Assouad type dimensions and homogeneity of fractals

Jonathan M. Fraser

The University of St Andrews
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People working in dimension theory and fractal geometry are often concerned with the rigorous computation of the dimensions of abstract classes of fractal sets.
However, fractals and dimensions often crop up in a wide variety of contexts, with links and applications being found in diverse areas of mathematics, for example, geometric measure theory, dynamical systems, probability theory and differential equations.
Figure: A Kakeya needle set.
Dynamical systems

Figure: Chaotic solution to the Lorenz system.
Probability theory

Figure: Fractal percolation.
**Figure:** Kelvin-Helmholtz instability
Consider a unit line segment.
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How many balls of diameter $\frac{1}{8}$ do we need to cover it?
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In general we will need roughly $r^{-1}$ balls of diameter $r$ to cover the line segment . . .

. . . and the ‘dimension’ of the line segment is 1.
Let \((X, d)\) be a compact metric space. For any non-empty subset \(F \subseteq X\) and \(r > 0\), let \(N_r(F)\) be the smallest number of open sets with diameter less than or equal to \(r\) required to cover \(F\).
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In fact, solving for \(\dim F\) formally yields the upper and lower box dimensions.
The Assouad dimension

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Robinson: Dimensions, Embeddings, and Attractors
Heinonen: Lectures on Analysis on Metric Spaces

Jonathan M. Fraser  Assouad type dimensions
The Assouad dimension was introduced by Assouad in the 1970s. An important tool in the study of quasi-conformal mappings, embeddability problems and PDEs. In fact the initial motivation was to prove the following theorem: a metric space can be quasisymmetrically embedded into some Euclidean space if and only if it has finite Assouad dimension.
The Assouad dimension was introduced by Assouad in the 1970s. It is an important tool in the study of quasi-conformal mappings, embeddability problems and PDEs. In fact, the initial motivation was to prove the following theorem: a metric space can be quasisymmetrically embedded into some Euclidean space if and only if it has finite Assouad dimension.

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The Assouad dimension of a non-empty subset $F$ of $X$ is defined by

$$\dim_A F = \inf \left\{ \alpha : \text{there exists constants } C, \rho > 0 \text{ such that,} \right.$$ 

for all $0 < r < R \leq \rho$, we have

$$\sup_{x \in F} N_r(B(x, R) \cap F) \leq C \left( \frac{R}{r} \right)^\alpha \right\}.$$
The lower dimension

We will also be concerned with the natural dual to Assouad dimension, which we call the *lower dimension*. 

\[ \dim_{\text{L}} F = \sup \{ \alpha : \text{there exist constants } C, \rho > 0 \text{ such that, for all } 0 < r < R \leq \rho, \text{ we have} \inf_{x \in F} N_r(B(x, R) \cap F) \geq C(R/r)\alpha \} \]

This quantity was introduced by Larman in the 1960s, where it was called the *minimal dimensional number*. It has also been referred to by other names, for example: the *lower Assouad dimension* by K¨aenm¨aaki, Lehrb¨ack and Vuorinen and the *uniformity dimension* (Tuomas Sahlsten, personal communication).
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For a totally bounded subset $F$ of a metric space, we have

$$\dim_L F \leq \dim_B F \leq \overline{\dim}_B F \leq \dim_A F.$$
Relationships between dimensions

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$$\dim_L F \leq \dim_B F \leq \overline{\dim}_B F \leq \dim_A F.$$ 

The lower dimension is in general not comparable to the Hausdorff dimension or packing dimension. However, if $F$ is compact, then

$$\dim_P F \leq \dim_L F \leq \dim_H F \leq \overline{\dim}_B F \leq \dim_A F.$$
## Basic properties

<table>
<thead>
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<th>Property</th>
<th>$\dim_H$</th>
<th>$\dim_P$</th>
<th>$\dim_B$</th>
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Basic properties: products

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\[ \dim_B X + \dim_B Y \leq \dim_B (X \times Y) \leq \dim_B X + \overline{\dim}_B Y \]

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Basic properties: products

The Assouad dimension and lower dimension are also a natural ‘dimension pair’.

**Theorem (Assouad ’77-’79, F. ’13)**

*For metric spaces $X$ and $Y$, we have*

\[
\dim_L X + \dim_L Y \leq \dim_L (X \times Y) \leq \dim_L X + \dim_A Y
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**Note:** there are many natural ‘product metrics’ to impose on the product space $X \times Y$, but any reasonable choice is bi-Lipschitz equivalent to the metric $d_{X \times Y}$ on $X \times Y$ defined by

$$
  d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.
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Iterated function systems

Iterated function systems (IFSs) provide many of the basic ‘toy models’ of fractals. It is a natural way of creating the self-similarity seen in many examples of real life and theoretical fractals.
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Let \((X, d)\) be a compact metric space. An iterated function system (IFS) is a finite collection \(\{S_i\}_{i \in I}\) of contracting self maps on \(X\). It is a fundamental result in fractal geometry that for every IFS there exists a unique non-empty compact set \(F\), called the attractor, which satisfies

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If an IFS consists solely of similarity transformations, then the attractor is called a self-similar set. Likewise, if \(X\) is a Euclidean space and the mappings are all translate linear (affine) transformations, then the attractor is called self-affine.
Iterated function systems

Figure: Left: The self-similar Sierpiński Triangle. Right: The self-affine Barnsley Fern.
Self-similar sets

Self-similar sets are in a certain sense the most basic type of fractal. Let $c_i \in (0, 1)$ denote the contraction ratio for the similarity map $S_i$. Then the solution $s$ to the famous Hutchinson-Moran formula

$$\sum_{i \in I} c_i^s = 1$$

is known as the similarity dimension of the system and is the ‘best guess’ for the Hausdorff dimension of the attractor $F$. 
It is well-known that any self-similar set (regardless of overlaps) satisfies:

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Olsen (’12) asked if the Assouad dimension of a self-similar set with overlaps can ever exceed the upper box dimension.
Answer:
Self-similar sets with overlaps

Answer: Yes!
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**Theorem (F. ’13)**

Any self-similar set satisfies

\[ \dim_L F = \dim_H F = \dim_B F = \overline{\dim}_B F = \dim_P F \leq \dim_A F \]

and the final inequality can be strict.
Proof

We need to prove two things:

(1) Any self-similar set satisfies \( \dim_L F = \dim_H F \)

and

(2) There exists a self-similar set with \( \overline{\dim}_B F < \dim_A F \)
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We will now prove (2) by constructing an example.
Let $\alpha, \beta, \gamma \in (0, 1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps $S_1, S_2, S_3$ on $[0, 1]$ as follows

$$S_1(x) = \alpha x, \quad S_2(x) = \beta x \quad \text{and} \quad S_3(x) = \gamma x + (1 - \gamma).$$
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Let $F$ be the self-similar attractor of $\{S_1, S_2, S_3\}$. We will now prove that $\dim_A F = 1$ and, in particular, the Assouad dimension is independent of $\alpha, \beta, \gamma$ provided they are chosen with the above property.
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**Proposition**

Let $X \subset \mathbb{R}$ be compact and let $F$ be a compact subset of $X$. Let $T_k$ be a sequence of similarity maps defined on $\mathbb{R}$ and suppose that $T_k(F) \cap X \to_{d_H} \hat{F}$ for some non-empty compact set $\hat{F}$. Then $\dim_A \hat{F} \leq \dim_A F$. The set $\hat{F}$ is called a weak tangent to $F$. 
Proof

We will now show that $[0, 1]$ is a weak tangent to $F$ in the above sense. Let $X = [0, 1]$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let $T_k$ be defined by

$$T_k(x) = \beta^{-k}x.$$
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Since

$$E_k := \{\alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, \ldots, \infty\}\} \cap [0, 1] \subset T_k(F) \cap [0, 1]$$

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Assouad type dimensions
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\[
m \log \alpha + n \log \beta = (n \log \alpha) \left(\frac{m}{n} + \frac{\log \beta}{\log \alpha}\right)
\]
and Dirichlet’s Theorem gives that there exists infinitely many \(n\) such that
\[
\left| \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right| < \frac{1}{n^2}
\]
for some \(m\).
Proof

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$$\{m \log \alpha + n \log \beta : m \in \mathbb{N}, n \in \mathbb{Z}\}$$

is dense in $(-\infty, 0)$. We have

$$m \log \alpha + n \log \beta = (n \log \alpha) \left( \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right)$$

and Dirichlet’s Theorem gives that there exists infinitely many $n$ such that

$$\left| \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right| < 1/n^2$$

for some $m$. Since $\log \beta/\log \alpha$ is irrational, we may choose $m, n$ to make

$$0 < |m \log \alpha + n \log \beta| < \frac{|\log \alpha|}{n}$$

with $n$ arbitrarily large.
Proof

To see why \( \{\alpha^m\beta^n : m \in \mathbb{N}, n \in \mathbb{Z}\} \cap [0, 1] = [0, 1] \) we apply Dirichlet’s Theorem in the following way. It suffices to show that

\[ \{m \log \alpha + n \log \beta : m \in \mathbb{N}, n \in \mathbb{Z}\} \]

is dense in \((-\infty, 0)\). We have

\[ m \log \alpha + n \log \beta = (n \log \alpha) \left( \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right) \]

and Dirichlet’s Theorem gives that there exists infinitely many \( n \) such that

\[ \left| \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right| < \frac{1}{n^2} \]

for some \( m \). Since \( \log \beta / \log \alpha \) is irrational, we may choose \( m, n \) to make

\[ 0 < |m \log \alpha + n \log \beta| < \frac{|\log \alpha|}{n} \]

with \( n \) arbitrarily large. We can thus make \( m \log \alpha + n \log \beta \) arbitrarily small and this gives the result.
If we choose $\alpha, \beta, \gamma$ such that $s < 1$, then

$$\dim_L F = \dim_H F = \dim_B F \leq s < 1 = \dim_A F.$$
The dimension theory of certain classes of planar self-affine sets, commonly referred to as self-affine carpets, has attracted enormous attention in the literature in the last 30 years.
The dimension theory of certain classes of planar self-affine sets, commonly referred to as self-affine carpets, has attracted enormous attention in the literature in the last 30 years.

This began with the Bedford-McMullen carpets with numerous generalisations being introduced by, for example, Lalley-Gatzouras (’92), Barański (’07), Feng-Wang (’05) and F. (’12).
**Figure:** A self-affine Bedford-McMullen carpet with $m = 4$, $n = 5$. The shaded rectangles on the left indicate the 6 maps in the IFS.
Self-affine carpets

Bedford-McMullen

Gatzouras-Lalley

Barański

Feng-Wang

Assouad type dimensions
Self-affine carpets

Some notation . . .
Self-affine carpets

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Assouad type dimensions
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$S_1$
Self-affine carpets

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Assouad type dimensions
Self-affine carpets
Self-affine carpets

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Assouad type dimensions
Self-affine carpets

\[ C_1 = \{ 1, 2, \times, 3, 4 \} \]
Self-affine carpets

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Assouad type dimensions
Theorem (Mackay ’11)
Let $F$ be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then
\[ \dim_A F = s_1 + \max_{i \in C_1} t_{1,i} \]
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Let $F$ be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

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Theorem (F. ’13)
Let $F$ be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

$$\dim_L F = s_1 + \min_{i \in C_1} t_{1,i}$$
Theorem (F. ’13)

Let $F$ be a self-affine carpet in the Barański class (and not in the Lalley-Gatzouras class). Then

$$\dim A F = \max_{j=1,2} \max_{i \in C_j} \left( s_j + t_{j,i} \right)$$

and

$$\dim L F = \min_{j=1,2} \min_{i \in C_j} \left( s_j + t_{j,i} \right)$$
Thank you!

