Fourier transforms of measures supported on graphs

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joint work with **Tuomas Orponen** and **Tuomas Sahlsten**

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$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-(s+\varepsilon)/2}, \xi \in \mathbb{R}^2 \quad\Longrightarrow\quad I_s(\mu) < \infty \quad\Longrightarrow\quad \dim_{\mathrm{H}} K \geq s.$$

Fourier analysis and Hausdorff dimension

This motivates...

Definition (Fourier dimension)

$$\dim_{\mathrm{F}} K := \sup\{s \leq 2 : \exists\, \mu \in \mathcal{P}(K) \text{ with } |\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}, \xi \in \mathbb{R}^2\}$$

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Definition (Round sets)

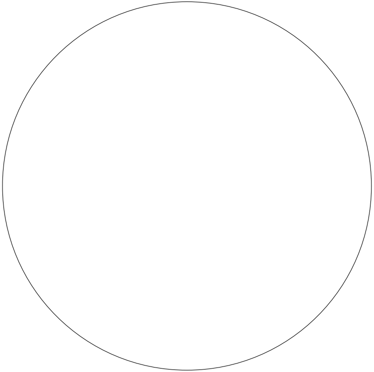
If $\dim_{\mathrm{F}} K = \dim_{\mathrm{H}} K$, we say that K is **round**.

- Round sets are also known as **Salem sets**.

Finding round sets

Finding round sets

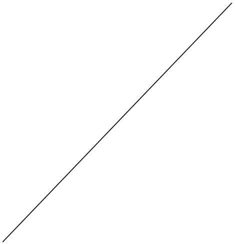
Unit circle S^1 is round...



...since $\widehat{\mathcal{H}^1 \llcorner S^1}$ decays like $|\xi|^{-1/2}$.

Finding round sets II

A line $L \subset \mathbb{R}^2$ is **not** round...



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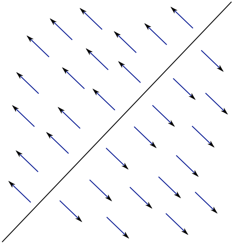

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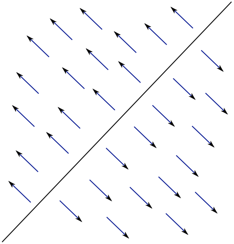

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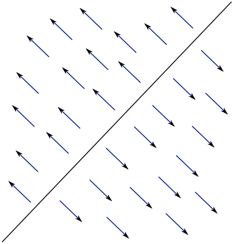

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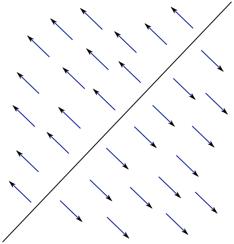

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'Non-trivial' round sets are hard to construct deterministically, but

- there are some examples by Kahane and Kaufman; in particular some sets arising from Diophantine approximation are round.

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Theorem (Kahane 1986)

Let $\omega \colon [0,\infty) \to \mathbb{R}$ *be* 1*-dimensional Brownian motion, and let* $K \subset [0,\infty)$ *be compact. Then the image* $\omega(K) \subset \mathbb{R}$ *is a.s. round, with*

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Analogous result also holds for fractional Brownian motion.

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> "*...proving almost sure roundedness for specific random sets is never easy and it remains an open program for most natural random sets: level sets and graphs of random functions in particular.*"

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Later (2006), Shieh and Xiao explicitly ask:

> *"Are the graph and level sets of a stochastic process such as fractional Brownian motion Salem sets?"*

Finding round sets V

The conjecture is partially confirmed for level sets:

Theorem (Fouché and Mukeru 2013)

Let $\omega : [0, \infty) \to \mathbb{R}$ *be* 1*-dimensional fractional Brownian motion. Then for* $a \in \mathbb{R}$*, the level set*

$$L_\omega(a) = \{0 \le t \le 1 : \omega(t) = a\}$$

is round with positive probability and

$$\dim_{\mathrm{F}} L_\omega(a) = \dim_{\mathrm{H}} L_\omega(a) = \frac{1}{2}.$$

Graphs of Brownian motion

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The Hausdorff dimension part is classical:

Theorem (Taylor 1953, Adler 1977)

Let $\omega : [0, \infty) \to \mathbb{R}$ *be the* 1*-dimensional fractional Brownian motion with Hurst exponent* $0 < H < 1$*. Then, the graph*

$$G_\omega := \{(t, \omega(t)) : t \in [0, \infty)\} \subset \mathbb{R}^2$$

a.s. satisfies

$$\dim_{\mathrm{H}} G_\omega = 2 - H.$$

- In particular, $\dim_{\mathrm{H}} G_\omega > 1$ a.s.

Fourier dimension of graphs

We proved:

Theorem (J. F., T. Orponen and T. Sahlsten. 2013)

Let $E \subset \mathbb{R}$ *be a set, and let* $f \colon E \to \mathbb{R}$ *be a function. Then*

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Combining this with Taylor's and Adler's results answers Kahane's, Shieh's and Xiao's questions on random graphs in the negative:

Corollary

*The Brownian graphs G_ω are a.s. **not** round/Salem.*

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- **Marstrand's projection theorem**: if a planar set K has Fourier dimension $s \in [0, 1]$, then all projections of K onto lines have Fourier dimension $\geq s$ (folklore).
- **Falconer's distance set conjecture**: if a planar set K has Fourier dimension $s > 1$, then the distance set of K, namely

$$\Delta(K) = \{|x - y| : x, y \in K\},$$

  has positive length (P. Mattila).

Proof II

For us, the key is proving a Fourier analytic version of Marstrand's slicing theorem.

- **Marstrand's slicing theorem**: if a planar set K has $\dim_{\mathrm{F}} K > 1$, then in every direction there are Leb positively many lines ℓ with

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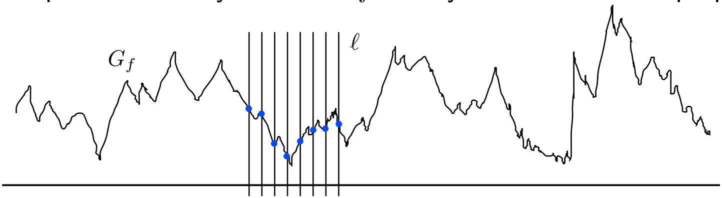
- Graphs of arbitrary functions f clearly do not have this property:



Hence, they can have Fourier dimension *at most one*!

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- **The solution**: invent a Fourier analytic proof for Marstrand's slicing theorem.

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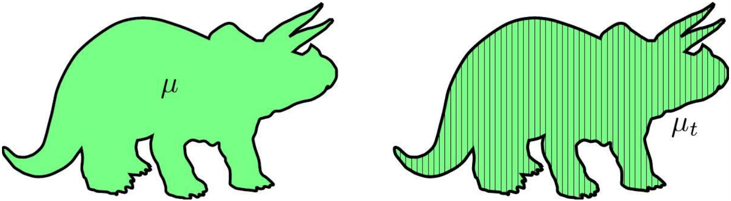
- Choose $\mu \in \mathcal{P}(K)$ with $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-(s+\varepsilon)/2}$ for some $s > 1$.
- **Slice** the measure μ with vertical lines $L_t = \{(t,y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$ to obtain 'sliced measures' μ_t, supported on $K \cap L_t$.



Easy but important: $\mu_t \neq 0$ for Leb positively many t.
(this requires the decay assumption of $\widehat{\mu}$ and Plancherel's formula)

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$$\Longrightarrow I_{s-1}(\mu_t) < \infty \quad \text{for Leb a.e. } t \in \mathbb{R}.$$

$$\Longrightarrow \dim_{\mathrm{H}}[K \cap L_t] \geq s-1 > 0 \quad \text{for Leb pos. many } t.$$

Q.E.D.

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But the question of finite stability is still open!

Further results

Not being able to solve the first open question, we considered the following variant:

Question

What is the Fourier dimension of the graph of a typical function $f \in C[0,1]$?

- Here we mean *typical* in the sense of Baire category.

Further results II

We proved:

Theorem (J. F., T. Orponen and T. Sahlsten. 2013)

The typical function $f \in C[0,1]$ has the following property. If $\mu \in \mathcal{P}(G_f)$, then

$$\limsup_{|\xi| \to \infty} |\widehat{\mu}(\xi)| \geq \frac{1}{5}.$$

In particular, $\dim_{\mathrm{F}} G_f = 0$.

- The constant $1/5$ is not sharp (optimal constant unknown).
- Hausdorff dimension $\dim_{\mathrm{H}} G_f \geq 1$ for any $f \in C[0,1]$, so our result implies that the graph of a typical function is *not* round/Salem!

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