Assouad type dimensions and homogeneity of fractals

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People working in dimension theory and fractal geometry are often concerned with the rigorous computation of the dimensions of abstract classes of fractal sets.

However, fractals and dimensions often crop up in a wide variety of contexts, with links and applications being found in diverse areas of mathematics, for example, geometric measure theory, dynamical systems, probability theory and differential equations.

Geometric measure theory

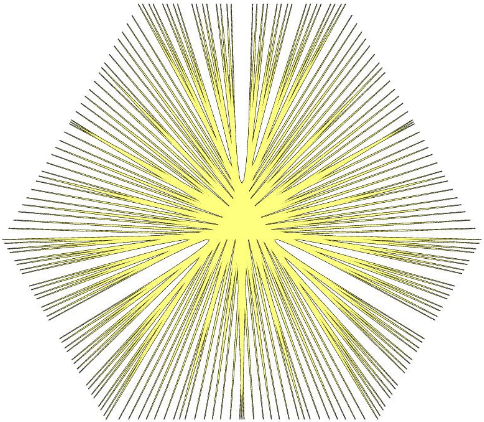

Figure: A Kakeya needle set.

Dynamical systems

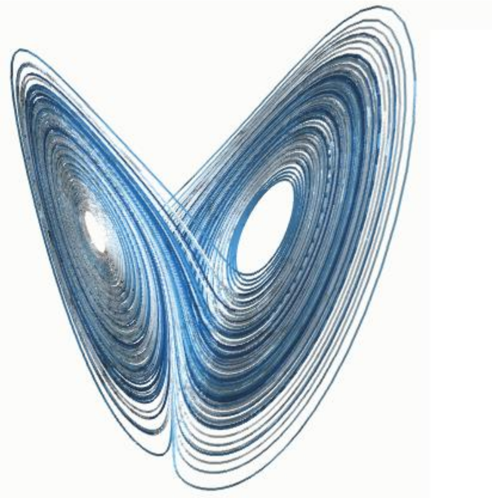

Figure: Chaotic solution to the Lorenz system.

Probability theory

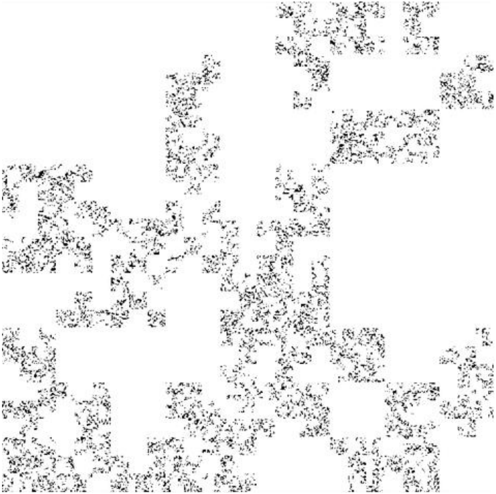

Figure: Fractal percolation.

Differential equations: fluid dynamics

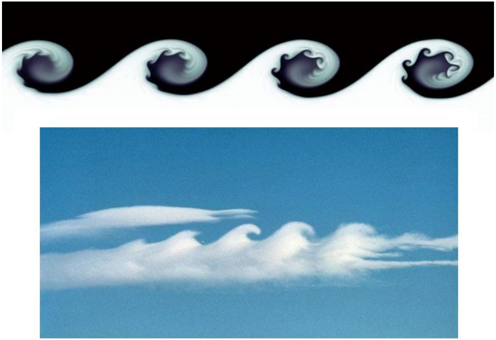

Figure: Kelvin-Helmholtz instability

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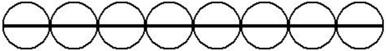

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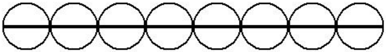

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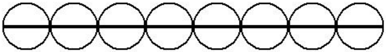

How many balls of diameter $1/8$ do we need to cover it? $\ldots 8 = (1/8)^{-1}$

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... and the 'dimension' of the line segment is 1.

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Let (X, d) be a compact metric space. For any non-empty subset $F \subseteq X$ and $r > 0$, let $N_r(F)$ be the smallest number of open sets with diameter less than or equal to r required to cover F.

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In fact, solving for $\dim F$ formally yields the upper and lower box dimensions.

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  Robinson: *Dimensions, Embeddings, and Attractors*
  Heinonen: *Lectures on Analysis on Metric Spaces*.

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- The Assouad dimension gives 'coarse but local' information about a set, unlike the Hausdorff dimension which gives 'fine but global' information.

The Assouad dimension

The *Assouad dimension* of a non-empty subset F of X is defined by

$$\dim_{\mathrm{A}} F \;=\; \inf \Bigg\{ \;\alpha \;\;:\; \text{there exists constants } C,\, \rho > 0 \text{ such that,} \\ \text{for all } 0 < r < R \leqslant \rho, \text{ we have} \\ \sup_{x \in F} N_r\big(B(x,R) \cap F\big) \;\leqslant\; C \left(\frac{R}{r}\right)^{\alpha} \;\Bigg\}.$$

The lower dimension

We will also be concerned with the natural dual to Assouad dimension, which we call the *lower dimension*.

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This quantity was introduced by Larman in the 1960s, where it was called the *minimal dimensional number*. It has also been referred to by other names, for example: the *lower Assouad dimension* by Käenmäki, Lehrbäck and Vuorinen and the *uniformity dimension* (Tuomas Sahlsten, personal communication).

Relationships between dimensions

For a totally bounded subset F of a metric space, we have

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The lower dimension is in general not comparable to the Hausdorff dimension or packing dimension. However, if F is compact, then

$$\begin{array}{ccccccccc} & & & & \dim_{\mathrm{P}} F & & & & \\ & & & \rotatebox{30}{$\leqslant$} & & \rotatebox{-30}{$\leqslant$} & & & \\ \dim_{\mathrm{L}} F & \leqslant & \dim_{\mathrm{H}} F & & & & \overline{\dim}_{\mathrm{B}} F & \leqslant & \dim_{\mathrm{A}} F. \\ & & & \rotatebox{-30}{$\leqslant$} & & \rotatebox{30}{$\leqslant$} & & & \\ & & & & \underline{\dim}_{\mathrm{B}} F & & & & \end{array}$$

Basic properties

Property	$\dim_H$	$\dim_P$	$\underline{\dim}_B$	$\overline{\dim}_B$	$\dim_L$	$\dim_A$
Monotone	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$
Finitely stable	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$
Countably stable	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$
Lipschitz stable	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$
Bi-Lipschitz stable	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Stable under taking closures	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Open set property	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$
Measurable	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$

Basic properties: products

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$$\begin{aligned}\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y \;&\leqslant\; \dim_{\mathrm{H}}(X \times Y) \;\leqslant\; \dim_{\mathrm{H}} X + \dim_{\mathrm{P}} Y \\ &\leqslant\; \dim_{\mathrm{P}}(X \times Y) \;\leqslant\; \dim_{\mathrm{P}} X + \dim_{\mathrm{P}} Y,\end{aligned}$$

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The Assouad dimension and lower dimension are also a natural 'dimension pair'.

Theorem (Assouad '77-'79, F. '13)

For metric spaces X and Y, we have

$$\dim_L X + \dim_L Y \leqslant \dim_L(X \times Y) \leqslant \dim_L X + \dim_A Y$$

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Note: there are many natural 'product metrics' to impose on the product space $X \times Y$, but any reasonable choice is bi-Lipschitz equivalent to the metric $d_{X\times Y}$ on $X \times Y$ defined by

$$d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) = \max\{d_X(x_1, x_2),\ d_Y(y_1, y_2)\}.$$

Iterated function systems

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Let (X, d) be a compact metric space. An iterated function system (IFS) is a finite collection $\{S_i\}_{i \in \mathcal{I}}$ of contracting self maps on X. It is a fundamental result in fractal geometry that for every IFS there exists a unique non-empty compact set F, called the *attractor*, which satisfies

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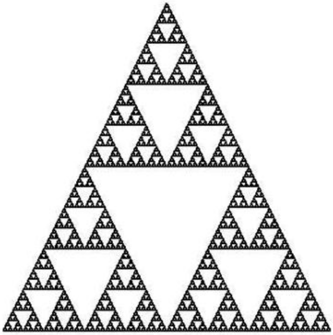
If an IFS consists solely of *similarity* transformations, then the attractor is called a *self-similar set*. Likewise, if X is a Euclidean space and the mappings are all translate linear (*affine*) transformations, then the attractor is called *self-affine*.

Iterated function systems



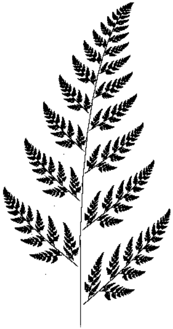


Figure: Left: The self-similar Sierpiński Triangle. Right: The self-affine Barnsley Fern.

Self-similar sets

Self-similar sets are in a certain sense the most basic type of fractal. Let $c_i \in (0,1)$ denote the contraction ratio for the similarity map S_i. Then the solution s to the famous Hutchinson-Moran formula

$$\sum_{i \in \mathcal{I}} c_i^s = 1$$

is known as the *similarity dimension* of the system and is the 'best guess' for the Hausdroff dimension of the attractor F.

Self-similar sets with overlaps

It is well-known that any self-similar set (regardless of overlaps) satisfies:

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Olsen ('12) asked if the Assouad dimension of a self-similar set with overlaps can ever exceed the upper box dimension.

Self-similar sets with overlaps

Answer:

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Theorem (F. '13)

Any self-similar set satisfies

$$\dim_L F \;=\; \dim_H F \;=\; \underline{\dim}_B F \;=\; \overline{\dim}_B F \;=\; \dim_P F \;\leqslant\; \dim_A F$$

and **the final inequality can be strict**.

Proof

We need to prove two things:

(1) Any self-similar set satisfies $\dim_{\mathrm{L}} F \; = \; \dim_{\mathrm{H}} F$

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(2) There exists a self-similar set with $\overline{\dim}_{\mathrm{B}} F \; < \; \dim_{\mathrm{A}} F$

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We will now prove (2) by constructing an example.

Proof

Let $\alpha, \beta, \gamma \in (0,1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps S_1, S_2, S_3 on $[0,1]$ as follows

$$S_1(x) = \alpha x, \qquad S_2(x) = \beta x \qquad \text{and} \qquad S_3(x) = \gamma x + (1-\gamma).$$

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Proposition

Let $X \subset \mathbb{R}$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$ for some non-empty compact set $\hat{F}$. Then $\dim_A \hat{F} \leqslant \dim_A F$. The set $\hat{F}$ is called a weak tangent to F.

Proof

We will now show that $[0, 1]$ is a weak tangent to F in the above sense. Let $X = [0, 1]$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let T_k be defined by

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Since

$$E_k \ := \ \left\{ \alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, \dots, \infty\} \right\} \cap [0,1] \ \subset \ T_k(F) \cap [0,1]$$

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Proof

To see why $\overline{\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \mathbb{Z}\}} \cap [0,1] = [0,1]$ we apply Dirichlet's Theorem in the following way.

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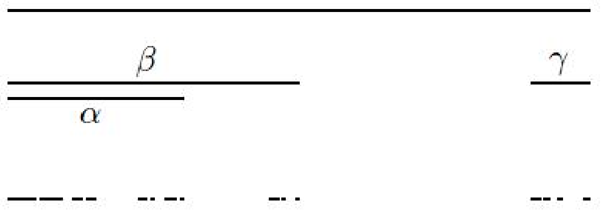
with n arbitrarily large. We can thus make $m \log \alpha + n \log \beta$ arbitrarily small and this gives the result.

Proof

If we choose α, β, γ such that $s < 1$, then

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F \leqslant s < 1 = \dim_{\mathrm{A}} F.$$



Self-affine carpets

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Self-affine carpets

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This began with the Bedford-McMullen carpets with numerous generalisations being introduced by, for example, Lalley-Gatzouras ('92), Barański ('07), Feng-Wang ('05) and F. ('12).

Self-affine carpets

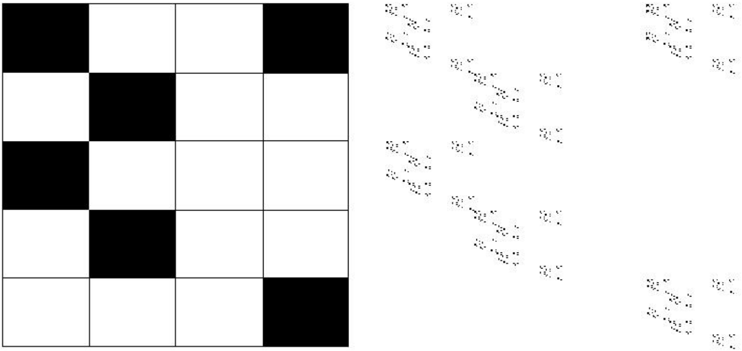
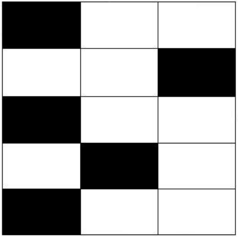
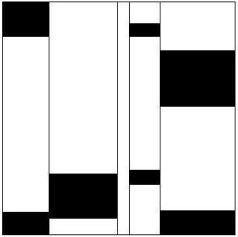


Figure: A self-affine Bedford-McMullen carpet with $m = 4$, $n = 5$. The shaded rectangles on the left indicate the 6 maps in the IFS.

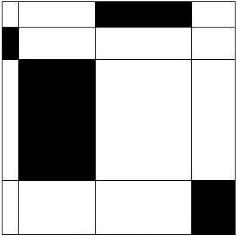
Self-affine carpets



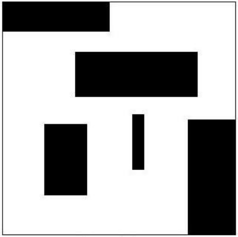
Bedford-McMullen



Gatzouras-Lalley



Barański

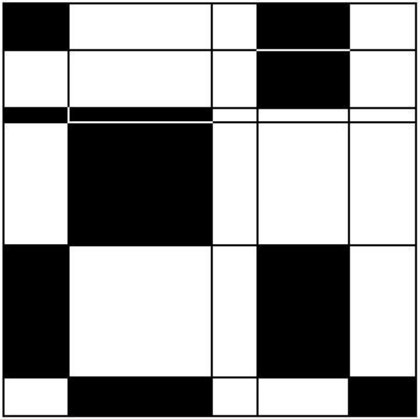


Feng-Wang

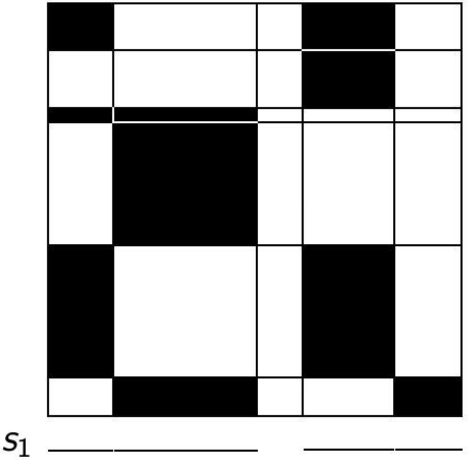
Self-affine carpets

Some notation . . .

Self-affine carpets



Self-affine carpets



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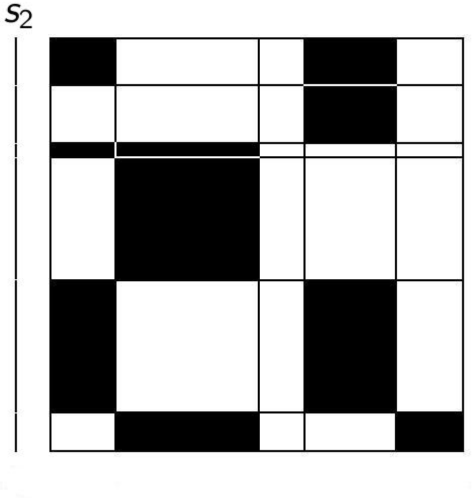

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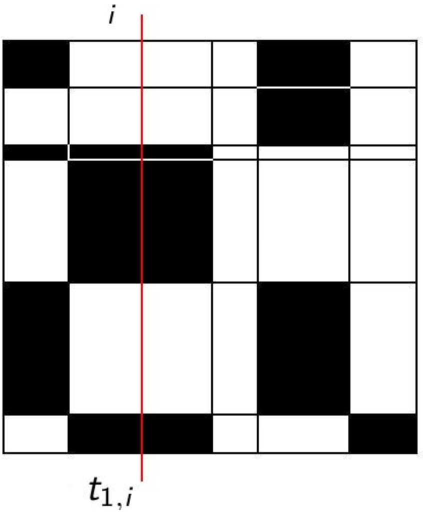

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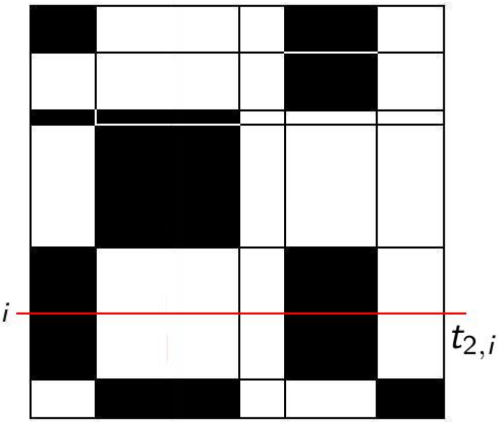
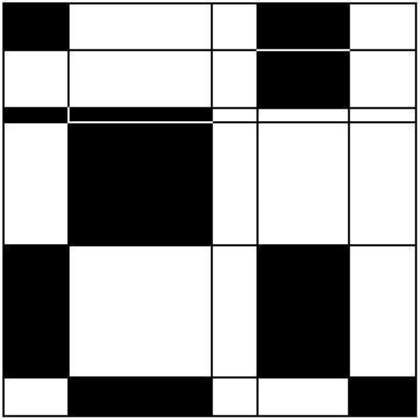
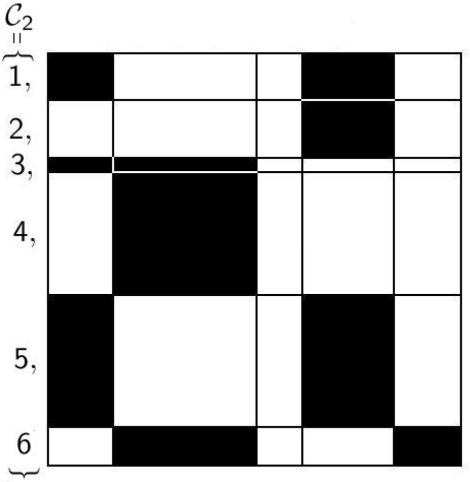


Self-affine carpets



$\mathcal{C}_1 = \{1, \quad 2, \quad \times \quad 3, \quad 4\}$

Self-affine carpets



Self-affine carpets

Theorem (Mackay '11)

Let F be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

$$\dim_A F \;=\; s_1 + \max_{i \in \mathcal{C}_1} t_{1,i}$$

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Theorem (F. '13)

Let F be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

$$\dim_L F = s_1 + \min_{i \in \mathcal{C}_1} t_{1,i}$$

Self-affine carpets

Theorem (F. '13)

Let F be a self-affine carpet in the Barański class (and not in the Lalley-Gatzouras class). Then

$$\dim_A F = \max_{j=1,2} \max_{i \in \mathcal{C}_j} \left(s_j + t_{j,i}\right)$$

and

$$\dim_L F = \min_{j=1,2} \min_{i \in \mathcal{C}_j} \left(s_j + t_{j,i}\right)$$

Thank you!

Main references

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