$, then equation (4) can be written more compactly as

$$C(t') = \sum_{\tau=0}^{t'} C(t'-\tau)\tilde{P}(\tau) + \delta_{t',0}, \tag{5}$$

where $\delta_{t',0}$ is the Kronecker delta.

We are interested only in the long time limit of the ACF. Hence, we can treat time as a continuous variable and solve equation (5) by applying the Laplace transform $\mathcal{L}\{f(\tau)\}(s) = \int_0^\infty f(\tau)\exp(-s\tau)d\tau$, utilising also the convolution theorem .Taking these steps we obtain

$$C(s) = \frac{1}{1-\tilde{P}(s)}, \tag{6}$$

where $\tilde{P}(s) = \mathcal{L}\left\{\tilde{P}(\tau)\right\}(s) = \int_0^\infty \tilde{P}(\tau)\exp(-s\tau)d\tau$. We will assume that $\tilde{P}(\tau) \propto \tau^{-a}$ for any $\tau \in [1,\infty)$, i.e. the asymptotic power-law behaviour ($\tau \gg 1$) will be assumed to remain accurate for all values of τ. Under this assumption,

$$\tilde{P}(\tau) = \begin{cases} A\tau^{-a}, & \tau \in [1,\infty), \\ 0, & \tau \in [0,1). \end{cases}, \tag{7}$$

where $A = 1/\int_1^\infty \tau^{-a}d\tau = a-1$. The Laplace transform of equation (7) is,

$$\tilde{P}(s) = (a-1)E_a(s), \tag{8}$$

where $E_a(s)$ denotes the exponential integral function defined as,

$$E_a(s) = \int_1^\infty \exp(-st)\,t^{-a}dt \;/;\; \mathrm{Re}(s) > 0. \tag{9}$$

The inversion of the Laplace transform after the substitution of equation (8) in equation (6) is not possible analytically. However, we can easily derive the correlation function in the Fourier space (known as the power spectral density function) $\mathcal{F}\{C(t')\}(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty C(t')\cos(2\pi f)dt'$ by the use of the identity (Jeffrey and Zwillinger 2007, p. 1129),

$$\mathcal{F}\{C(t')\} = \frac{1}{\sqrt{2\pi}}\left[C(s \to 2\pi i f) + C(s \to -2\pi i f)\right]. \tag{10}$$

relating the Fourier cosine transform $\mathcal{F}\{g(t)\}(f)$, of a function $g(t)$, to its Laplace transform $g(s)$, to obtain,

$$C(f) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-(a-1)E_a(2if\pi)} + \frac{1}{1-(a-1)E_a(-2if\pi)}\right) \tag{11}$$

From equation (11) we can readily see that as $f \to 0_+$ (equivalently $t' \to \infty$), $C(f) \to \infty$. To derive the asymptotic behaviour of $C(f)$ we expand about $f \to 0_+$ (up to linear order) using

$$E_a(2if\pi) = ai^{a+1}(2\pi)^{a-1}f^{a-1}\Gamma(-a) - \frac{2i\pi f}{a-2} + \frac{1}{a-1} + O(f^2) \tag{12}$$

to obtain

$$\begin{aligned} C(f) = &-\frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + (2^{a+1}\pi^a(if)^a - a(2i\pi)^a f^a)\,\Gamma(2-a)} \\ &+ \frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + (2^{a+1}\pi^a(-if)^a - a(-2i\pi)^a f^a)\,\Gamma(2-a)}. \end{aligned} \tag{13}$$

After some algebraic manipulation, for $f \to 0$ equation (13) yields

$$C(f) = Af^{a-3}, \tag{14}$$

where

$$A = -\frac{2^{a+\frac{1}{2}}(a-2)^2\pi^{a-\frac{3}{2}}\sin\left(\frac{\pi a}{2}\right)\Gamma(1-a)}{(a-1)}. \tag{15}$$

Therefore, for $2 < a < 3$ we see that the Fourier transform of the correlation function behaves as,

$$C(f) \propto f^{a-3}. \tag{16}$$

If $a = 3$, then the instances of the Gamma function appearing on the RHS of equation (13) diverge. Therefore, for $a = 3$ we need to use a different series expansion around $f \to 0_+$. Namely,

$$E_3(2\pi i f) = \frac{1}{2} - 2i\pi f + \pi^2 f^2 (2\log(2i\pi f) + 2\gamma - 3) + O\left(f^5\right), \tag{17}$$

where γ stands for the Euler's constant. The substitution of equation (17) into equation (11) leads to

$$\begin{aligned} C(f) = -\operatorname{Re}\Big\{ &\left[2\log(\pi f) - 2\gamma + 3 - \log(4)\right] \Big/ \Big[\sqrt{2\pi}(2i\pi f \log(\pi f) \\ &+ \pi f(2i\gamma + \pi + i(\log(4) - 3)) - 2) \\ &\times (\pi(3i - 2i\gamma + \pi) f - 2i\pi f \log(2\pi f) - 2)\Big] \Big\}, \end{aligned} \tag{18}$$

and thus,

$$\begin{aligned} C(f) = \Big(&-8\gamma^3\pi^2 f^2 - 2\pi^2 f(f(-6\log(\pi)(\log\left(16\pi^3\right) \\ &- 2\gamma\log\left(4\pi f^2\right)) + \left(12\gamma^2 + \pi^2\right)\log(\pi f) + 9(3 - 4\gamma)\log(2\pi f)) \\ &+ 4f\log^3(f) + 6f(2\gamma - 3 + \log(4) + 2\log(\pi))\log^2(f) \\ &+ 6f\left(\gamma\log(16) + (\log(2\pi) - 3)\log\left(4\pi^2\right)\right)\log(f) + 4f\log(2\pi)((\log(2) - 3)\log(2) \\ &+ \log(\pi)\log(4\pi)) - 4\log(2\pi f)) - 4\gamma^2\pi^2 f^2(\log(64) - 9) - 2\gamma(\pi^2 f(f\left(\pi^2 + 27 + 12\log^2(2)\right) \\ &- 4) + 4) + \pi^2 f\left(f\left(27 - \pi^2(\log(4) - 3) + \log(8)\log(16)\right) - 12\right) - 8\log(2\pi f) + 12\Big) \\ &\Big/ \Big(\sqrt{2\pi}(4\pi^2 f^2 \log(\pi f)(\log(4\pi f) + 2\gamma - 3) + \pi^2 f(f(4\gamma^2 + \pi^2 + (\log(4) - 3)^2 \\ &+ 4\gamma(\log(4) - 3)) - 4) + 4)^2\Big). \end{aligned} \tag{19}$$

As $f \to 0$ we have,

$$C(f) \sim |\log(f)| \tag{20}$$

Finally, if $a > 3$, then equation (11) for $f \to 0$ tends to a constant, and thus, Y_t behaves as white noise. Consequently, if the variance of $\tilde{P}(\tau)$ is finite, then Y_t is for large values of t' is uncorrelated.

To summarise, the spectral density function for $f \ll 1$ is,

$$C(f) \overset{f \ll 1}{\propto} \begin{cases} f^{a-3}, & 2 < a < 3 \\ |\log(f)|, & a = 3 \\ \text{const.}, & a > 3 \end{cases}. \tag{21}$$

The inversion of the Fourier (cosine) transform in equation (21) yields,

$$C(t') \propto t'^{2-a}/;\ 2 < a \leq 3 \wedge t' \gg 1. \tag{22}$$

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