Scaling scenery of $(\times m, \times n)$ invariant measures

Jonathan M. Fraser
The University of Warwick, UK

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joint work with **Andrew Ferguson** and **Tuomas Sahlsten**

University of BRISTOL

THE UNIVERSITY OF WARWICK

האוניברסיטה העברית בירושלים
The Hebrew University of Jerusalem

My coauthors

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- Käenmäki-Sahlsten-Shmerkin (2014): applications to Marstrand's conical density theorems, rectifiability and porosity.

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and define the **magnification operator** $M : \Xi \to \Xi$ by
$$M(x,\mu) = (T_{D_1(x)}(x), \mu^{D_1(x)}).$$

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- It is also possible to use more general (regular) filtrations than dyadic. Then the dynamics is described by a Markov process (a **CP chain**) and the **CP distribution** is the stationary measure for this chain.

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Condition (2) seems strange at first sight, but is essential to carry geometric information from the micromeasure back to μ.

In 'nice' situations, (2) does not cause any problems in the proofs and often $Q_q = Q$ for all $q \in \mathbb{N}$.

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Proposition (Hochman-Shmerkin 2012)

Let μ be a self-similar measure in $\mathbb{R}^d$ satisfying the strong separation condition. Then μ generates an ergodic CP chain Q for the dyadic partition operator supported on the dyadic micromeasures of μ such that the dyadic micromeasures ν are of the form

$$\nu = \mu(B)^{-1} S(\mu|_B)$$

for some Borel-set B with $\mu(B) > 0$ and some similitude S of $\mathbb{R}^d$. Moreover, the original measure can be recovered from a given micromeasure ν as $\mu = \nu(B')^{-1} S'(\nu|'_B)$, for some Borel-set B' and similitude S'.

Example: dimensions of projections

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- For a CP distribution Q, write

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(3) $E(\pi) = \min\{k, \dim \mu\}$ for a.e. $\pi \in \Pi_{d,k}$.

Example: a projection theorem for self-similar sets

Theorem (Hochman-Shmerkin 2012)

Let μ be a self-similar measure in $\mathbb{R}^d$ satisfying the SSC and such that the IFS satisfies the **minimality assumption**. Then, for all $\pi \in \Pi_{d,k}$,

$$\dim \pi\mu = \min\{k, \dim \mu\}.$$

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It is often more convenient from a dynamical point of view to think of $[0,1]$ as the unit circle $\mathbb{T}$ and $[0,1]^2$ as the 2-torus $\mathbb{T}^2$.

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$$S_{i,j} = \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix},$$

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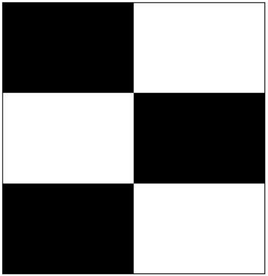

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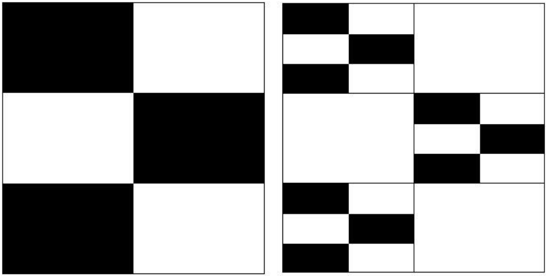

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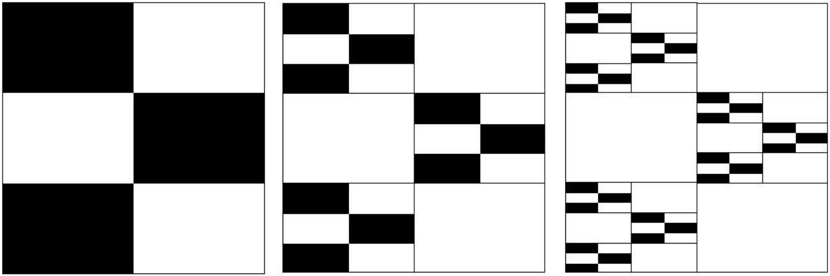

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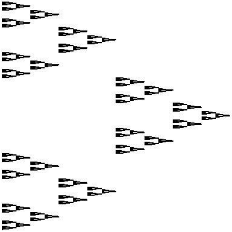

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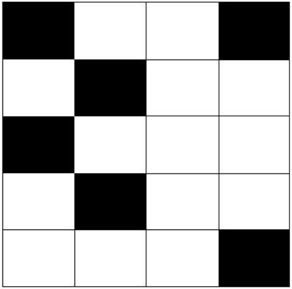

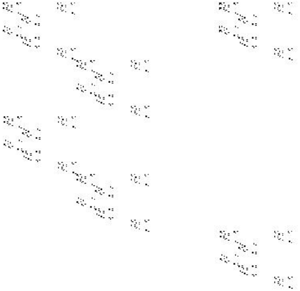

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Bernoulli measures on Bedford-McMullen carpets are good examples to work with as they display many of the interesting features of $T_{m,n}$ invariant measures, whilst being very explicit and neat to write down.

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- This product form of the tangents was exploited by Mackay (2011) and F (2013) when computing the Assouad dimension of Bedford-McMullen carpets.

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- Measure component $\tilde{Q}$ is the distribution of the random measure

$$S_t(\pi_1\mu \times \mu_x),$$

where $x \sim \pi_1\mu$ and $\mu_x \in \mathcal{P}([0,1])$ is the conditional measure of μ with respect to the fibre $\pi_1^{-1}\{x\}$ and S_t is the unique affine map which sends $[0,1]^2$ to $[0,1/n^{t/2}] \times [0,n^{t/2}]$ and $t \in [0,1)$ is drawn according to Lebesgue in the 'irrational case' and according to a uniform measure on a periodic orbit in the 'rational case'.

Application I: Projections

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Furstenberg's Conjecture (from the 1960s)

If $X, Y \subset [0,1]$ are closed and T_2 and T_3 invariant respectively. Then

$$\dim \pi(X \times Y) = \min\{1, \dim(X \times Y)\}, \quad \pi \in \Pi_{2,1} \setminus \{\pi_1, \pi_2\}.$$

Application I: Projections

Furstenberg's Conjecture (from the 1960s)

If $X, Y \subset [0,1]$ are closed and T_2 and T_3 invariant respectively. Then

$$\dim \pi(X \times Y) = \min\{1, \dim(X \times Y)\}, \quad \pi \in \Pi_{2,1} \setminus \{\pi_1, \pi_2\}.$$

Solved:

Theorem (Hochman-Shmerkin 2012)

If $\mu, \nu \in \mathcal{P}([0,1])$ are T_m and T_n invariant respectively and $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\dim \pi_*(\mu \times \nu) = \min\{1, \dim(\mu \times \nu)\}, \quad \pi \in \Pi_{2,1} \setminus \{\pi_1, \pi_2\}.$$

Obtained by constructing an ergodic CP distribution for $\mu \times \nu$.

Application I: Projections

Conjecture

Suppose μ is a $T_{m,n}$ invariant measure and $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$, then

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Theorem (Ferguson-Jordan-Shmerkin 2010)

Suppose K is a Bedford-McMullen carpet with $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\dim \pi(K) = \min\{1, \dim K\}, \quad \pi \in \Pi_{2,1} \setminus \{\pi_1, \pi_2\}.$$

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Theorem (Ferguson, F, Sahlsten, 2013)

The conjecture above holds for $T_{m,n}$ invariant *Bernoulli* measures.

Recall…

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Theorem (Hochman-Shmerkin 2012)

Suppose μ generates an ergodic CP distribution Q. Then

(1) The map $E : \Pi_{d,k} \to \mathbb{R}$ is **lower semicontinuous**.

(2) $\dim \pi\mu \geq E(\pi)$ for any $\pi \in \Pi_{d,k}$.

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- In our setting, after suitable reparametrisation of $\Pi_{2,1}$, the map E is invariant under the **irrational** $\frac{\log m}{\log n}$ **rotation** of the circle, so E is constant as a lower semicontinuous function on $\Pi_{2,1} \setminus \{\pi_1, \pi_2\}$.

Application II: Distance sets

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The **distance set** of $K \subset \mathbb{R}^d$ is

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Distance set conjecture (Falconer, 1980s)

Suppose $K \subset \mathbb{R}^d$ is Borel and $\dim K \geq d/2$. Then $\dim D(K) = 1$.
Moreover, if $\dim K > d/2$, then $\mathcal{L}^1(D(K)) > 0$.

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Many people have been involved in the study of this conjecture.

- Bourgain (2003) found a small constant $\varepsilon > 0$ with

$$\dim D(K) \geq \frac{1}{2} + \varepsilon$$

  whenever $K \subset \mathbb{R}^2$ with $\dim K \geq 1$.
- Erdogan (2006) proved $\dim K > d/2 + 1/3$ in $\mathbb{R}^d$ yields positive measure for $D(K)$.
- Orponen (2011) proved $\dim D(K) = 1$ if K is a planar self-similar set with $\mathcal{H}^1(K) > 0$.

Application II: Distance sets

Theorem (Ferguson, F, Sahlsten, 2013)

If μ on $\mathbb{R}^2$ generates an ergodic CP distribution and $\mathcal{H}^1(\operatorname{spt}\mu) > 0$, then

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If K is a Bedford-McMullen carpet with $\dim K \geq 1$, then $\dim D(K) = 1$.

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Corollary (Ferguson, F, Sahlsten, 2013)

If K is a Bedford-McMullen carpet with $\dim K \geq 1$, then $\dim D(K) = 1$.

- Using standard dimension approximation theorems via Bedford-McMullen carpets, this yields results for other **Lalley-Gatzouras** and **Barański** type self-affine carpets as well.

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- Scaling scenery of more general $T_{m,n}$ invariant measures (Gibbs measures with strong mixing properties?)
- Scaling scenery of general self-affine/other non-conformal sets
- Conformal Hausdorff dimension of self-affine carpets
- Applications of scaling scenery to other problems in geometric measure theory

Thank you!

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