Fractal Geometry and Dimension Theory

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Fractals

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(2) they display some sort of 'self-similarity'

(3) classical techniques in (smooth) geometry are not sufficient to describe them

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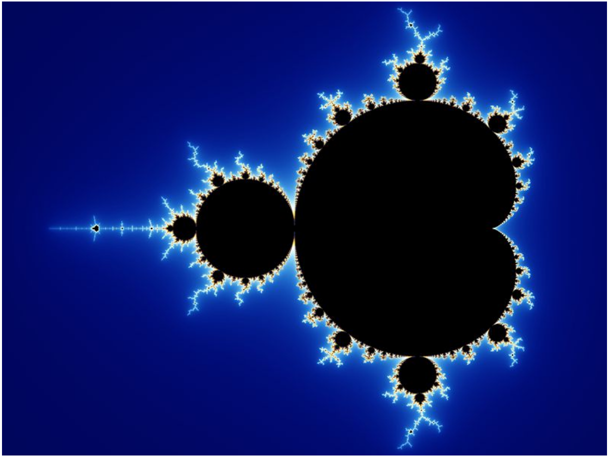
Fractal geometry is the study of fractals and is mainly concerned with examining their geometrical properties in a rigorous framework.

Some examples



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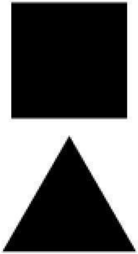

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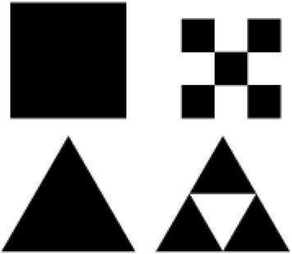

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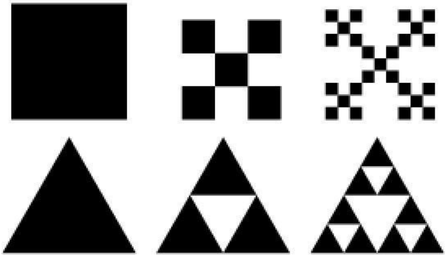

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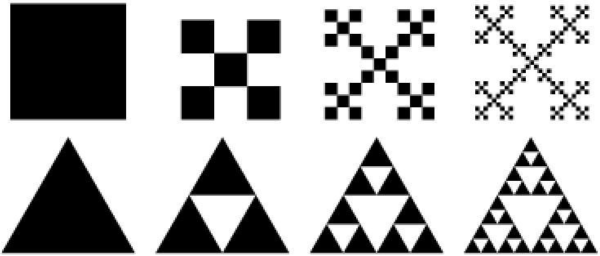

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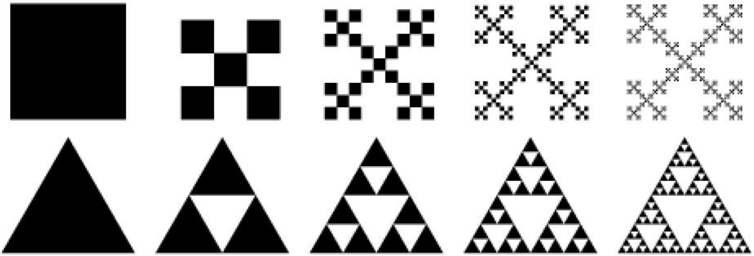

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Fractals at St Andrews

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(4) probability theory: random fractals and random measures

An example problem

A central theme to the geometry of fractals is dimension. Several different notions of dimension are used to study fractals and they attempt to describe how an object fills up space on small scales.

These dimensions are a natural way of extending the familiar notions of 1,2 and 3 dimensional objects to more general situations; fractals often have fractional dimensions!

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However, this limit may not exist and so we take lower and upper limits. These dimensions are actually called the lower and upper box dimensions of E

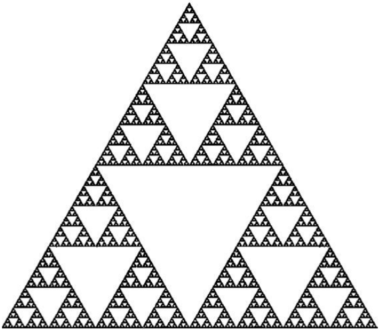
$$\underline{\dim}_{\mathrm{B}} E = \liminf_{r \to 0} \frac{\log N_r(E)}{-\log r}$$

and

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r}.$$

An example problem

We may wish to compute the box dimensions of certain families of fractals sets. What about the classic Sierpiński Triangle?



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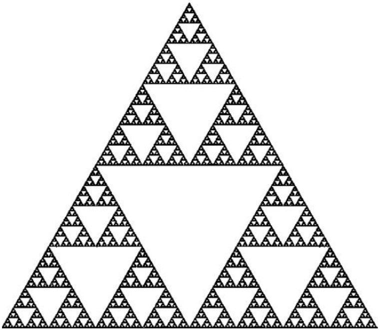

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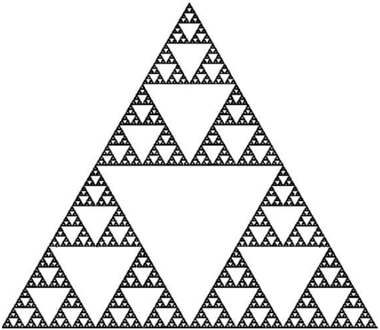

OK, it's $\log 3 / \log 2$. Developing new techniques to compute the box (and other) dimensions of complicated classes of fractals is on the forefront of current research.

Hope to see you in St Andrews sometime!