Assoud type dimensions and homogeneity of fractals

Jonathan M. Fraser

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- The Assouad dimension gives 'coarse but local' information about a set, unlike the Hausdorff dimension which gives 'fine but global' information.

The Assouad dimension

Let (X, d) be a metric space and for any non-empty subset $F \subseteq X$ and $r > 0$, let $N_r(F)$ be the smallest number of open sets with diameter less than or equal to r required to cover F.

The Assouad dimension

Let (X, d) be a metric space and for any non-empty subset $F \subseteq X$ and $r > 0$, let $N_r(F)$ be the smallest number of open sets with diameter less than or equal to r required to cover F. The *Assouad dimension* of a non-empty subset F of X is defined by

$$\dim_{\mathrm{A}} F = \inf \Bigg\{ \alpha : \text{there exists constants } C, \rho > 0 \text{ such that, for all } 0 < r < R \leqslant \rho, \text{ we have } \sup_{x \in F} N_r\big(B(x,R) \cap F\big) \leqslant C\left(\frac{R}{r}\right)^{\alpha} \Bigg\}.$$

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This quantity was introduced by Larman in the 1960s, where it was called the *minimal dimensional number*. It has also been referred to by other names, for example: the *lower Assouad dimension* by Käenmäki, Lehrbäck and Vuorinen and the *uniformity dimension* (Tuomas Sahlsten, personal communication).

Relationships between dimensions

For a totally bounded subset F of a metric space, we have

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The lower dimension is in general not comparable to the Hausdorff dimension or packing dimension. However, if F is compact, then

$$\begin{array}{ccccccccc} & & & & \dim_{\mathrm{P}} F & & & & \\ & & & \rotatebox{45}{$\leqslant$} & & \rotatebox{-45}{$\leqslant$} & & & \\ \dim_{\mathrm{L}} F & \leqslant & \dim_{\mathrm{H}} F & & & & \overline{\dim}_{\mathrm{B}} F & \leqslant & \dim_{\mathrm{A}} F. \\ & & & \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & & \\ & & & & \underline{\dim}_{\mathrm{B}} F & & & & \end{array}$$

Basic properties

Property	$\dim_H$	$\dim_P$	$\underline{\dim}_B$	$\overline{\dim}_B$	$\dim_L$	$\dim_A$
Monotone	✓	✓	✓	✓	×	✓
Finitely stable	✓	✓	×	✓	×	✓
Countably stable	✓	✓	×	×	×	×
Lipschitz stable	✓	✓	✓	✓	×	×
Bi-Lipschitz stable	✓	✓	✓	✓	✓	✓
Stable under taking closures	×	×	✓	✓	✓	✓
Open set property	✓	✓	✓	✓	×	✓
Measurable	✓	×	✓	✓	✓	✓

Basic properties: products

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$$\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y \leqslant \dim_{\mathrm{H}}(X \times Y) \leqslant \dim_{\mathrm{H}} X + \dim_{\mathrm{P}} Y$$

$$\leqslant \dim_{\mathrm{P}}(X \times Y) \leqslant \dim_{\mathrm{P}} X + \dim_{\mathrm{P}} Y,$$

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Basic properties: products

The Assouad dimension and lower dimension are also a natural 'dimension pair'.

Theorem (Assouad '77-'79, F. '13)

For metric spaces X and Y, we have

$$\dim_L X + \dim_L Y \leqslant \dim_L(X \times Y) \leqslant \dim_L X + \dim_A Y$$

$$\leqslant \dim_A(X \times Y) \leqslant \dim_A X + \dim_A Y.$$

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Note: there are many natural 'product metrics' to impose on the product space $X \times Y$, but any reasonable choice is bi-Lipschitz equivalent to the metric $d_{X\times Y}$ on $X \times Y$ defined by

$$d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) = \max\{d_X(x_1, x_2),\ d_Y(y_1, y_2)\}.$$

Self-similar sets with overlaps

It is well-known that any self-similar set (regardless of overlaps) satisfies:

$$\dim_{\mathrm{H}} F \; = \; \underline{\dim}_{\mathrm{B}} F \; = \; \overline{\dim}_{\mathrm{B}} F \; = \; \dim_{\mathrm{P}} F \; \leqslant \; s$$

where s is the similarity dimension.

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Olsen ('12) asked if the Assouad dimension of a self-similar set with overlaps can ever exceed the upper box dimension.

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Answer:

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Theorem (F. '13)

Any self-similar set satisfies

$$\dim_L F \;=\; \dim_H F \;=\; \underline{\dim}_B F \;=\; \overline{\dim}_B F \;=\; \dim_P F \;\leqslant\; \dim_A F$$

and **the final inequality can be strict**.

Proof

We need to prove two things:

(1) Any self-similar set satisfies $\dim_L F = \dim_H F$

and

(2) There exists a self-similar set with $\overline{\dim}_B F < \dim_A F$

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We will now prove (2) by constructing an example.

Proof

Let $\alpha, \beta, \gamma \in (0,1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps S_1, S_2, S_3 on $[0,1]$ as follows

$$S_1(x) = \alpha x, \qquad S_2(x) = \beta x \qquad \text{and} \qquad S_3(x) = \gamma x + (1-\gamma).$$

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Let F be the self-similar attractor of $\{S_1, S_2, S_3\}$. We will now prove that $\dim_A F = 1$ and, in particular, the Assouad dimension is independent of α, β, γ provided they are chosen with the above property.

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Let F be the self-similar attractor of $\{S_1, S_2, S_3\}$. We will now prove that $\dim_A F = 1$ and, in particular, the Assouad dimension is independent of α, β, γ provided they are chosen with the above property. We will use the following proposition due to Mackay and Tyson.

Proposition

Let $X \subset \mathbb{R}$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$ for some non-empty compact set $\hat{F}$. Then $\dim_A \hat{F} \leqslant \dim_A F$. The set $\hat{F}$ is called a weak tangent to F.

Proof

We will now show that $[0, 1]$ is a weak tangent to F in the above sense. Let $X = [0, 1]$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let T_k be defined by

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and Dirichlet's Theorem gives that there exists infinitely many n such that

$$\left|\frac{m}{n} + \frac{\log \beta}{\log \alpha}\right| < 1/n^2$$

for some m.

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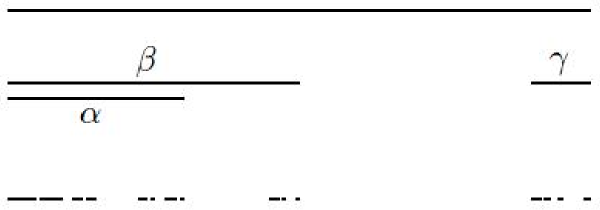
with n arbitrarily large. We can thus make $m \log \alpha + n \log \beta$ arbitrarily small and this gives the result.

Proof

If we choose α, β, γ such that $s < 1$, then

$$\dim_L F = \dim_H F = \dim_B F \leqslant s < 1 = \dim_A F.$$



Self-affine carpets

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This began with the Bedford-McMullen carpets with numerous generalisations being introduced by, for example, Lalley-Gatzouras ('92), Barański ('07), Feng-Wang ('05) and F. ('12).

Self-affine carpets

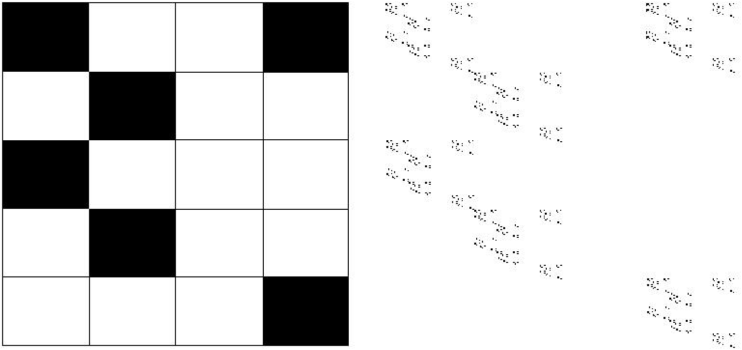
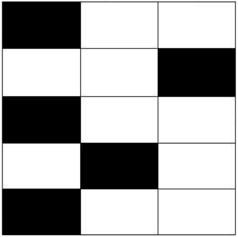
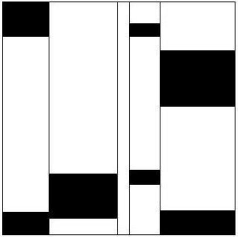


Figure: A self-affine Bedford-McMullen carpet with $m = 4$, $n = 5$. The shaded rectangles on the left indicate the 6 maps in the IFS.

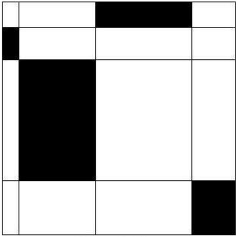
Self-affine carpets



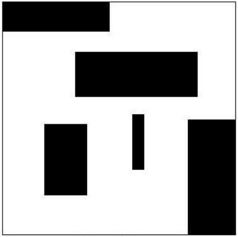
Bedford-McMullen



Gatzouras-Lalley



Barański

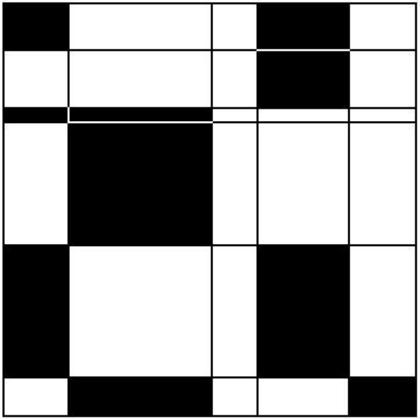


Feng-Wang

Self-affine carpets

Some notation . . .

Self-affine carpets



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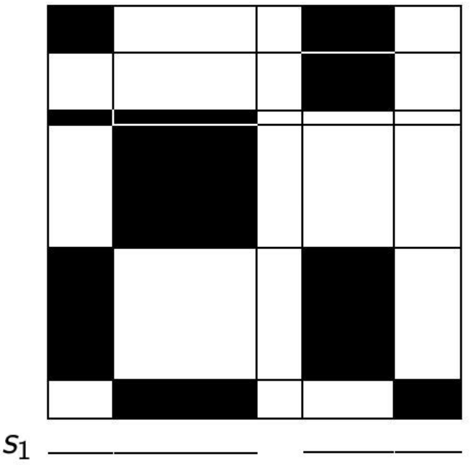

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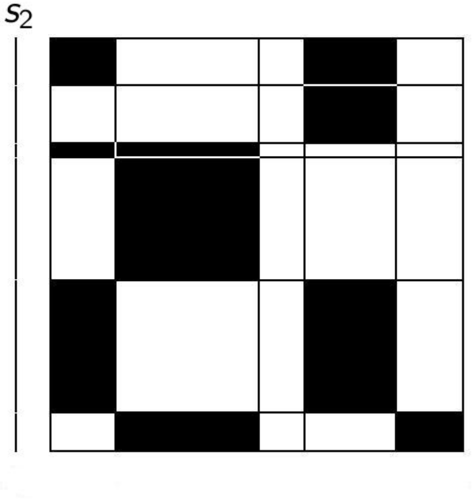

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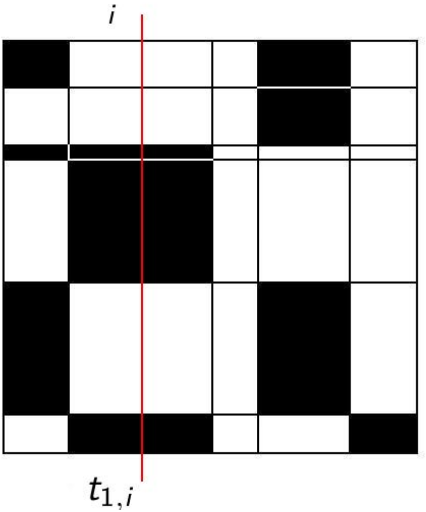
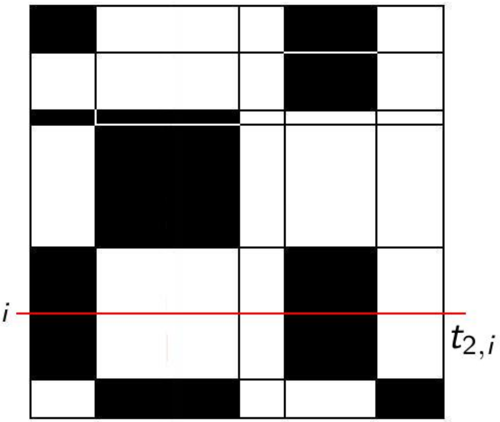


Self-affine carpets



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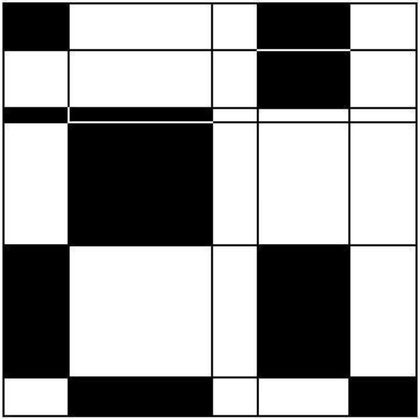
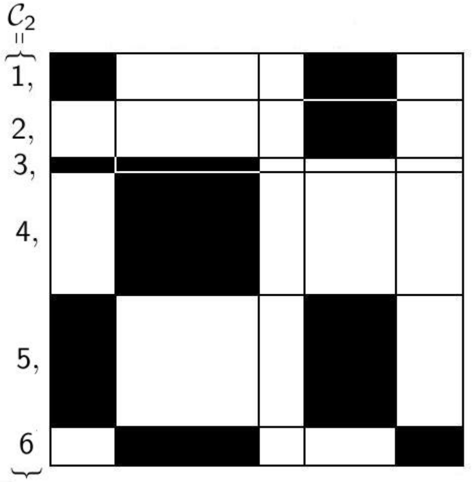


$\mathcal{C}_1 = \{1, \quad 2, \quad \times \quad 3, \quad 4\}$

Self-affine carpets



Self-affine carpets

Theorem (Mackay '11)

Let F be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

$$\dim_A F \ = \ s_1 \ + \ \max_{i \in \mathcal{C}_1} \ t_{1,i}$$

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Theorem (F. '13)

Let F be a self-affine carpet in the Bedford-McMullen or Lalley-Gatzouras class. Then

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Self-affine carpets

Theorem (F. '13)

Let F be a self-affine carpet in the Barański class (and not in the Lalley-Gatzouras class). Then

$$\dim_A F \;=\; \max_{j=1,2} \; \max_{i \in \mathcal{C}_j} \; \left(s_j \,+\, t_{j,i} \right)$$

and

$$\dim_L F \;=\; \min_{j=1,2} \; \min_{i \in \mathcal{C}_j} \; \left(s_j \,+\, t_{j,i} \right)$$

Outline of the proof: upper bound for Assouad dimension

We rely on a delicate covering argument. In particular, we wish to show that for all $\varepsilon > 0$ any **approximate square** $Q(\mathbf{i}, R)$ of radius R can be covered by no more than

$$C_\varepsilon \left(\frac{R}{r}\right)^{s+\varepsilon}$$

balls of radius r, where $s = \max_{j=1,2} \max_{i\in\mathcal{C}_j} \left(s_j + t_{j,i}\right)$ is the target dimension.

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balls of radius r, where $s = \max_{j=1,2} \max_{i \in \mathcal{C}_j} \left(s_j + t_{j,i} \right)$ is the target dimension.

We need to split $Q(\mathbf{i}, R)$ into three parts and cover each part separately using a different technique.

Outline of the proof: lower bound for Assouad dimension

The key idea is to construct a weak tangent which is the appropriate product of an orthogonal projection of F with a self-similar slice of F.

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This is essentially the same argument that was used in Mackay '11, however, there are some extra difficulties due to the fact that all of the maps are not 'lined up' as in the Lalley-Gatzouras case. Mackay is able to choose one map to 'follow into the construction', whereas in certain cases we have to choose one map for a long time and then switch to a map which we can 'follow in' to find the 'correct tangent'. This is so the map we ultimately choose is 'lined up' in the correct way.

Outline of the proof: upper bound for lower dimension

Since lower dimension is a natural dual to Assouad dimension and tends to ‘mirror’ the Assouad dimension in many ways, one might expect, given that weak tangents provide a very natural way to find *lower bounds* for Assouad dimension, that weak tangents might provide a way of giving *upper bounds* for lower dimension.

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"Let $X \subset \mathbb{R}^2$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}^2$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$. Then $\dim_L \hat{F} \geqslant \dim_L F$."

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However, it is easy to see that this is false as one can often find weak tangents with isolated points, and hence lower dimension equal to zero, even if the original set has positive lower dimension.

Outline of the proof: upper bound for lower dimension

As such we need to modify the definition of weak tangent in the following way.

Proposition (very weak tangents)

Let $X \subset \mathbb{R}^n$ be compact and let F be a compact subset of X. Let T_k be a sequence of bi-Lipschitz maps defined on $\mathbb{R}^n$ with Lipschitz constants $a_k, b_k \geqslant 1$ such that

$$a_k|x-y| \;\leqslant\; |T_k(x)-T_k(y)| \;\leqslant\; b_k|x-y| \qquad (x,y \in \mathbb{R}^n)$$

and

$$\sup_k b_k/a_k \;=\; C_0 \;<\; \infty$$

Also assume that there exists a uniform constant $\theta \in (0,1]$ such that for all $r \in (0,1]$ and $\hat{x} \in \hat{F}$, there exists $\hat{y} \in \hat{F}$ such that $B(\hat{y}, r\theta) \subseteq B(\hat{x}, r) \cap X$; then

$$\dim_L F \;\leqslant\; \dim_L \hat{F}.$$

Outline of the proof: upper bound for lower dimension

We can now construct a very weak tangent to F with the correct dimension.

Outline of the proof: lower bound for lower dimension

We use a delicate covering argument similar to that used in the proof of the upper bound for Assouad dimension. Initially we expected the argument to be totally symmetrical, but surprisingly this turned out not to be the case and the argument requires an extra step.

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We do not have time to elaborate on this, but the basic problem is that when we split the approximate square up into three different bits to cover separately, one of the terms gives the wrong dimension. We must hence justify that the other two terms do not 'disappear', allowing us to drop the problematic term. The key idea is to iterate inside the approximate square using a subsystem where the maps are 'lined up' in the same way.

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This technique is reminiscent of that used by Ferguson-Jordan-Shmerkin ('10) when studying projections of self-affine carpets.

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Indeed this idea came from a conversation I had with Thomas in Hong Kong in December, so thanks again!

Thank you!

Main references

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