Random self-affine multifractal Sierpiński sponges in $\mathbb{R}^d$

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Abstract

I will first introduce the Hausdorff and packing multifractal spectrum functions, $f_{\mathsf{H},\mu}$ and $f_{\mathsf{P},\mu}$. I will briefly discuss the self-affine sets introduced by Bedford and McMullen and go on to mention various generalisations that have been made in recent years in particular the work of King and Olsen. Finally, I will discuss the recent work by Fraser and Olsen on the Hausdorff spectrum of random self-affine multifractal sponges in $\mathbb{R}^d$.

Lower and upper local dimension of μ at $x \in \mathsf{supp}\,\mu$

$$\underline{\dim}_{\mathsf{loc}}\,\mu(x) = \liminf_{\delta\to 0}\,\frac{\log\mu\big(B(x,\delta)\big)}{\log\delta}$$

and

$$\overline{\dim}_{\mathsf{loc}}\,\mu(x) = \limsup_{\delta\to 0}\,\frac{\log\mu\big(B(x,\delta)\big)}{\log\delta}$$

Multifractal decomposition sets

$$\Delta_\alpha = \Big\{x \in \mathsf{supp}\,\mu : \dim_{\mathsf{loc}}\mu(x) = \alpha\Big\}$$

for $\alpha \geq 0$.

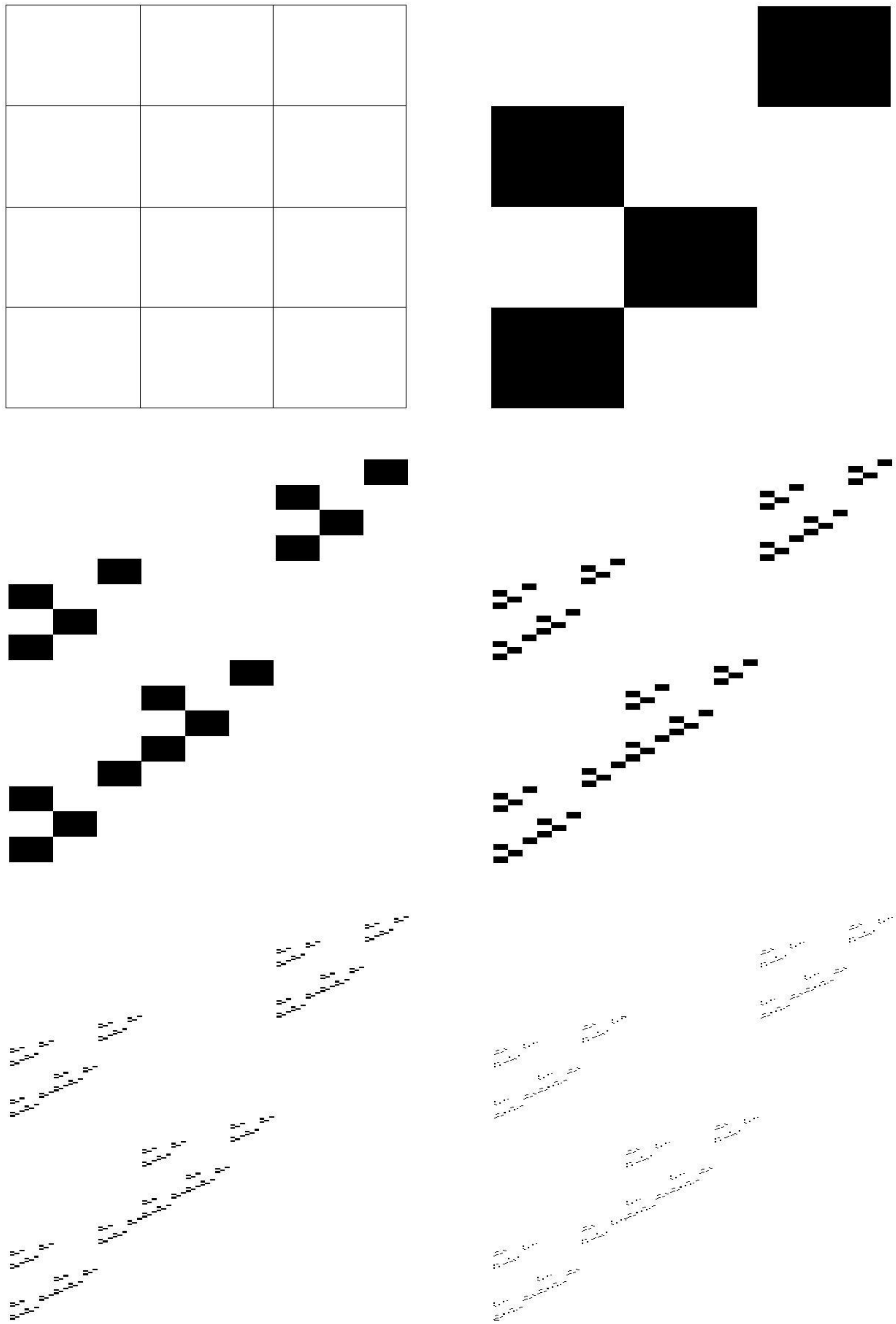
Hausdorff and packing multifractal spectrum functions, $f_{\mathsf{H},\mu}$ and $f_{\mathsf{P},\mu}$ of μ

$$f_{\mathsf{H},\mu}(\alpha) = \dim_{\mathsf{H}}\Delta_\alpha$$

and

$$f_{\mathsf{P},\mu}(\alpha) = \dim_{\mathsf{P}}\Delta_\alpha$$

for $\alpha \geq 0$.



A self-affine Bedford-McMullen carpet

We define

$$\beta_{i_1\dots i_d},\ \beta_{i_1\dots i_{d-1}},\ \dots,\ \beta_{i_1 i_2},\ \beta_{i_1},\ \beta : \mathbb{R} \to \mathbb{R}$$

inductively as follows,

(0) For $q \in \mathbb{R}$ and $(i_1, \dots, i_d) \in D$ let $\beta_{i_1\dots i_d}(q) = 0$

(1) For $q \in \mathbb{R}$ and $(i_1, \dots, i_{d-1}) \in \pi_{d-1}(D)$ define $\beta_{i_1\dots i_{d-1}}(q)$ by

$$\sum_{\substack{i_d \\ (i_1,\dots,i_d)\in\pi_d(D)}} p_{D,\mathbf{p},d}(i_d \mid i_1, \dots, i_{d-1})^q n_d^{\beta_{i_1\dots i_d}(q)-\beta_{i_1\dots i_{d-1}}(q)} = 1$$

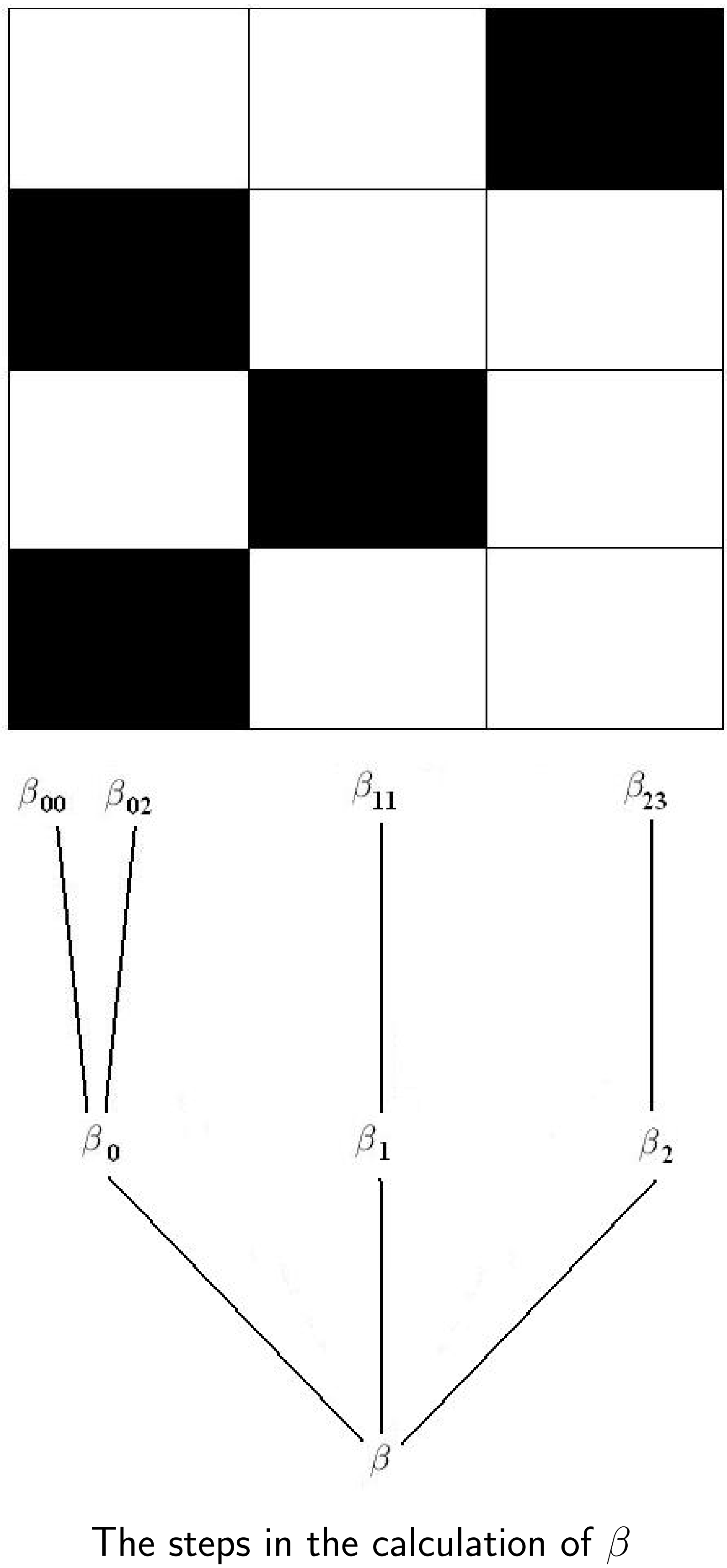
(2) For $q \in \mathbb{R}$ and $(i_1, \dots, i_{d-2}) \in \pi_{d-2}(D)$ define $\beta_{i_1\dots i_{d-2}}(q)$ by

$$\sum_{\substack{i_{d-1} \\ (i_1,\dots,i_{d-1})\in\pi_{d-1}(D)}} p_{D,\mathbf{p},d-1}(i_{d-1} \mid i_1, \dots, i_{d-2})^q n_{d-1}^{\beta_{i_1\dots i_{d-1}}(q)-\beta_{i_1\dots i_{d-2}}(q)} = 1$$

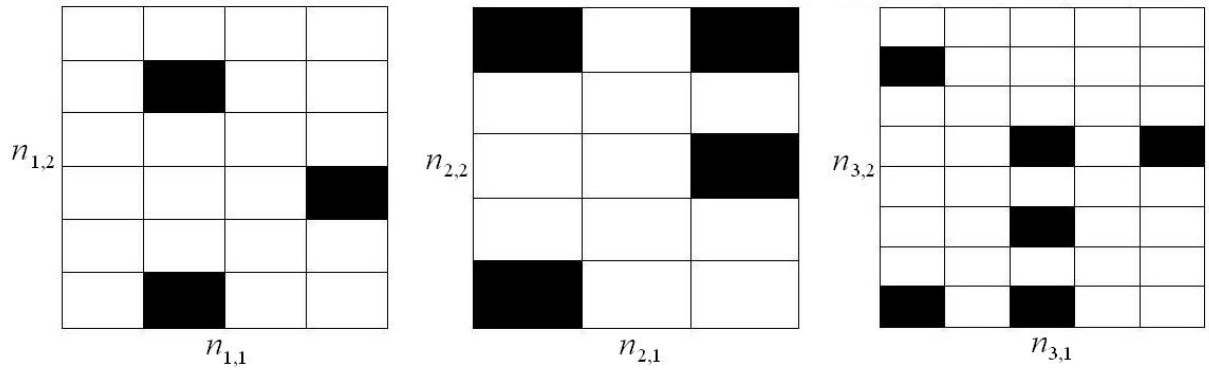
$$\vdots$$

(d) For $q \in \mathbb{R}$ define $\beta(q)$ by

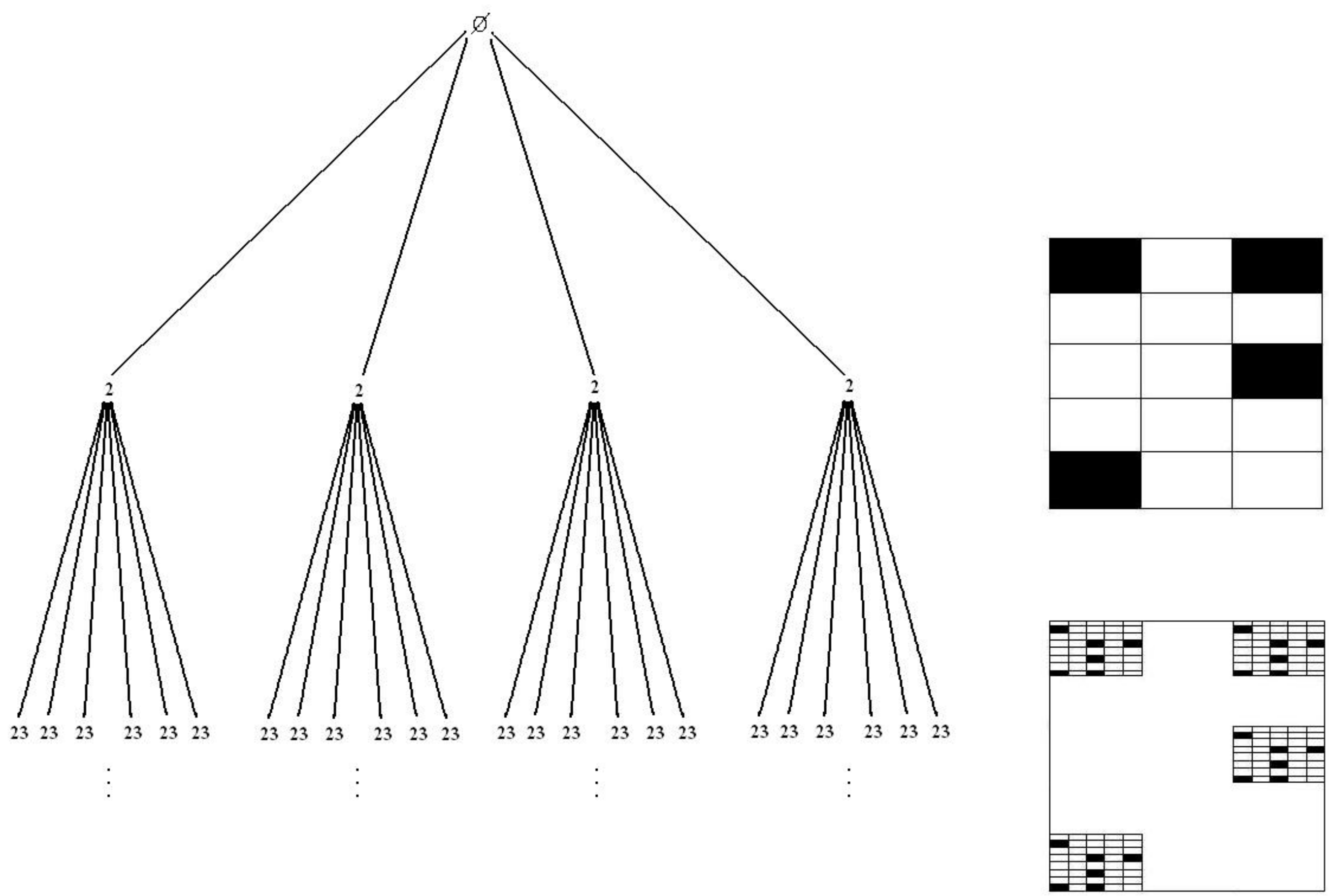
$$\sum_{\substack{i_1 \\ i_1\in\pi_1(D)}} p_1(i_1)^q n_1^{\beta_{i_1}(q)-\beta(q)} = 1$$



The steps in the calculation of β



Three options at each step in the construction



The first two steps in the construction of a random carpet based on the realisation $\omega = (2, 3, \dots)$

For $l = 1, \ldots, d$ define $\nu_l = n_{1,l}^{p_1} \cdots n_{N,l}^{p_N}$ and for $i = 1, \ldots, N$ we define

$$\beta_{i,i_1 \ldots i_d}, \; \beta_{i,i_1 \ldots i_{d-1}}, \; \ldots, \; \beta_{i,i_1 i_2}, \; \beta_{i,i_1}, \; \beta_i : \mathbb{R} \to \mathbb{R}$$

inductively as follows,

(0) For $q \in \mathbb{R}$ and $(i_1, \ldots, i_d) \in D_i$ let $\beta_{i,i_1 \ldots i_d}(q) = 0$

(1) For $q \in \mathbb{R}$ and $(i_1, \ldots, i_{d-1}) \in \pi_{d-1}(D_i)$ define $\beta_{i,i_1 \ldots i_{d-1}}(q)$ by

$$\sum_{\substack{i_d \\ (i_1,\ldots,i_d) \in \pi_d(D_i)}} p_{D_i,\mathbf{p}_i,d}(i_d \mid i_1, \ldots, i_{d-1})^q \nu_d^{\beta_{i,i_1 \ldots i_d}(q) - \beta_{i,i_1 \ldots i_{d-1}}(q)} = 1$$

(2) For $q \in \mathbb{R}$ and $(i_1, \ldots, i_{d-2}) \in \pi_{d-2}(D_i)$ define $\beta_{i,i_1 \ldots i_{d-2}}(q)$ by

$$\sum_{\substack{i_{d-1} \\ (i_1,\ldots,i_{d-1}) \in \pi_{d-1}(D_i)}} p_{D_i,\mathbf{p}_i,d-1}(i_{d-1} \mid i_1, \ldots, i_{d-2})^q \nu_{d-1}^{\beta_{i,i_1 \ldots i_{d-1}}(q) - \beta_{i,i_1 \ldots i_{d-2}}(q)} = 1$$

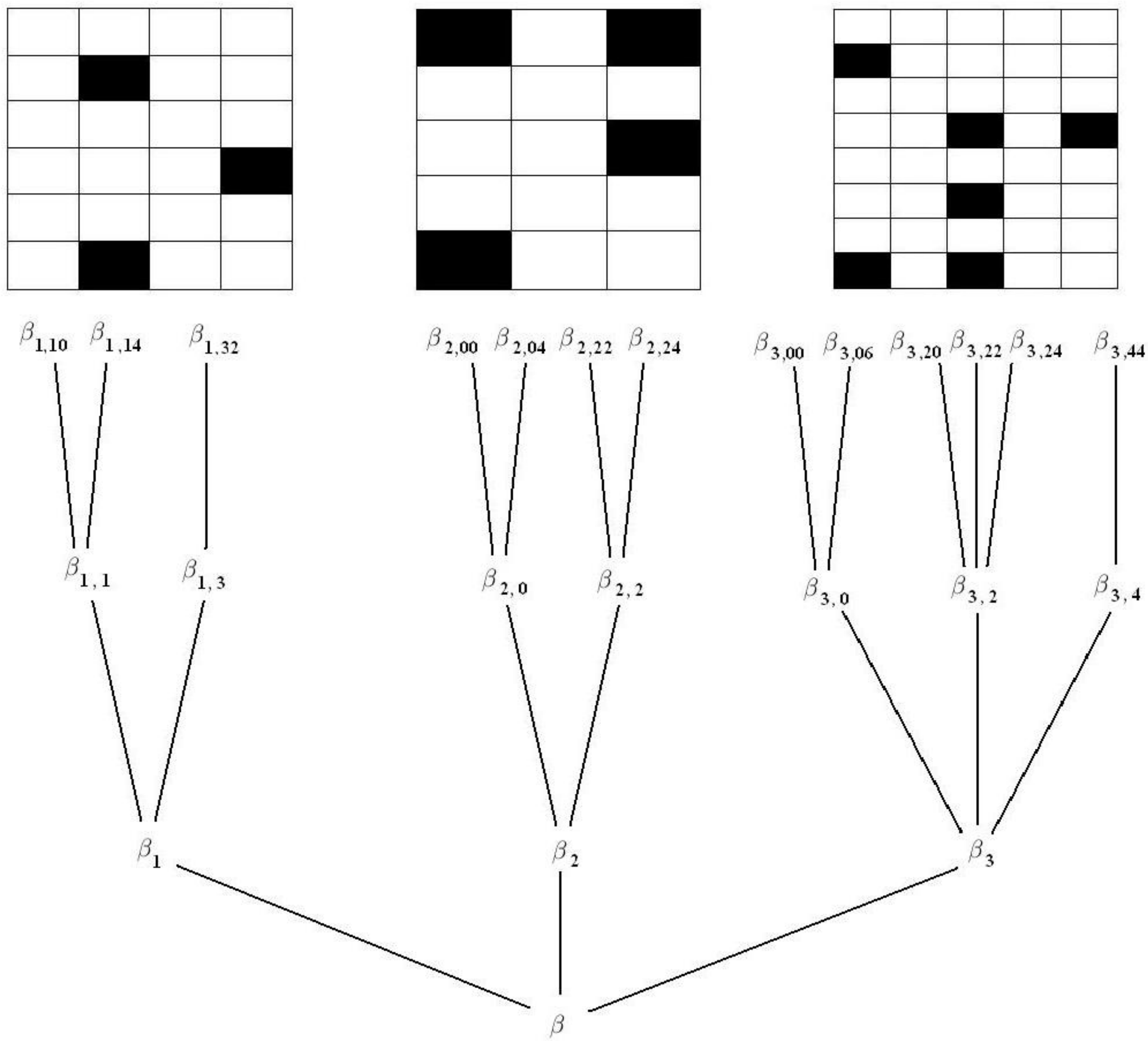
$\vdots$

(d) For $q \in \mathbb{R}$ define $\beta_i(q)$ by

$$\sum_{\substack{i_1 \\ i_1 \in \pi_1(D)}} p_1(i_1)^q \nu_1^{\beta_{i,i_1}(q) - \beta_i(q)} = 1$$

Finally, define $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta = \sum_i p_i \beta_i$$



The steps in the calculation of β

Theorem 1.1. *(F and Olsen*, Indiana Univ. Math. J., *(to appear)).*

Assuming the VSSC is satisfied for $D_1, \ldots, D_N$ *and* $\nu_1 < \nu_2 < \cdots < \nu_d$ *we have*

(1) For almost all realisations ω *we have:*

$$f_{\mathsf{H},\mu_\omega}(\alpha) = \dim_{\mathsf{H}} \left\{ x \in K_\omega \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu_\omega(B(x,r))}{\log r} = \alpha \right\} = \beta^*(\alpha)$$

for all $\alpha \in (\alpha_{\min}, \alpha_{\max})$ *(here* β^* *denotes the Legendre transform of* β *defined by* $\beta^*(\alpha) = \inf_q (q\alpha + \beta(q))$*).*

(2) For almost all realisations ω *we have:*

$$\left\{ x \in K_\omega \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu_\omega(B(x,r))}{\log r} = \alpha \right\} = \emptyset$$

for all $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.

Theorem 1.2. *(F and Olsen*, Indiana Univ. Math. J., *(to appear)).*

Assuming that $\nu_1 < \nu_2 < \cdots < \nu_d$ *we have*

(1) For almost all realisations ω *we have:*

$$f^s_{\mathsf{H},\mu_\omega}(\alpha) = \dim_{\mathsf{H}} \left\{ x \in K_\omega \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu_\omega(Q(x,r))}{\log r} = \alpha \right\} = \beta^*(\alpha)$$

for all $\alpha \in (\alpha_{\min}, \alpha_{\max})$ *(here* β^* *denotes the Legendre transform of* β *defined by* $\beta^*(\alpha) = \inf_q (q\alpha + \beta(q))$*).*

(2) For almost all realisations ω *we have:*

$$\left\{ x \in K_\omega \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu_\omega(Q(x,r))}{\log r} = \alpha \right\} = \emptyset$$

for all $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.

Open(ish) Questions

1. Can we remove the assumption that $\nu_1 < \nu_2 < \cdots < \nu_d$?

   Yes - see paper.

2. Can we compute the almost sure Hausdorff spectrum for a more general random model?

   Watch this space!

3. Do our results still hold if the VSSC is not satisfied? In particular, do the results of Olsen on deterministic self-affine sponges hold without the VSSC?

   Probably! The VSSC was removed by Jordan and Rams in the case $d = 2$.

4. What is the packing spectrum of a determinsitic self-affine sponge? This is not known even in $\mathbb{R}^2$.

   No idea! We have

$$\beta^*(\alpha) = f_{\mathsf{H},\mu}(\alpha) \leqslant f_{\mathsf{P},\mu}(\alpha) \leqslant \gamma^*(\alpha)$$

References

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