Dimension and measure for typical random fractals

Jonathan M. Fraser

St Andrews

Deterministic iterated function systems

Let (X, d) be a compact metric space. A (deterministic) iterated function system (IFS) is a finite set of contraction mappings on X. Given such an IFS, $\{S_1, \dots, S_m\}$, it is a fundamental result of Hutchinson that there exists a unique non-empty compact set F satisfying

$$F = \bigcup_{i=1}^{m} S_i(F)$$

which is called the *attractor* of the IFS.

Deterministic iterated function systems

Let (X, d) be a compact metric space. A (deterministic) iterated function system (IFS) is a finite set of contraction mappings on X. Given such an IFS, $\{S_1, \ldots, S_m\}$, it is a fundamental result of Hutchinson that there exists a unique non-empty compact set F satisfying

$$F = \bigcup_{i=1}^{m} S_i(F)$$

which is called the *attractor* of the IFS.

This construction can be randomised in many natural ways.

Random iterated function systems

We adopt the following randomisation:

(1) Let $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$ be a finite collection of deterministic IFSs

(2) Each deterministic IFS takes the form $\mathbb{I}_i = \{S_{i,j}\}_{j \in \mathcal{I}_i}$ for a finite index set, $\mathcal{I}_i$, and each map, $S_{i,j}$, is a contracting bi-Lipschitz self-map on X

(3) Let $D = \{1, \dots, N\}$, $\Omega = D^{\mathbb{N}}$ and let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Define the attractor of $\mathbb{I}$ corresponding to ω by

$$F_\omega = \bigcap_k \bigcup_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_k, i_k}(K).$$

Random iterated function systems

We adopt the following randomisation:

(1) Let $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$ be a finite collection of deterministic IFSs

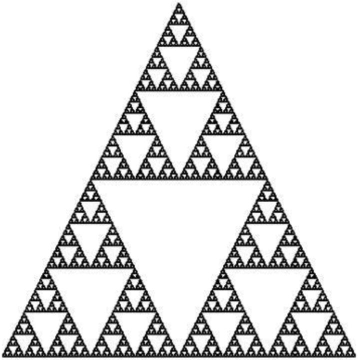
(2) Each deterministic IFS takes the form $\mathbb{I}_i = \{S_{i,j}\}_{j \in \mathcal{I}_i}$ for a finite index set, $\mathcal{I}_i$, and each map, $S_{i,j}$, is a contracting bi-Lipschitz self-map on X

(3) Let $D = \{1, \dots, N\}$, $\Omega = D^{\mathbb{N}}$ and let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Define the attractor of $\mathbb{I}$ corresponding to ω by

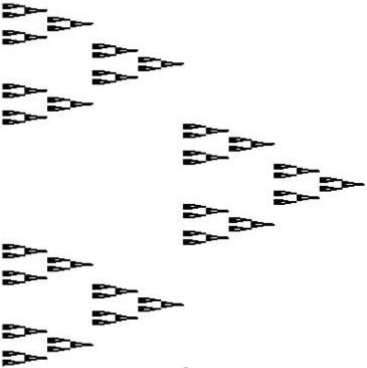
$$F_\omega = \bigcap_k \bigcup_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_k, i_k}(K).$$

We have thus defined a continuum of random attractors $\{F_\omega\}_{\omega \in \Omega}$ and by 'randomly choosing' $\omega \in \Omega$, we 'randomly choose' an attractor F_ω.

An example

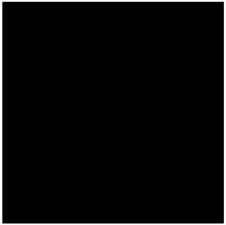


1

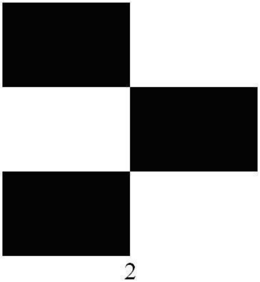


2

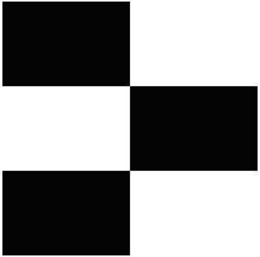
An example



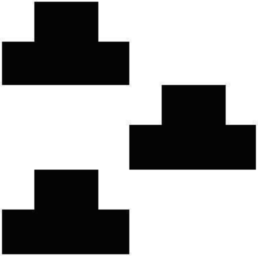
An example



An example

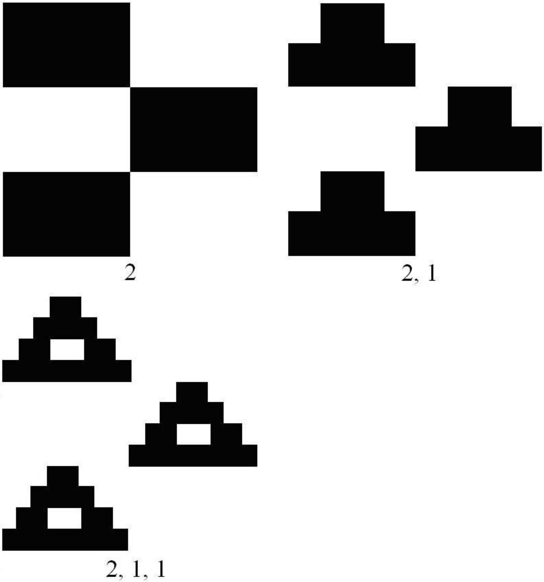


2

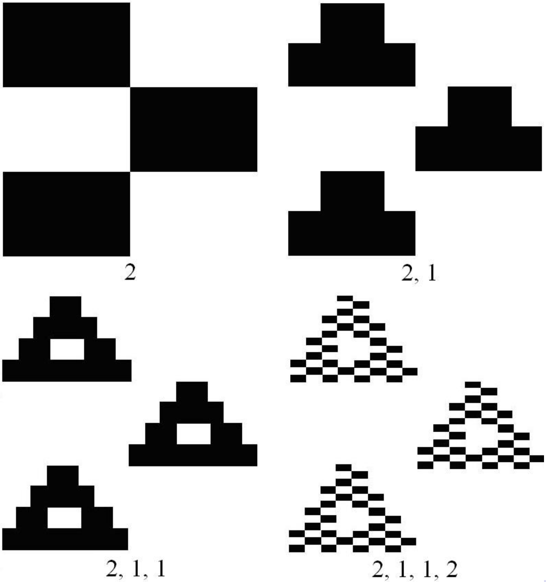


2, 1

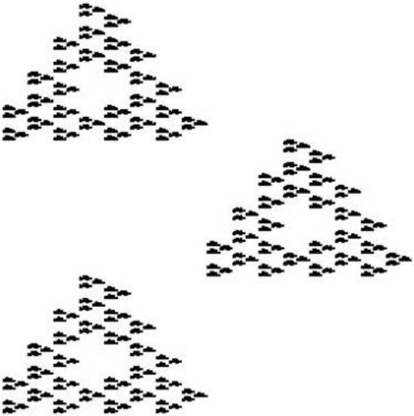
An example



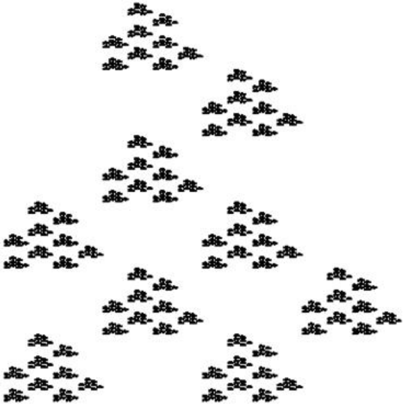
An example



An example



(2,1,1,2,2,1,2,1)



(1,2,1,2,1,2,1,2)

What does a *generic* attractor look like?

The most common approach to answering this question is *probabilistic*.

What does a *generic* attractor look like?

The most common approach to answering this question is *probabilistic*.

(1) Associate a probability vector, $\mathbf{p} = (p_1, \ldots, p_N)$, with D

What does a *generic* attractor look like?

The most common approach to answering this question is *probabilistic*.

(1) Associate a probability vector, $\mathbf{p} = (p_1, \ldots, p_N)$, with D

(2) Choose each entry in ω randomly and independently with respect to $\mathbf{p}$

What does a *generic* attractor look like?

The most common approach to answering this question is *probabilistic*.

(1) Associate a probability vector, $\mathbf{p} = (p_1, \dots, p_N)$, with D

(2) Choose each entry in ω randomly and independently with respect to $\mathbf{p}$

(3) This induces a probability measure, $\mathbb{P}$, on Ω given by

$$\mathbb{P} = \prod_{\mathbb{N}} \sum_{i=1}^{N} p_i \, \delta_i$$

What does a *generic* attractor look like?

The most common approach to answering this question is *probabilistic*.

(1) Associate a probability vector, $\mathbf{p} = (p_1, \dots, p_N)$, with D

(2) Choose each entry in ω randomly and independently with respect to $\mathbf{p}$

(3) This induces a probability measure, $\mathbb{P}$, on Ω given by

$$\mathbb{P} = \prod_{\mathbb{N}} \sum_{i=1}^{N} p_i \, \delta_i$$

(4) We say a property of the random attractors is generic if it occurs for $\mathbb{P}$-almost all $\omega \in \Omega$

Random self-similar sets

Theorem (Hambly '97; Barnsley, Hutchinson, Stenflo '05)

Let $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$ *be an RIFS consisting of similarity maps on* $\mathbb{R}^n$ *with associated probability vector* $\mathbf{p} = (p_1, \dots, p_N)$*. Assume that* $\mathbb{I}$ *satisfies the UOSC and let* s *be the solution of*

$$\prod_{i=1}^{N} \left(\sum_{j \in \mathcal{I}_i} Lip(S_{i,j})^s \right)^{p_i} = 1. \tag{1}$$

Then, for $\mathbb{P}$*-almost all* $\omega \in \Omega$, $\dim_H F_\omega = \dim_B F_\omega = \dim_P F_\omega = s$.

Equation (1) should be viewed as a randomised version of Hutchison's formula.

Random self-similar sets

Theorem (Hambly '97; Barnsley, Hutchinson, Stenflo '05)

Let $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$ *be an RIFS consisting of similarity maps on* $\mathbb{R}^n$ *with associated probability vector* $\mathbf{p} = (p_1, \dots, p_N)$*. Assume that* $\mathbb{I}$ *satisfies the UOSC and let* s *be the solution of*

$$\prod_{i=1}^{N} \left(\sum_{j \in \mathcal{I}_i} Lip(S_{i,j})^s \right)^{p_i} = 1. \tag{1}$$

Then, for $\mathbb{P}$*-almost all* $\omega \in \Omega$, $\dim_H F_\omega = \dim_B F_\omega = \dim_P F_\omega = s$.

Equation (1) should be viewed as a randomised version of Hutchison's formula.

Key tool in the proof: $\mathbb{P}$ is ergodic with respect to the left shift on Ω.

A different approach...

Let (K, d) be a complete metric space.

A different approach...

Let (K, d) be a complete metric space.

A set $N \subseteq K$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in K \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A different approach...

Let (K, d) be a complete metric space.

A set $N \subseteq K$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in K \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A set M is said to be *of the first category*, or, *meagre*, if it can be written as a countable union of nowhere dense sets.

A different approach...

Let (K, d) be a complete metric space.

A set $N \subseteq K$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in K \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A set M is said to be *of the first category*, or, *meagre*, if it can be written as a countable union of nowhere dense sets.

We think of a meagre set as being *small* and the complement of a meagre set as being *big*.

A different approach...

Let (K, d) be a complete metric space.

A set $N \subseteq K$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in K \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A set M is said to be *of the first category*, or, *meagre*, if it can be written as a countable union of nowhere dense sets.

We think of a meagre set as being *small* and the complement of a meagre set as being *big*.

A set $T \subseteq K$ is *residual* or *co-meagre*, if $K \setminus T$ is meagre.

A different approach...

Let (K, d) be a complete metric space.

A set $N \subseteq K$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in K \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A set M is said to be *of the first category*, or, *meagre*, if it can be written as a countable union of nowhere dense sets.

We think of a meagre set as being *small* and the complement of a meagre set as being *big*.

A set $T \subseteq K$ is *residual* or *co-meagre*, if $K \setminus T$ is meagre.

A property is called *typical* if the set of points which have the property is residual.

A different approach...

We will now ask the question: what does F_ω look like for a *typical* $\omega \in \Omega$?

A different approach...

We will now ask the question: what does F_ω look like for a *typical* $\omega \in \Omega$?

Note: We turn Ω into a complete metric space by equipping it with the metric d_Ω where, for $u = (u_1, u_2, \dots) \neq v = (v_1, v_2, \dots) \in \omega$,

$$d_\Omega(u, v) = 2^{-k}$$

where $k = \min\{n \in \mathbb{N} : u_n \neq v_n\}$.

Results: good behaviour

Theorem (F)

Let $G : (0, \infty) \to (0, \infty)$ *be a gauge function.*

Results: good behaviour

Theorem (F)

Let $G : (0, \infty) \to (0, \infty)$ *be a gauge function.*

(1) *If* $\inf_{u \in \Omega} \mathcal{H}^G(F_u) = 0$*, then for a typical* $\omega \in \Omega$*, we have*

$$\mathcal{H}^G(F_\omega) = 0$$

Results: good behaviour

Theorem (F)

Let $G : (0, \infty) \to (0, \infty)$ *be a gauge function.*

(1) *If* $\inf_{u \in \Omega} \mathcal{H}^G(F_u) = 0$*, then for a typical* $\omega \in \Omega$*, we have*

$$\mathcal{H}^G(F_\omega) = 0$$

(2) *If* G *is doubling and* $\sup_{u \in \Omega} \mathcal{P}^G(F_u) = \infty$*, then for a typical* $\omega \in \Omega$*, we have*

$$\mathcal{P}^G(F_\omega) = \infty$$

Results: good behaviour

Theorem (F)

(3) *The typical Hausdorff dimension is infimal, i.e., for a typical* $\omega \in \Omega$*, we have*

$$\dim_H F_\omega = \inf_{u \in \Omega} \dim_H F_u$$

Results: good behaviour

Theorem (F)

(3) *The typical Hausdorff dimension is infimal, i.e., for a typical $\omega \in \Omega$, we have*

$$\dim_H F_\omega = \inf_{u \in \Omega} \dim_H F_u$$

(4) *The typical packing dimension and upper box dimension are supremal and, in fact, for a typical $\omega \in \Omega$, we have*

$$\overline{\dim}_B F_\omega = \dim_P F_\omega = \sup_{u \in \Omega} \overline{\dim}_B F_u = \sup_{u \in \Omega} \dim_P F_u$$

Results: good behaviour

Theorem (F)

(3) *The typical Hausdorff dimension is infimal, i.e., for a typical* $\omega \in \Omega$, *we have*

$$\dim_H F_\omega = \inf_{u \in \Omega} \dim_H F_u$$

(4) *The typical packing dimension and upper box dimension are supremal and, in fact, for a typical* $\omega \in \Omega$, *we have*

$$\overline{\dim}_B F_\omega = \dim_P F_\omega = \sup_{u \in \Omega} \overline{\dim}_B F_u = \sup_{u \in \Omega} \dim_P F_u$$

(5) *The typical lower box dimension is infimal, i.e, for a typical* $\omega \in \Omega$, *we have*

$$\underline{\dim}_B F_\omega = \inf_{u \in \Omega} \underline{\dim}_B F_u$$

Results: good behaviour

Perhaps surpisingly, the above results hold without assuming any separation conditions and the maps can be arbitrary bi-Lipschitz contractions.

Results: bad behaviour

(1) If $0 < \inf_{u \in \Omega} \mathcal{H}^G(F_u) < \infty$ or $0 < \sup_{u \in \Omega} \mathcal{P}^G(F_u) < \infty$, then we do not know what the typical measures are.

Results: bad behaviour

(1) If $0 < \inf_{u \in \Omega} \mathcal{H}^G(F_u) < \infty$ or $0 < \sup_{u \in \Omega} \mathcal{P}^G(F_u) < \infty$, then we do not know what the typical measures are.

(2) The typical packing and Hausdorff measures are not consistently infimal or supremal.

Results: bad behaviour

(1) If $0 < \inf_{u \in \Omega} \mathcal{H}^G(F_u) < \infty$ or $0 < \sup_{u \in \Omega} \mathcal{P}^G(F_u) < \infty$, then we do not know what the typical measures are.

(2) The typical packing and Hausdorff measures are not consistently infimal or supremal.

(3) The infimal Hausdorff dimension is not (in general) the minimum of the Hausdorff dimensions of the attractors of the individual determinsitc IFS. So we cannot usually compute the typical dimensions explicitly.

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

(1) Assume $X = [0, 1]^d$

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

(1) Assume $X = [0, 1]^d$

(2) Assume all the maps are similarities

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

(1) Assume $X = [0, 1]^d$

(2) Assume all the maps are similarities

(3) Assume the UOSC

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

(1) Assume $X = [0, 1]^d$

(2) Assume all the maps are similarities

(3) Assume the UOSC

This is a *much* simpler setting,

Can we do better in a simpler setting?

Since the above results are obtained in very general circumstances, perhaps we can say something more precise in a simpler setting.

(1) Assume $X = [0,1]^d$

(2) Assume all the maps are similarities

(3) Assume the UOSC

This is a *much* simpler setting, but we can obtain more precise results!

Results in the self-similar setting

For each $i \in D$, let s_i be the solution of

$$\sum_{j \in \mathcal{I}_i} \mathrm{Lip}(S_{i,j})^{s_i} = 1$$

and write $s_{\min} = \min_{i \in D} s_i$ and $s_{\max} = \max_{i \in D} s_i$.

Theorem (F)

(1) $\sup_{\omega \in \Omega} \dim_P F_\omega = \sup_{\omega \in \Omega} \overline{\dim}_B F_\omega = s_{\max}$

(2) $0 < \sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega) < \infty$

(3) $\inf_{\omega \in \Omega} \dim_H F_\omega = \inf_{\omega \in \Omega} \underline{\dim}_B F_\omega = s_{\min}$

(4) $0 < \inf_{\omega \in \Omega} \mathcal{H}^{s_{\min}}(F_\omega) < \infty$

Results in the self-similar setting

Write $\mathcal{H}_{\min} = \inf_{\omega \in \Omega} \mathcal{H}^{s_{\min}}(F_\omega)$ and $\mathcal{P}_{\max} = \sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega)$.

Theorem

(1) *If* $s_{\min} = s_{\max} = s$*, then for a typical* $\omega \in \Omega$*,*

$$\dim_H F_\omega = \dim_P F_\omega = s$$

and

$$0 < \mathcal{H}^s(F_\omega) = \mathcal{H}_{\min} \leqslant \mathcal{P}_{\max} = \mathcal{P}^s(F_\omega) < \infty$$

(2) *If* $s_{\min} < s_{\max}$*, then for a typical* $\omega \in \Omega$*,*

$$\dim_H F_\omega = s_{\min} < s_{\max} = \dim_P F_\omega,$$

$$\mathcal{H}^{s_{\min}}(F_\omega) = \infty$$

and

$$\mathcal{P}^{s_{\max}}(F_\omega) = 0$$

Future work

(1) Is the typical measure always extremal?

Future work

(1) Is the typical measure always extremal?

(2) Can we say anything more about the typical dimensions in the general case? i.e., can we compute the extremal dimensions explicitly?

Future work

(1) Is the typical measure always extremal?

(2) Can we say anything more about the typical dimensions in the general case? i.e., can we compute the extremal dimensions explicitly?

(3) Can we obtain similar results for a more general random model?

Thank you!