# The horizon problem for prevalent surfaces

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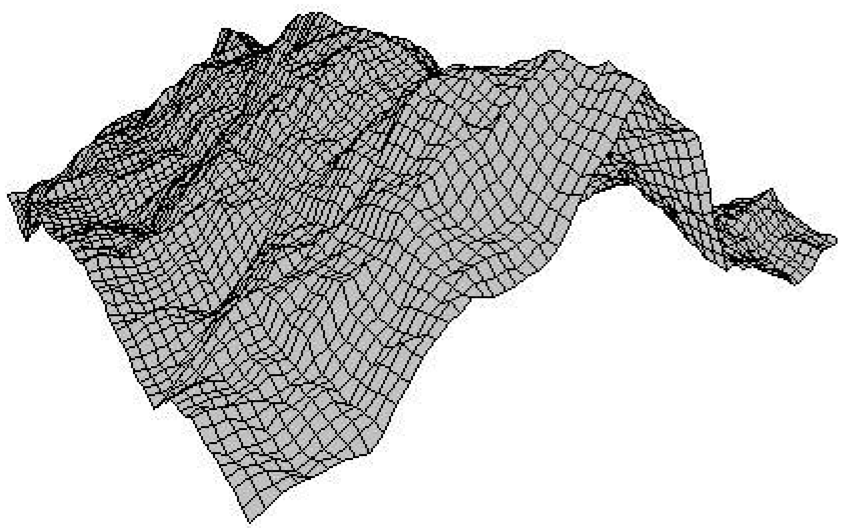
**Abstract**

I will introduce and study the 'horizon problem' for continuous surfaces. That is, given a (fractal) surface, what can be said about the relationship between the dimension of the surface and the dimension of the horizon? It becomes quickly apparent that one cannot say anything about this relationship in general, however, it is possible to say something in the 'generic' case for a suitable definition of the word 'generic'.

## The horizon problem for prevalent surfaces

$$C[0,1]^d = \{f : [0,1]^d \to \mathbb{R} \mid f \text{ is continuous}\}$$



A fractal surface

The horizon function, $H(f) \in C[0,1]$, of $f \in C[0,1]^2$ is defined by

$$H(f)(x) = \sup_{y \in [0,1]} f(x,y)$$

We will say that $f \in C[0,1]^2$ satisfies the *horizon property (for box dimension)* if

$$\dim_{\mathsf{B}} G_{H(f)} = \dim_{\mathsf{B}} G_f - 1$$

**Prevalence**.

Let $X$ be a completely metrizable topological vector space. A set $F \subseteq X$ is *prevalent* if the following conditions are satisfied:

1) $F$ is a Borel set;

2) There exists a Borel measure $\mu$ on $X$ and a compact set $K \subseteq X$ such that $0 < \mu(K) < \infty$ and

$$\mu\big(X \setminus (F + x)\big) = 0$$

for all $x \in X$.

The complement of a prevalent set is called a *shy* set.

**Remark:** $(C[0,1]^2, d_\infty)$ *is a completely metrizable topological vector space.*

**Box dimensions**.

Let $F \subset \mathbb{R}^d$ be bounded

$$\underline{\dim}_{\mathsf{B}} F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and

$$\overline{\dim}_{\mathsf{B}} F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

where $N_\delta(F)$ is the number of cubes in a $\delta$-mesh which intersect $F$.

**Theorem 1.1** (Falconer and F, '11).

*A prevalent function* $f \in (C[0,1]^2, d_\infty)$ *satisfies:*

$$\dim_{\mathsf{B}} G_f = 3$$

*and*

$$\dim_{\mathsf{B}} G_{H(f)} = 2.$$

Let $\alpha \in [2,3)$ and let

$$C_\alpha[0,1]^2 \;=\; \left\{ f \in C[0,1]^2 : \overline{\dim}_{\mathsf{B}} G_f \leqslant \alpha \right\}$$

**Proposition 1.2** (Falconer and F, '11). $C_\alpha[0,1]^2$ *is a completely metrizable topological vector space when equipped with a (natural) metric* $d_\alpha$.

**Remark**: *The topology generated by* $d_\alpha$ *is strictly finer than that generated by* $d_\infty$.

**Theorem 1.3** (Falconer and F, '11).

*A prevalent function* $f \in (C_\alpha[0,1]^2, d_\alpha)$ *satisfies:*

$$\dim_{\mathsf{B}} G_f = \alpha$$

*and*

$$\alpha - 1 \leqslant \underline{\dim}_{\mathsf{B}} G_{H(f)} \leqslant \overline{\dim}_{\mathsf{B}} G_{H(f)} \leqslant 2.$$

**A (very short) note on the proof.**

**Proposition 1.4** (Falconer and F, '11)**.**

*Let* $f, g \in C[0,1]^d$*. Then*

$$\max\left\{\underline{\dim}_{\mathsf{B}} G_f, \underline{\dim}_{\mathsf{B}} G_g\right\}$$

$$\leqslant \underline{\dim}_{\mathsf{B}} G_{f+\lambda g} \leqslant \overline{\dim}_{\mathsf{B}} G_{f+\lambda g}$$

$$= \max\left\{\overline{\dim}_{\mathsf{B}} G_f, \overline{\dim}_{\mathsf{B}} G_g\right\}$$

*for* $\mathcal{L}^1$*-almost all* $\lambda \in \mathbb{R}$*.*

*Proof.* Lots of box-counting and the Borel-Cantelli Lemma. □

Let $\alpha \in [2,3]$, let $\psi \in C[0,1]$ satisfy

$$\dim_{\mathsf{B}} G_\psi = \alpha - 1$$

and define $\Psi \in C[0,1]^2$ by

$$\Psi(x,y) = \psi(x).$$

Finally, let

$$K = \left\{\lambda\Psi : \lambda \in \mathbb{R}\right\} \subset C[0,1]^2$$

and

$$\mu = \mathcal{L}_K$$

**Question: Can we obtain sharper results on the box dimension of the horizon of a prevalent surface?**

Let $\alpha \in [2,3)$ and let

$$U_\alpha[0,1]^2 = \Big\{ f \in C_\alpha[0,1]^2 : \dim_{\mathsf{B}} G_f = \alpha$$

and

$$\alpha - 1 \leqslant \underline{\dim}_{\mathsf{B}} G_{H(f)} \leqslant \overline{\dim}_{\mathsf{B}} G_{H(f)} < 2 \Big\}.$$

and

$$L_\alpha[0,1]^2 = \Big\{ f \in C_\alpha[0,1]^2 : \dim_{\mathsf{B}} G_f = \alpha$$

and

$$\alpha - 1 < \underline{\dim}_{\mathsf{B}} G_{H(f)} \leqslant \overline{\dim}_{\mathsf{B}} G_{H(f)} \leqslant 2 \Big\}.$$

**Theorem 1.5** (Falconer and F, '11)**.**

*(1)* $U_\alpha[0,1]^2$ *is* *<u>not</u>* *a prevalent subset of* $(C_\alpha[0,1]^2, d_\alpha)$*;*

*(2)* $L_\alpha[0,1]^2$ *is* *<u>not</u>* *a prevalent subset of* $(C_\alpha[0,1]^2, d_\alpha)$*.*

**Discussion and questions.**

1) What about Hausdorff dimension and packing dimension?

We have recently obtained the following result:

**Theorem 1.6** (F and Hyde, '11)**.**

*A prevalent function* $f \in (C[0,1]^2, d_\infty)$ *satisfies:*

$$\dim_{\mathsf{H}} G_f = \dim_{\mathsf{P}} G_f = 3$$

*and*

$$\dim_{\mathsf{H}} G_{H(f)} = \dim_{\mathsf{P}} G_{H(f)} = 2.$$

2) How about surfaces with Hausdorff or packing dimension less than 3?

It can be shown that for $\alpha < 3$ the set

$$\Big\{ f \in C[0,1]^2 : \dim_{\mathsf{H}} G_f \leqslant \alpha \Big\}$$

is not a vector space because it is not closed under addition (see [MW]) and so we cannot conduct the same analysis as we did in the box dimension case.

3) Is it true that for all $f, g \in C[0,1]^d$ we have

$$\dim_{\mathsf{P}} G_{f+g} \leqslant \max\{\dim_{\mathsf{P}} G_f, \dim_{\mathsf{P}} G_g\}?$$

## References

[FF] K. J. Falconer and J. M. Fraser. The horizon problem for prevalent surfaces, *preprint* (2011).

[FH] J. M. Fraser and J. T. Hyde. The Hausdorff dimension of graphs of prevalent continuous functions, *preprint* (2011).

[MW] R. D. Mauldin and S. C. Williams. On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.*, **298**, (1986), 793–803.