Understanding Coleman’s Theory of Integration
Marc Masdeu-Sabaté
April 28, 2008

1 Results in \( p \)-adic analytic geometry

1.1 Affinoids
Consider \( \mathbb{C}_p \), and fix an absolute value \(|·|\) on it. Fix \( K \subseteq \mathbb{C}_p \) a complete subfield. Write \( R \) for the maximal order of \( K \), and \( p \) for the maximal ideal of \( R \). Let \( \mathbb{F} = R/p \) be the residue field of \( K \) (\( \mathbb{F} \) is an algebraic extension of \( \mathbb{F}_p \)).

Given an affinoid \( X \) over \( K \), denote by \( A(X) \) the algebra of rigid analytic functions on \( X \) over \( K \). We have that \( \text{Sp} A(X) = X \).

For \( f \in A(X) \), and \( x \in X \), write \( |f(x)| \) for the absolute value of the image of \( f \) in \( A(X)/x \). Set also:
\[
\|f\|_X \overset{\text{def}}{=} \sup_{x \in X} |f(x)|
\]
and define \( A_0(X) \overset{\text{def}}{=} \{ f \in A(X) \mid \|f\|_X \leq 1 \} \).

We have that \( \|f\|_X \) is a seminorm (called the spectral norm) on \( A(X) \), and that \( A_0(X) \) is a sub-\( R \)-algebra of \( A(X) \). The spectral norm is a norm if \( X \) is reduced, and then \( A(X) \) is complete with respect to this norm.

Set also \( \tilde{A}(X) \overset{\text{def}}{=} A_0(X)/pA_0(X) \), and \( \tilde{X} \overset{\text{def}}{=} \text{Spec} \tilde{A}(X) \). Then \( \tilde{X} \) is a scheme of finite type over \( \mathbb{F} \) if \( A_0(X) \) is of topological finite type over \( R \) (true if \( K = \mathbb{C}_p \), or \( K \) a DVR). In general, \( \tilde{X} \text{red} \) is of finite type.

Definition 1.1. We say that \( X \) has good reduction if \( A_0(X) \) is of topological finite type over \( R \) and \( \tilde{X} \) is smooth over \( \mathbb{F} \).

Lemma 1.2. Suppose that \( r \colon Y \to X \) is a morphism of affinoids over \( K \), such that the image of \( \tilde{Y} \) is a dense open subset of \( \tilde{X} \). Let \( f \in A(X) \). Then \( \|f\|_X = \|f \circ r\|_Y \).

Definition 1.3. A Tate \( R \)-algebra is an \( R \)-algebra of the form
\[
R\langle x_1, \ldots, x_n \rangle/I
\]
for some finitely-generated ideal \( I \) of \( R\langle x_1, \ldots, x_n \rangle \), the ring of restricted power series in \( x_1, \ldots, x_n \) (which is actually the completion of \( R[x_1, \ldots, x_n] \) over \( R \)).

Definition 1.4. The annihilator in \( A \) of \( r \in R \) is:
\[
\text{Ann}_A(r) \overset{\text{def}}{=} \{ a \in A \mid ra = 0 \}
\]

Definition 1.5. Given a homomorphism \( A \to B \) of Tate \( R \)-algebras, we say that \( B \) is \( R \)-torsion free over \( A \) if:
\[
\text{Ann}_B(r) = \text{Ann}_A(r) \cdot B
\]
for all \( r \in R \).
Let any Tate $R$-algebra $A$, we set $\tilde{A} \overset{\text{def}}{=} A/pA$.

**Definition 1.6.** We say that $B$ is **formally smooth over** $A$ if $\tilde{B}$ is smooth over $\tilde{A}$ and $B$ is $R$-torsion free over $A$.

The proof of the following statement is omitted, as it is not needed in the sequel.

**Proposition 1.7.** The following are equivalent:

1. $B$ is formally smooth over $A$,
2. $\tilde{B}$ is smooth over $\tilde{A}$ and $B$ is flat over $A$,
3. $B/rB$ is smooth over $A/rA$ for all $r \in R$.

The following theorem is what is needed to prove Theorem 1.12.

**Theorem 1.8.** Suppose that there is a commutative diagram of Tate $R$-algebras

\[
\begin{array}{ccc}
D & \leftarrow & B \\
\uparrow & & \uparrow \\
C & \leftarrow & A
\end{array}
\]

such that $\tilde{C} \rightarrow \tilde{D}$ is surjective and $B$ is formally smooth over $A$. Suppose that there is a homomorphism $s: \tilde{B} \rightarrow \tilde{C}$ making the reduction of the diagram commutative. Then there is a lifting $\overline{s}: B \rightarrow C$ of $s$ which makes the original diagram commutative.

**Proof.** We will proceed by proving several lemmas that will patch together to get our result.

**Lemma 1.9.** Suppose that $A \rightarrow B$ is a surjective homomorphism of Tate $R$-algebras. Then its kernel is finitely generated.

**Proof.** First, note that WLOG can assume that $A = R_m$ (in general, $A$ is a quotient of it, so there is no harm in replacing it). The hypothesis says that $B$ is a quotient of $R_n$ (for some $n \geq 0$), with finitely generated kernel $J$:

\[
0 \rightarrow J \rightarrow R_n \rightarrow B \rightarrow 0
\]

Let now $h: R_n \rightarrow R_m$ be a homomorphism such that the following commutes:

\[
\begin{array}{ccc}
R_n & \rightarrow & B \\
\downarrow & & \downarrow \\
R_m & \rightarrow & B
\end{array}
\]

Take now $x'_1, \ldots, x'_m$ lifts (in $R_n$) of the images of $x_1, \ldots, x_m \in R_n$ in $B$. Then the kernel of $R_m \rightarrow B$ is generated by $h(J)$, together with the set of $\{x_i - h(x'_i) \mid 1 \leq i \leq m\}$, so it’s finitely generated. \(\square\)

As $B$ is **topologically of finite type** over $R$, a fortiori it is so over $A$. Hence there is a surjection $A_n \rightarrow B$, for some $n$. By the previous lemma, there exist $G_1, \ldots, G_m \in A_n$, such that

\[
B \simeq A_n/(G_1, \ldots, G_m)
\]

as an $A$-algebra.

Let $G \overset{\text{def}}{=} (G_1, \ldots, G_m) \in A_n^m$, and let $g$ denote the composition $A_n \rightarrow B \rightarrow D$, and $\tilde{V}$ the composition $\tilde{A}_n \rightarrow \tilde{B} \rightarrow \tilde{C}$. The fact that $\tilde{C} \rightarrow \tilde{D}$ is surjective implies that $\tilde{C} \rightarrow \tilde{D}$, and so $D \simeq C/I$, for some ideal $I$. 

2
Lemma 1.10. There exists a $V : A_n \to C$ which lifts $\tilde{V}$, and such that $V \equiv g \pmod{I}$.

![Diagram]

Proof. As $C \to D$, there exists a hom. $g' : A_n \to C$ such that

$$g' \equiv g \pmod{I}$$

(just take, if $X = (x_1, \ldots, x_n)$, $g'(X)$ to be any lift of $g(X)$, and extend to all $A_n$ in the natural way). In the same way, $\tilde{V}$ can be lifted to $V' : A_n \to C$. Then one has:

$$V'(X) - g'(X) \in (p + I)^n \subseteq C^n$$

Now let $a \in p^n \subseteq C^n$, and $b \in I^n \subseteq C^n$ be such that:

$$V'(X) - g'(X) = a - b$$

Set then $d \overset{\text{def}}{=} V'(X) - a = g'(X) - b$, and clearly $\tilde{d} = \tilde{V}$, and $d \equiv g \pmod{I}$.

Hence we may take $V$ to be the unique homomorphism $A_n \to C$ such that $V(X) = d$. □

The homomorphism $V$ is a first approximation to the lifting we are after.

We need to construct a sequence of approximations that tends to our desired lift. As our algebra is complete, we will be then get the lift by taking the limit.

Lemma 1.11. There exists an $n \times m$ matrix $N$, and $m \times m$ matrices $M$ and $Q$ over $A_n$ such that:

$$G(X + NG) = G^tMG + QG$$

where the coordinates of $Q$ are in $pA_n$.

Let now $V_0 = V$, and define recursively $V_k$ by setting

$$V_{k+1}(X) \overset{\text{def}}{=} V_k(X) + N(V_k(X)) G(V_k(X))$$

As $V_{k+1}(X) \in C^m$, it determines a unique homomorphism $V_{k+1} : A_n \to C$. From the previous lemma, $V_{k+1} - V_k \to 0$. As Tate algebras are complete, the limit of these will do. □

1.2 Lifting Morphisms

If $h : X \to Y$ is a morphism of affinoids over $K$, denote by $\tilde{h} : \tilde{X} \to \tilde{Y}$ its reduction. Given $\tilde{h}$, we say that $h$ lifts $\tilde{h}$.

Theorem 1.12. Suppose $K = \mathbb{C}_p$ or $K$ a DVR. Suppose that there is a commutative diagram of reduced affinoids over $K$:

![Diagram]
such that $W \to Y$ is a closed immersion, and $\tilde{X}$ is smooth over $\tilde{Z}$. Suppose that $h: \tilde{Y} \to \tilde{X}$ is a morphism commuting with the reduction of the given diagram. Then there is a lifting $\tilde{h}: Y \to X$ of $h$ commuting with the diagram:

\[
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
\]

Suppose now that $S$ is a scheme over a field $F$, and $\sigma: F \to F$ is an automorphism of $F$. Let $S^\sigma$ be the scheme over $F$ obtained by base change via $\sigma$. We have a commutative diagram:

\[
\begin{array}{ccc}
S^\sigma & \longrightarrow & S \\
\downarrow & & \downarrow \\
\text{Spec}(F) & \longrightarrow & \text{Spec}(F)
\end{array}
\]

Given a form (a function or a differential) $f$ on $S$, one denotes by $f^\sigma$ its pullback via $\sigma$ to $S^\sigma$. The resulting map $f \mapsto f^\sigma$ is $\sigma$-linear (but not linear, in general).

Let now $X$ be an affinoid over $F = K$, and let $S = \text{Spec}(A(X))$ (over $\text{Spec}(K)$), and let $\sigma$ be a continuous automorphism of $K$. Let then $X^\sigma$ be the affinoid characterized by $\text{Spec}(A(X^\sigma)) = S^\sigma$, as schemes over $K$.

Next, consider the case $F = \mathbb{F}$, and let $\sigma$ be the Frobenius automorphism of $\mathbb{F}$. For each $n \in \mathbb{Z}^+$, there is a canonical morphism $\phi: S \to S^\sigma^n$, called the Frobenius morphism, and characterized by the equation $\phi^* f^\sigma^n = f^{p^n}$ (for $f \in \mathcal{O}_S(U)$). If $S$ is of finite type over $\mathbb{F}$, then there exists some positive integer $n$ such that $S \simeq S^\sigma^n$. Fix an $F$-isomorphism $\rho: S^\sigma^n \to S$, and then the morphism $\rho \circ \phi: S \to S$ is called a Frobenius endomorphism of $S$.

Now suppose that $X$ is an affinoid over $K$, and $\sigma$ is a continuous automorphism of $K$, which restricts to the Frobenius automorphism $\sigma$ on $\mathbb{F}$. As $\tilde{X}$ is of finite type over $\mathbb{F}$, then $\tilde{X}$ has Frobenius endomorphisms, and an endomorphism of $X$ lifting one of those is called a Frobenius endomorphism of $X$. Such an endomorphism is actually $K$-linear (and not just $\sigma$-linear). We have the following corollary:

**Corollary 1.13.** Suppose that $X$ is a reduced affinoid over $K$ with good reduction. Then:

1. $X$ possesses a Frobenius endomorphism.
2. There is a morphism from $X$ to $X^\sigma$ lifting the robenius morphism $\tilde{X} \to \tilde{X}^\sigma$.
3. $X \simeq X^\sigma^n$ for some positive integer $n$.

Let $X$ be as in the previous corollary, and let $\phi$ be a Frobenius endomorphism of $X$. In each residue class $U$ of $X$ there is a unique point $\varepsilon_U$ such that

\[
\phi^m(\varepsilon_U) = \varepsilon_U
\]

for some positive integer $m$. It can be computed/defined as: think of $U$ as a point in $\tilde{X}(\overline{\mathbb{F}})$ (over the algebraic closure of $\mathbb{F}$). Then there is some $m$ such that $\tilde{\phi}^m(U) = U$, because $U$ is defined over some finite extension of $\mathbb{F}_p$. Then

\[
\varepsilon_U = \lim_{n \to \infty} \phi^{mn}(x)
\]

for any $x \in U$. This point $\varepsilon_U$ is called a **Teichmüller point** of $\phi$. 


1.3 Differentials

Suppose that $X$ is an affinoid over $K$. Let $\Omega^1_{X/K}$ be the module of rigid differentials on $X$, and $d: A(X) \to \Omega^1_{X/K}$ the natural derivation. We define $\Omega^i_{X/K}$ as the $i$-th exterior power of $\Omega^1_{X/K}$.

If $W$ is any rigid space over $K$, we can make a natural complex of rigid sheaves $(\Omega^\bullet_{W/K}, d)$ on $W$. A closed differential will then be an element $\omega \in H^0(W, \Omega^1_{W/K})$ such that $d\omega = 0$.

**Proposition 1.14.** Let $X$ be a connected reduced affinoid with good reduction over $K$. Let $D = \text{red}^{-1} \Delta$, where $\Delta$ is the diagonal in $\tilde{X} \times \tilde{X}$. Then $D$ has a natural structure of rigid analytic space, and we let $A(D)$ be the ring of rigid analytic functions on $D$. Consider $p_1, p_2: D \to X$ the two natural projections. Suppose that $\omega$ is closed on $X$. Then:

$$p_1^*\omega - p_2^*\omega \in dA(D)$$

**Proof.** Let $C$ be a cover of $\tilde{X}$ by affine opens such that $Y \in C$ may be expressed as a finite unramified covering of an affine open subset of $\mathbb{A}^d_K$, where $d$ is the dimension of $X$. For each $Y \in C$, the inverse image $\tilde{Y} = \text{red}^{-1} Y$ has a natural structure of an affinoid over $K$ such that $\tilde{\tilde{Y}} = Y$.

Fix $Y \in C$. There exist functions $\tilde{x}_1, \ldots, \tilde{x}_n$ on $Y$, which are local parameters at each point of $Y$ (by how we are taking our $Y$). Let $x_1, \ldots, x_n$ be liftings to $\tilde{Y}$. Then $x_1, \ldots, x_n$ are also local parameters everywhere on $\tilde{Y}$. So we may write:

$$\omega = f_1 dx_1 + \cdots + f_n dx_n$$

for some $f_i \in A(Y)$.

The idea of the proof can be seen in the case $n = 1$. In that case, write $\omega = f(x)dx$, and then $p_1^*\omega - p_2^*\omega = f(x)dx - f(y)dy$. We want to “integrate” this. So let $h \overset{\text{def}}{=} x - y$, and rewrite the previous expression as $f(y + h)d(y + h) - f(y)dy$. Now expand $f$ around $y$, noting that $h$ is divisible by $p$ (because $h = x - y$ vanishes on the diagonal). Write then:

$$p_1^*\omega - p_2^*\omega = \sum_{i=1}^{\infty} \frac{f^{(i)}(y)}{i!} h^n dy + \sum_{i=0}^{\infty} \frac{f^{(i)}(y)}{i!} h^n dh$$

Now, just check that if one defines $F \overset{\text{def}}{=} \sum_{i=1}^{\infty} \frac{f^{(i-1)}(y)}{i!} h^i$, then $dF$ is the desired expression.

Following we write the general case, which is just the same, but messier. So let now $x = (x_1, \ldots, x_n)$, and let $C = p_1^*x - p_2^*x$. Clearly $C \in T^n$, where $T$ is the ideal of $A_0(X) \otimes A_0(X) \subseteq A(D)$ consisting of functions which vanish on $\Delta$ (the diagonal on $X \times X$). Set then

$$F_Y \overset{\text{def}}{=} \sum_I \frac{1}{I!} (p_2^* F_I) C^I$$

where $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $I > 0$, $I \neq 0$, $I! = i_1! \cdots i_n!$, and

$$F_I \overset{\text{def}}{=} \frac{d^1}{dx_1} \cdots \frac{d^{i_k-1}}{dx_{i_k}} f_k$$

where $k$ is such that $i_k > 0$ and $i_j = 0$ for $j > k$. It is a well-known fact that, if $f \in A(Y)$ and $J \in \mathbb{Z}^n$, $J \geq 0$, then:

$$\left| \frac{1}{J!} \left. \frac{d^J}{dx^J} f \right|_Y \right| \leq |f|_Y$$
Hence:
\[
\frac{|F_I|}{|I|} \leq \frac{\max_j |f_j|}{|i_k|}
\]
and so \(F_Y \in A(D_Y)\).

Now we can compute \(dF_Y\) and prove that, on \(\overline{Y}\),
\[
dF_Y = p_1^* \omega - p_2^* \omega
\]
Next, we check that the \(F_Y\) glue together into a function \(F \in B(D)\) as required.

**Corollary 1.15.** Suppose that \(f_1, f_2: X' \to X\) are morphisms of reduced connected affinoids with good reduction, such that \(\tilde{f}_1 = \tilde{f}_2\). Let \(\omega\) be a closed 1-form on \(X\). Then:
1. \(f_1^* \omega - f_2^* \omega \in dA(X')\)
2. Suppose that \(\lambda\) is a function on \(X(\mathbb{C}_p)\), analytic on each residue class of \(X\), and such that \(d\lambda = \omega\). Then:
\[
f_1^* \lambda - f_2^* \lambda \in A(X')
\]

Let now \(V\) be a proper scheme of finite type over \(R\), and let \(\tilde{V}\) be its special fiber. Let \(W \subseteq \tilde{V}\) be an affine open set. Consider:
\[
\overline{W} \overset{\text{def}}{=} \{ x \in V_K | x \text{ is closed and } \tilde{x} \in W \}
\]
Then \(\overline{W}\) has a natural structure of affinoid over \(K\). If \(V\) is smooth, then \(\overline{W}\) has good reduction, and \(\overline{W} = W\). The set \(\overline{W}\) is called a **Zariski affinoid open set** of \(V\).

**Definition 1.16.** Suppose that \(V_K\) is smooth. A **differential of the second kind** on \(V_K\) is an element \(\omega \in \Omega^1_{V_K/K}(U)\), for some dense open \(U\) of \(V_K\), such that:
1. \(d\omega = 0\).
2. there exists a Zariski open covering \(\mathcal{C}\) of \(V_K\) such that for each \(W \in \mathcal{C}\),
\[
\text{Res}^U_{\mathcal{C} \cap W}(W) \in \text{Res}^W_{\mathcal{C} \cap W}(\Omega^1_{V_K/K}(W)) + d\mathcal{O}_{V_K}(U \cap W)
\]

**Definition 1.17.** Suppose that \(V\) is smooth and proper over \(R\). We say that **Frobenius acts properly on** \(V\) if for each Frobenius endomorphism \(\phi\) of \(\tilde{V}\) there is a polynomial \(Z(T) \in \mathbb{C}_p[T]\) such that:
1. No root of \(Z(T)\) in \(\mathbb{C}_p\) is a root of unity.
2. For each Zariski affinoid open \(W\) of \(V\) such that \(\phi \tilde{W} = \tilde{W}\), there is a lifting \(\overline{\phi}: W \to W\) of the restriction of \(\phi\) to \(\tilde{W}\) such that
\[
Z(\overline{\phi}^*)\omega \in dA(W)
\]
for each algebraic differential of the second kind \(\omega\) on \(V_K\) regular on \(W\).

**Remark.** If (ii) holds for one lifting \(\overline{\phi}\), then it holds for all, thanks to the previous corollary.

**Theorem 1.18.** Suppose that \(K\) is a DVR and that \(V\) is a smooth projective scheme over \(R\). Then any Frobenius endomorphism acts properly on \(V\).
2 $p$-adic Abelian Integrals

Here the integrals are constructed, following the Dwork principle.

**Theorem 2.1.** Let $X$ be a smooth connected affinoid over $K$ with good reduction $	ilde{X}$. Let $\omega$ be a closed one-form on $X$. Let $\phi$ be a Frobenius endomorphism of $X$, and suppose that $P(T)$ is a polynomial over $\mathbb{C}_p$ such that

$$P(\phi^* \omega) \in dA(X)$$

and such that no root of $P(T)$ is a root of unity. Then there exists a locally analytic function $f_\omega$ on $X(\mathbb{C}_p)$ unique up to an additive constant such that:

1. $df_\omega = \omega$
2. $P(\phi^*) f_\omega \in A(X)$.

**Proof.** We copy the proof in the original paper, but for the special case of $P(T) = T - a$, with $a \in \mathbb{C}_p$ (the degree of $P(T)$ is $n = 1$). This is (hopefully) enough to see the ideas behind it.

So assume that $\phi^* \omega - a \omega \in dA(X)$.

We first prove uniqueness: suppose that one has two solutions to the problem. Then their difference would be a locally analytic function $g$ satisfying $dg = 0$ and $\phi^* g - a g \in A(X)$. We will see that $g$ is constant. As $dg = 0$, $g$ must be locally constant, and thus $\phi^* g - a g$ is locally constant as well. As $X$ is connected, then $\phi^* g - a g = C$, for some constant $C \in \mathbb{C}_p$. We will prove that $g(x) = C/(1 - a)$ for all $x \in X$.

Let $U$ be a residue class of $X$, and let $\varepsilon = \varepsilon_U$ be a Teichmüller point of $\phi$ in $U$, of period $m$ (so that $\phi^m(\varepsilon) = \varepsilon$). Then one can check (by induction, for example) that:

$$\phi^{mk}(x) = \varepsilon$$

Take now $k = m$ and evaluate at $\varepsilon$, to get:

$$(1 - a^m) g(\varepsilon) = C \frac{1 - a^m}{1 - a}$$

As $1 - a^m$ is invertible, this implies that $g(\varepsilon) = C/(1 - a)$ as we want.

Now let $x \in U$ be arbitrary. As $g$ is locally constant and $\phi^{mk}(x) \to \varepsilon$, there is some integer $k$ such that:

$$g(\phi^{mk}(x)) = g(\varepsilon)$$

Again by Equation 1, we get:

$$a^{mk} g(x) = a^{mk} C/(1 - a)$$

So $g(x) - C/(1 - a)$ is in the kernel of $a^{mk}$, thought as acting on $\mathbb{C}_p$. But here is the trick to cancel this: for each integer $r$, there is some element $y_r \in U$ such that $\phi^{mr}(y_r) = x$. Using Equation 1 again, we deduce that:

$$g(x) - C/(1 - a) = a^r (g(y_r) - C/(1 - a))$$

so $g(x) - C/(1 - a)$ is both in the kernel of $a^{mk}$ and in the image of $a^{mr}$ for all $r \geq 0$, and only $0$ is there:

$$\ker(a^{mk}) \cap (\bigcap_{r \geq 0} a^{mr}(\mathbb{C}_p)) = \{0\}$$
so \( g \) is constant.

Next, we prove existence: for this, write first \( \phi^* \omega = a \omega + dh \) for some \( h \in A(X) \). We can surely integrate \( \omega \) locally, but we need to do it in a coherent way so that the second condition in the theorem is satisfied. Once again, the Teichmüller points will save the day. If \( U \) is a residue class of \( X \) and \( \varepsilon \in U \) is the corresponding Teichmüller point for \( \phi \), write \( m \) for the minimal positive integer such that \( \phi^m(\varepsilon) = \varepsilon \). Then define \( f_U \) to be the local integral to \( \omega \), normalized such that:

\[
f_U(\varepsilon) = \frac{1}{1 - a^m} \sum_{i=0}^{m-1} a^i h \left( \phi^{m-(i+1)}(\varepsilon) \right)
\]

and define \( f \) by \( f|_U \overset{\text{def}}{=} f_U \). We can then compute \((\phi^* f)(\varepsilon)\) and show that it equals \( a f(\varepsilon) + h(\varepsilon) \) as we wanted. \( \square \)

**Corollary 2.2.** The function \( f_\omega \) is analytic on each residue class of \( X \).

**Corollary 2.3.** The function \( f_\omega \) depends modulo constants only on \( \omega \) and not on the choice of \( P \).

**Corollary 2.4.** Let \( \omega' \) be a closed one-form on \( X \) such that \( P(\phi^*)\omega' \in dA(X) \). Then:

1. \( f_{\omega + \omega'} = f_\omega + f_{\omega'} \) (modulo constants in \( \mathbb{C}_p \)),
2. if \( \omega \) is exact, then \( f_\omega \in A(X) \).

**Corollary 2.5.** The function \( f_\omega \) is independent (up to constants) of the choice of \( \phi \).

Let now \( \sigma \) be a continuous automorphism of \( \mathbb{C}_p \). Let \( \omega^\sigma \) denote the pullback of \( \omega \) to \( X^\sigma \). Let \( f_\omega^\sigma \) be the function on \( X^\sigma(\mathbb{C}_p) \) defined by:

\[
f_\omega^\sigma(x) \overset{\text{def}}{=} \sigma f_\omega(\sigma^{-1}(x))
\]

**Corollary 2.6.** The differential \( \omega^\sigma \) satisfies the hypotheses of the theorem over \( X^\sigma \), and \( f_\omega^\sigma = f_\omega^\sigma \) up to constants. In particular, if \( \sigma \) fixes \( K \), then \( f_\omega^\sigma = f_\omega \) up to constants.

**Proposition 2.7.** Suppose that \( F: \tilde{X}' \to X \) is a morphism of smooth affinoids with good reduction over \( K \). Let \( \omega' = F^* \omega \). Then there exists a Frobenius endomorphism \( \phi' \) of \( \tilde{X}' \) and a polynomial \( P'(T) \) in \( \mathbb{C}_p[T] \) such that

\[
P'(\phi'^*) \omega' \in dA(X')
\]

and such that no root of \( P'(T) \) is a root of unity. Moreover, if \( f_{\omega'} \) is a solution of the Theorem with \( \omega' \) in place of \( \omega \), then \( f_{\omega'} = F^* f_\omega \) up to constants.

**Proof.** The key observation to be made is that there exists Frobenius endomorphisms \( \phi: X \to X \) and \( \phi': \tilde{X}' \to \tilde{X}' \) compatible with \( F \) on the reductions. That is, such that the following commutes:

\[
\begin{array}{ccc}
\tilde{X}' & \overset{\tilde{F}}{\longrightarrow} & \tilde{X} \\
\downarrow \phi' & & \downarrow \phi \\
\tilde{X}' & \overset{\tilde{F}}{\longrightarrow} & \tilde{X}
\end{array}
\]
Then there is (by what we have seen so far) a polynomial \( P(T) \), without roots of unity, such that:

\[ P(\phi^*)\omega \in dA(X) \]

From this, we deduce that:

\[ P(\phi^{*'})\omega' \in dA(X') \]

Also, we deduce (because \( \phi \circ F \) and \( F \circ \phi' \) to the same morphism) that:

\[ F^*(\phi^{*'})^k f_\omega - (\phi^{*'})^k F^* f_\omega \in A(X') \quad \text{for all } k > 0 \]

and hence:

\[ P(\phi^{*'}) F^* - F^* P(\phi^*) \in A(X') \]

Now apply the uniqueness of \( f_\omega' \) to conclude the result. \( \square \)

Next, we will describe how to integrate differentials \( \omega \) of the second kind on \( V_K \), where \( V \) is a smooth, proper, connected scheme of finite type over \( R \).

Let \( D \) be the collection of Zariski affinoid opens \( X \) in \( V_K \) such that, on \( X \),

\[ \omega - dg_X \in \Omega^1_K(X) \]

for some \( g_X \) in \( K(V_K) \) (the function field of \( V_K \)). Note that \( D \) is a covering, because \( \omega \) is of the second kind. Let \( \langle \omega \rangle_\infty \) be the support of the polar divisor of \( \omega \) on \( V_K \), and write \( V'_K \overset{\text{def}}{=} V_K - \langle \omega \rangle_\infty \).

Fix a Frobenius endomorphism \( \phi \) of \( \bar{V} \). Write \( D' \) for the subcollection of \( D \) consisting of those \( X \) such that \( \phi X = \bar{X} \) (note that \( D' \) is also a covering of \( V_K \) (why??)). Let \( Z(T) \) be a polynomial associated to \( V \) and \( \phi \) (as in Definition 1.17).

Fix now \( X \in D' \). Write for short \( g = g_X \), and set \( \nu = \nu_X = \omega - dg \). Let \( \overline{\phi} = \overline{\phi_X} \) be a lifting of the restriction of \( \phi \) to \( \bar{X} \). Hence:

\[ Z(\overline{\phi}^{*})\nu \in dA(X) \]

By Theorem 2.1, there exists \( f = f_X \), locally analytic on \( X \) and unique up to an additive constant such that \( df = \nu \), and \( Z(\overline{\phi}^{*}) f \in A(X) \).

Now, set \( h_X \overset{\text{def}}{=} f + g \), as a function on \( X - \langle \omega \rangle_\infty \).

**Claim.** The function \( h_X \) is independent of the choices of \( f \) and \( g \), up to an additive constant.

**Proof.** Suppose that \( g' \in K(V_K) \) is such that \( \omega - dg' = \nu' \in \Omega^1_K(X) \). It follows then that \( \nu' = \nu + d(g - g') \), and so in particular \( g - g' \in A(X) \). If now \( f' \) is a solution of \( df' = \nu' \), and \( Z(\overline{\phi}^{*}) f' \in A(X) \), then \( f' = f + (g - g') \), from a previous corollary (up to constants). This finishes the proof. \( \square \)

Finally, we need to patch together the local integrals \( h_X \):

**Lemma 2.8.** Let \( X, X' \in D' \). Then \( h_X - h_{X'} \) is constant on \( X \cap X' \).

**Proof.** Note first that \( X \cap X' \in D \), so it suffices to prove it in the case \( X' \subseteq X \). In this case, we may take \( g_X = g_X' \). Then \( \nu_X' \) is the restriction of \( \nu_X \) to \( X' \), and if we restrict \( f_X \) to \( X' \) we get a solution for our problem, hence \( h_{X'} = h_X \mid_{X'} \), as we wanted. \( \square \)
This makes the map \((X, X') \mapsto h_X - h_{X'}\) into a \(1\)-cocycle wrt the covering \(D'\) and the constant sheaf \(\mathbb{C}_p\). It is actually a coboundary, since any finite subcollection of \(D'\) has non-empty intersection.

We have proved:

**Theorem 2.9.** There exists a function \(f_\omega\) on \(V'_K(\mathbb{C}_p)\), unique up to an additive constant, such that:

1. \(df_\omega = \omega\),
2. For each \(X \in D'\), there exists a \(g \in K(V_K)\) such that \(f_\omega - g\) extends to a locally analytic function on \(X\), and

\[ Z(\phi^X_X)(f_\omega - g) \in A(X) \]

**Definition 2.10.** Given \(\omega\) and \(f_\omega\) as above, and given two points \(P, Q \in V'_K(\mathbb{C}_p)\), the integral of \(\omega\) from \(P\) to \(Q\) is defined as:

\[ \int_P^Q \omega \overset{\text{def}}{=} f_\omega(Q) - f_\omega(P) \]

**Proposition 2.11.** Let \(\omega\) and \(\omega'\) be two differentials of the second kind on \(V_K\). Then:

- If \(P, Q \notin (\omega)_{\infty} \cup (\omega')_{\infty}\), we have:

\[ \int_P^Q (\omega + \omega') = \int_P^Q \omega + \int_P^Q \omega' \]

- If \(\omega = dg\) for a meromorphic function \(g\) on \(V_K\), then:

\[ \int_P^Q \omega = g(Q) - g(P) \]

- Let \(g: W \to V\) be a morphism of smooth proper schemes over \(R\), on which Frobenius acts properly. Then, if \(g(Q), g(P) \notin (\omega)_{\infty}\), we have:

\[ \int_P^Q g^* \omega = \int_{g(P)}^{g(Q)} \omega \]

- If \(P, Q \notin (\omega)_{\infty}\), then:

\[ \left( \int_P^Q \omega \right)^\sigma = \int_{g(P)}^{g(Q)} \omega^\sigma \]

where the second integral is taken on \(V^\sigma\).

The following theorem, whose proof omit for now, is a strengthening of the change of variable formula from the previous proposition:

**Theorem 2.12 (Change of Variables).** Suppose that \(V\) and \(W\) are smooth proper schemes of finite type over a ring \(R\) on which Frobenius acts properly. Suppose \(f: V_K \to W_K\) is a rational map. Let \(\omega\) be a differential of the second kind on \(W_K\). Then:

\[ \int_P^Q f^* \omega = \int_{f(P)}^{f(Q)} \omega \]

for any \(P, Q \in V(\mathbb{C}_p)\) in the domain of regularity of \(f\) such that \(f(P), f(Q) \notin (\omega)_{\infty}\).
Corollary 2.13. The integral $\int_P^Q \omega$ doesn’t depend on the model $V$ for $V_K$.

Corollary 2.14. Suppose that $V_K$ is a variety over $K$ which may be completed to a smooth proper scheme $V$ of finite type over $R$ on which Frobenius acts properly. Let $\omega$ be a regular differential on $V_K$ of the second kind. Then for $P, Q \in V_K(\mathbb{C}_p)$ the integral $\int_P^Q \omega$ depends only on $V_K$ and not on its completion.

Let now $G$ be a connected commutative group scheme over $R$, which is an extension of an abelian scheme $A$ by a vector group $B$: 

$$0 \to B \to G \to A \to 0$$

Let $O$ be the origin on $G$.

**Theorem 2.15.** Let $\omega$ be an invariant differential on $G$, and let 

$$\lambda_{\omega}(Q) \overset{\text{def}}{=} \int_Q^O \omega$$

where $Q \in G_K(\mathbb{C}_p)$. (this is well defined by a previous corollary). Then:

1. $\lambda_{\omega}$ is a homomorphism from $G_K(\mathbb{C}_p)$ into $\mathbb{C}_p$
2. $\lambda_{\omega}$ is locally analytic, and $d\lambda_{\omega} = \omega$.

**Proof.** Let $T_a: G \to G$ denote translation by $a \in G_K(\mathbb{C}_p)$. Then $T_a^*\omega = \omega$, and so by the change of variables formula:

$$\int_O^P \omega = \int_O^P T_a^*\omega = \int_Q^{P+Q} \omega = \int_Q^{P+Q} \omega - \int_Q^O \omega$$

which implies $\lambda(P) = \lambda(P + Q) - \lambda(Q)$.

The second statement was known already from the previous results. \qed

In particular, we get the addition theorem:

**Theorem 2.16.** Let $C$ be a complete curve over $K$ with a smooth proper model over $R$, on which Frobenius acts properly. Consider $D_1, D_2, D_3$ three divisors on $C$, such that $D_1 + D_2 \equiv D_3 + n[P]$. Then, for any differential $\omega$ of the first kind on $C$, we have:

$$\sum_{i=1}^n \int_P^{P_i} \omega + \sum_{i=1}^n \int_P^{Q_i} \omega = \sum_{i=1}^n \int_P^{R_i} \omega$$

**Proof.** Just take $G$ in the previous theorem to be the Néron model of the Jacobian of $C$. \qed