SOME VANISHING SUMS INVOLVING BINOMIAL COEFFICIENTS IN THE DENOMINATOR

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Abstract. We obtain expressions for sums of the form \( \sum_{j=0}^{m} (-1)^j \frac{d^n}{(n+j)^j} \) and deduce, for an even integer \( d \geq 0 \) and \( m = n > d/2 \), that this sum is 0 or \( \frac{1}{2} \) according as to whether \( d > 0 \) or not. Further, we prove for even \( d > 0 \) that \( \sum_{l=1}^{d} c_{l-1} \frac{(-1)^l \binom{n}{l}}{(l+1)(l+1)} = 0 \) where \( c_r = \frac{1}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^{d-1} \).

Similarly, we show when \( d > 0 \) is even that \( \sum_{r=0}^{d} a_r \frac{(-1)^r \binom{n}{r+1}}{(r+1)(r+1)} = 0 \) where \( a_r = \frac{(-1)^{d+r}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^d \).

Introduction

Identities involving binomial coefficients usually arise in situations where counting is carried out in two different ways. For instance, some identities obtained by William Horrace [1] using probability theory turn out to be special cases of the Chu-Vandermonde identities. Here, we obtain some generalizations of the identities observed by Horrace and give different types of proofs; these, in turn, give rise to some other new identities. In particular, we evaluate sums of the form \( \sum_{j=0}^{m} (-1)^j \frac{d^n}{(n+j)^j} \) and deduce that they vanish when \( d \) is even and \( m = n > d/2 \).

It is well-known [2] that sums involving binomial coefficients can usually be expressed in terms of the hypergeometric functions but it is more interesting if such a function can be evaluated explicitly at a given argument. Identities such as the ones we prove could perhaps be of some interest due to the explicit evaluation possible. The papers [3], [4] are among many which deal with identities for sums where the binomial coefficients occur in the denominator and we use similar methods here.

1. Horrace’s identities - other proofs and generalizations

We start with the identities in Horrace’s paper which he deduced using probability theory.

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Lemma 1.1. For $m \geq 1, n \geq 0$; we have
\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} = \frac{n}{n+m}; \text{ and }
\sum_{j=1}^{m} (-1)^j \binom{m}{j} = \frac{mn}{(n+m)(n+m-1)}.
\]

The lemma can be easily deduced by induction or using the method of [3].

Remark 1.2. We give another expression for the left hand sides of these identities. Recall the forward difference operator $\Delta$ defined on a function $f$ by $(\Delta f)(x) = f(x+1) - f(x)$. As usual, one defines $\Delta^{k+1}f = \Delta(\Delta^k f)$ etc. It is easily seen by induction on $m$ that
\[
(\Delta^m f)(x) = \sum_{r=0}^{m} (-1)^r \binom{m}{r} f(x + m - r).
\]

Now, the left hand side of the first identity of Lemma 1.1 is
\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} (n+j)
\]
which is $(\Delta^m g)(0)$ where
\[
g(x) = \frac{n!}{(m+1-x)(m+2-x)\cdots(m+n-x)}.
\]

Now, one can express $g(x)$ as a partial fraction $\sum_{i=1}^{n} \frac{a_i}{m+i-x}$. Also, each $a_j$ can be found by multiplying both sides by the product $(m+1-x)(m+2-x)\cdots(m+n-x)$ and evaluating at $x = m+j$; we have $a_j \prod_{i \not= j} (i-j) = n!$ for each $j \leq n$. Now, we compute $(\Delta^m g)(x) = \sum_{i=1}^{n} (\Delta^m g_i)(x)$ where $g_i(x) = \frac{a_i}{m+i-x}$. Computing, we see that
\[
(\Delta^m g)(0) = n! \sum_{i=1}^{n} \sum_{r=0}^{m} \prod_{j \leq n, j \not= i} \frac{1}{j-i} (-1)^r \binom{m}{r+i}.
\]
which easily simplifies to
\[
(\Delta^m g)(0) = n \sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1} \binom{n-1}{r+1} \binom{m}{r}}{r+i}.
\]

It is worth noting that although the left hand sides of these identities can be thought of as the action by the $(m+n)$-th difference operator, it does not give anything new and merely reproduces the left hand sides again. Now, by Lemma 1.1, we get $(\Delta^m g)(0) = \frac{n}{m+n}$ and we have the following corollary.

Corollary 1.3.
\[
\sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1} \binom{n-1}{r+1} \binom{m}{r}}{r+i} = \frac{1}{m+n}.
\]

Doing the same process with the second identity in Lemma 1.1, we have:
\[
\sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1} \binom{n-1}{r+1} \binom{m}{r}}{r+i} = \frac{mn}{(m+n)(m+n-1)}.
\]

As a matter of fact, the identity of Corollary 1.3 can be proved in a much more general form by another manner as follows.
Lemma 1.4. \[
\sum_{i_1, \ldots, i_k} (-1)^{i_1 + \cdots + i_k} \binom{n_1}{i_1} \cdots \binom{n_k}{i_k} \frac{1}{i_1 + i_2 + \cdots + i_k + 1} = \frac{1}{n_1 + n_2 + \cdots + n_k + 1}.
\]

Proof. Writing \((1 - t)^{n_1 + \cdots + n_k} = (1 - t)^{n_1} \cdots (1 - t)^{n_k}\) and integrating both sides from 0 to 1 after expanding the right side binomially, we have the identity asserted. \[
\square
\]

2. A Vanishing Theorem

A natural generalization of Lemma 1.1 would be to consider the sums of the form \(\sum_{j=1}^{m} (-1)^{j-1} j^d \binom{n}{j}\) for various \(d > 1\). We have the following result which first shows how the roles of \(m\) and \(n\) are interchanged and then implies a vanishing result when \(m = n\). In between, we also adopt a method used in [3] for evaluating sums where binomial coefficients appear in the denominator.

Theorem 2.1. Let \(\theta\) be a polynomial and let \(m + n > \deg(\theta)\). Then, the sum
\[
P_{m,n}(\theta) := \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \frac{\theta(j) \binom{n}{j}}{\binom{m+n}{j}}
\]
satisfies
\[
\binom{m+n}{n} P_{m,n}(\theta) = \sum_{j=0}^{m} (-1)^{j} \theta(j) \binom{m+n}{m-j} = \sum_{i=0}^{n} (-1)^{i} i (\theta(-i) \binom{m+n}{n-i} + \theta(0)).
\]
Further, if \(\theta\) is an even function and if \(m = n\), then \(P_{m,n}(\theta) = \theta(0)/2\).

In particular, for \(n > 2k \geq 0\), \(\sum_{j=0}^{n} (-1)^{j} j^k \binom{n}{j} = 0\) if \(k > 0\) and \(\frac{1}{2}\) if \(k = 0\).

Proof. Now \(P_{m,n}(\theta) = \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \theta(j) \binom{n}{j} = (\Delta^m \Phi)(0)\) where
\[
\Phi(x) = \frac{\theta(m-x) n!}{(m+1-x)(m+2-x) \cdots (m+n-x)}.
\]
Now, we divide \(\theta(x)\) by the polynomial \(\prod_{i=1}^{n} (x+i)\) and write
\[
\theta(x) = u(x) \prod_{i=1}^{n} (x+i) + v(x)
\]
and \(\deg(v) < n\).

Note that if \(u\) is not the zero polynomial, we have \(\deg(u) < m\) by hypothesis. In particular, \((\Delta^m u)\) is the zero polynomial.

Now, we expand in partial fractions as in Remark 1.2:
\[
\frac{v(m-x)n!}{(m+1-x)(m+2-x) \cdots (m+n-x)} = \sum_{r=1}^{n} \frac{c_r}{m+r-x}.
\]
The coefficients \(c_r\) are obtained easily as before; we get
\[
c_i = \frac{v(-i)n!}{(-1)^{i-1}(i-1)!(n-i)!}.
\]
Note that \( v(-i) = \theta(-i) \) for all \( i = 1, \ldots, n \). Thus,
\[
P_{m,n}(\theta) = (\Delta^m \Phi)(0) = (\Delta^m w)(0)
\]
where \( w(x) = \frac{v(m-x)!}{(m+1-x)(m+2-x)\cdots(m+n-x)} = \sum_{r=1}^{n} \frac{c_r}{m+r-x} \).

For \( i = 1, \ldots, n \) we evaluate \( (\Delta^m w)^{(n)})(0) = \sum_{r=0}^{n} (-1)^r \frac{m!}{r!} \) as in [3] as follows.
\[
\sum_{r=0}^{n} (-1)^r \frac{m!}{r!} = \sum_{r=0}^{n} (-1)^r \left( \begin{array}{c} m \\ r \end{array} \right) \int_0^1 (1-t)^{r+i-1} dt
\]
\[
= \int_0^1 t^{i-1}(1-t)^m dt = \beta(i, m+1) = \frac{(i-1)!m!}{(m+i)!}.
\]
Therefore,
\[
P_{m,n}(\theta) = \sum_{i=1}^{n} c_i \frac{(i-1)!m!}{(m+i)!} = \sum_{i=1}^{n} \frac{v(-i)!n!}{(i-1)!r-1!} \frac{(i-1)!m!}{(m+i)!}
\]
\[
= \frac{1}{(m+n) n} \sum_{i=1}^{n} (-1)^{i-1} v(-i) \left( \frac{n+m}{n-i} \right) = \frac{1}{(m+n) n} \sum_{i=1}^{n} (-1)^{i-1} \theta(-i) \left( \frac{n+m}{n-i} \right)
\]
because \( v(-i) = \theta(-i) \) for all \( i = 1, \ldots, n \), which is adding and subtracting the term corresponding to \( i = 0 \), we get the expression asserted in the theorem, viz.,
\[
P_{m,n}(\theta) = \frac{1}{(m+n) n} \sum_{i=0}^{n} (-1)^{i-1} \theta(-i) \left( \frac{n+m}{n-i} \right) + \theta(0).
\]
Adding this expression and the expression \( \frac{1}{(m+n) n} \sum_{j=0}^{n} (-1)^j \theta(j) \left( \frac{m+n}{m-j} \right) \), it is evident that when \( m = n \) and \( \theta(i) = \theta(-i) \) for all \( i \), the sum is \( \theta(0) \). Taking \( \theta(x) = x^{2k} \), the last statement follows. The proof is complete.

\[\square\]

**Remark 2.2.** It is important to note that although \( P_{m,n}(\theta) \) can be re-expressed as a multiple of \( \sum_{j=0}^{m} (-1)^j \theta(j) \left( \frac{m+n}{m-j} \right) \), and hence, can be viewed as the effect of the \((m+n)\)-th order difference operator on a certain function, this does not give any new information but merely reproduces the expression. Thus, it is indeed worthwhile to view \( P_{m,n}(\theta) \) rather as the effect of the \(m\)-th order difference operator on a certain function.

We proved the vanishing of \( P_{m,n}(\theta) \) when \( m = n \) and \( \theta(j) = j^{2k} \), but did not evaluate it for general \( m, n \). As we will see, a natural method to evaluate it is to evaluate and use the following sums:

**Proposition 2.3.** For \( m, n \geq 1, d \geq 0 \) we have
\[
T_d := \sum_{j=0}^{m} (-1)^j (j+1)(j+2) \cdots (j+d) \frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{d! \binom{n}{d+1}}{\binom{m+n}{d+1}}.
\]
We also have
\[
S_d := \sum_{j=0}^{m} (-1)^j j(j-1) \cdots (j-d+1) \frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{(-1)^d n \binom{m}{d} d!}{(d+1) \binom{m+n}{d+1}}.
\]
As usual, the convention is that the empty product (when \( d = 0 \) here) is understood to be equal to 1.
Proof. As we did in the proof of Theorem 2.1, we express the denominator \( \binom{n+j}{j} \) in terms of the beta function and evaluate the sums. We omit details. 

\[ \sum_{j=0}^{m} (-1)^j j^d \binom{m}{j} = \sum_{l=1}^{d} c_{l-1} (\frac{1}{l+1})^d \binom{m}{l} \]

where \( c_r = \frac{1}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^{d-1} \) for all \( 0 \leq r < d-1 \).

In particular, if \( d > 0 \) is even and \( < 2n \), then

\[ \sum_{l=1}^{d} c_{l-1} (\frac{1}{l+1})^d \binom{m}{l} = 0 \]

with \( c_l \)'s as above.

Similarly, we have

\[ \sum_{j=0}^{m} (-1)^j j^d \binom{m+j}{j} = \sum_{r=1}^{d} a_r (\frac{1}{r+1})^d \binom{m+n}{r} \]

where \( a_r = \frac{(-1)^{d+r}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^{d} \) for all \( 0 \leq r < d \).

In particular, if \( d > 0 \) is even and \( < 2n \), then

\[ \sum_{r=1}^{d} a_r (\frac{1}{r+1})^d \binom{m+n}{r} = 0 \]

with \( a_r \)'s as above.

Proof. Now \( \sum_{j=0}^{m} (-1)^j j^d \binom{n}{j} = \sum_{l=1}^{d} c_{l-1} S_l \) where \( S_l \) is as above and where \( c_l \)'s are defined by \( j^d = \prod_{k=0}^{d-1} c_k (j-1) \cdots (j-k) \).

If we write

\[ x^d = \prod_{k=0}^{d-1} c_k x(x-1) \cdots (x-k) \]

then it is easy to determine \( c_k \)'s recursively and we find that for \( 0 \leq r < d-1 \), we have

\[ r! c_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^{d-1} \]

Thus, Proposition 2.3 implies the first assertion.

Similarly, if we express \( x^d = \sum_{r=0}^{d} a_r (x+1)(x+2) \cdots (x+r) \), then we have

\[ \sum_{j=0}^{m} (-1)^j j^d \binom{n}{j} = \sum_{l=1}^{d} a_l T_l \]

We may compute the \( a_r \)'s recursively and find that for \( 0 \leq r < d \), we get

\[ (-1)^{d+r} r! a_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s+1)^d \]

\[ \square \]

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References


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