\( l \)-TOP: graph topologies induced by edge-lengths

Agelos Georgakopoulos

Mathematisches Seminar
Universität Hamburg

Vienna, 27.8.2008
Kirchhoff’s second law

In an electrical network, the net potential drop along a cycle is 0.
Kirchhoff’s second law

Theorems about cycles in finite graphs generalise to infinite ones if you consider topological circles in $|\Gamma|$. 

Kirchhoff’s second law: in an electrical network, the net potential drop along a cycle is 0.
Kirchhoff’s second law

Theorems about cycles in finite graphs generalise to infinite ones if you consider topological circles in $|\Gamma|$.

What about:

**Kirchhoff’s second law**: in an electrical network, the net potential drop along a cycle is 0.
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.
Infinite electrical networks

Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s $2$nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$
a function $r : E \to \mathbb{R}_+$ (the resistances)
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$

a function $r : E \to \mathbb{R}_+$ (the resistances)

a source and a sink $s, t \in V$
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$
a function $r : E \to \mathbb{R}_+$ (the *resistances*)
a source and a sink $s, t \in V$
a constant $I \in \mathbb{R}$ (the *intensity* of the current)
Theorem

The circles of a locally finite electrical network \( N \) satisfy Kirchhoff’s 2nd law if the sum of the resistances in \( N \) is finite.

Infinite electrical network:

A graph \( \Gamma = (V, E) \)

a function \( r : E \to \mathbb{R}_+ \) (the resistances)

a source and a sink \( s, t \in V \)

a constant \( I \in \mathbb{R} \) (the intensity of the current)

electrical current: the \( s-t \) flow \( i \) of strength \( I \) in \( \Gamma \) that minimises energy.
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$

- a function $r : E \to \mathbb{R}_+$ (the resistances)
- a source and a sink $s, t \in V$
- a constant $I \in \mathbb{R}$ (the intensity of the current)

electrical current: the $s$-$t$ flow $i$ of strength $I$ in $\Gamma$ that minimises energy.

$$\text{energy} := \sum_{e \in E} i^2(e) r(e)$$
Infinite electrical networks

Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Infinite electrical network:

A graph $\Gamma = (V, E)$

- a function $r : E \to \mathbb{R}_+^+$ (the resistances)
- a source and a sink $s, t \in V$
- a constant $I \in \mathbb{R}$ (the intensity of the current)

electrical current: the $s$-$t$ flow $i$ of strength $I$ in $\Gamma$ that minimises energy.

\[
\text{energy} := \sum_{e \in E} i^2(e)r(e)
\]

Kirchhoff’s 2nd law: $\sum_{\tilde{e} \in \tilde{C}} i(\tilde{e})r(e) = 0$ for every cycle $\tilde{C}$. 

---

Agelos Georgakopoulos  I-TOP
Our tool:

\( \ell\text{-TOP} \)
Our tool: \( \ell \)-\textit{TOP}.

Let \( \Gamma = (V, E) \) be any graph.

\( \ell \)-\textit{TOP} is the completion of the corresponding metric space.

Theorem (G '06 (easy))

If \( \sum_{e \in E(\Gamma)} \ell(e) < \infty \) then \( \ell \)-\text{TOP}(\Gamma) \approx |\Gamma| \).
Our tool:

\[ \ell \text{-TOP} \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
Our tool:

\[ \ell-TOP \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
- this induces a metric: \( d(v, w) := \inf \{ \ell(P) \mid P \text{ is a } v-w \text{ path} \} \)
Our tool:

\[ \ell\text{-TOP} \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
- this induces a metric: \( d(v, w) := \inf \{ \ell(P) \mid P \text{ is a } v-w \text{ path} \} \)
- let \( \ell\text{-TOP}(\Gamma) \) be the completion of the corresponding metric space
Our tool:

\[ \ell - TOP \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
- this induces a metric: \( d(v, w) := \inf \{ \ell(P) \mid P \text{ is a } v-w \text{ path} \} \)
- let \( \ell\text{-TOP}(\Gamma) \) be the completion of the corresponding metric space

\[ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{8} \\
\frac{1}{8} \\
\frac{1}{8} \\
\frac{1}{8} \\
\frac{1}{16} \\
\frac{1}{16} \\
\frac{1}{16} \\
\frac{1}{16} \\
\ldots
\end{array} \]
Our tool:

\[ \ell\text{-TOP} \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
- this induces a metric: \( d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path} \} \)
- let \( \ell\text{-TOP}(\Gamma) \) be the completion of the corresponding metric space
Our tool:

\[ \ell\text{-TOP} \]

- let \( \Gamma = (V, E) \) be any graph
- give each edge a length \( \ell(e) \)
- this induces a metric: \( d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v-w \text{ path}\} \)
- let \( \ell\text{-TOP}(\Gamma) \) be the completion of the corresponding metric space

**Theorem (G '06 (easy))**

If \( \sum_{e \in E(\Gamma)} \ell(e) < \infty \) then \( \ell\text{-TOP}(\Gamma) \approx |\Gamma| \).
Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.
Kirchhoff’s 2nd law

Theorem

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

Theorem (G ’06)

If $\sum_{e \in E(\Gamma)} \ell(e) < \infty$ then $\ell$-TOP($\Gamma$) $\approx |\Gamma|$.
Kirchhoff’s 2nd law

The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

**Theorem (G ’06)**

If $\sum_{e \in E(\Gamma)} \ell(e) < \infty$ then $\ell\text{-TOP} (\Gamma) \approx |\Gamma|$.

**Theorem (G ’07)**

Let $N$ be an electrical network on a graph $\Gamma$ with resistances $\ell : E \rightarrow R_+$. Then, the proper circles in $\ell\text{-TOP}(\Gamma)$ satisfy Kirchhoff’s 2nd law.
The circles of a locally finite electrical network $N$ satisfy Kirchhoff’s 2nd law if the sum of the resistances in $N$ is finite.

**Theorem (G ’06)**

If $\sum_{e \in E(\Gamma)} \ell(e) < \infty$ then $\ell\text{-TOP}(\Gamma) \approx |\Gamma|$.

**Theorem (G ’07)**

Let $N$ be an electrical network on a graph $\Gamma$ with resistances $\ell : E \to R_+$. Then, the proper circles in $\ell\text{-TOP}(\Gamma)$ satisfy Kirchhoff’s 2nd law.

A circle is *proper* if its edges: – have finite total resistance, and – form a dense subset.
The hyperbolic compactification

**Theorem (Gromov ’87)**

If $\Gamma$ is a hyperbolic graph then there is $\ell : E \rightarrow \mathbb{R}^+$ such that $\ell$-TOP($\Gamma$) is the hyperbolic compactification of $\Gamma$. 
The hyperbolic compactification

**Theorem (Gromov ’87)**

If $\Gamma$ is a hyperbolic graph then there is $\ell : E \to \mathbb{R}_+$ such that $\ell$-TOP($\Gamma$) is the hyperbolic compactification of $\Gamma$.

(because the *Floyd boundary* is a special case of $\ell$-TOP.)
The hyperbolic compactification

Theorem (Gromov ’87)

If $\Gamma$ is a hyperbolic graph then there is $\ell : E \rightarrow \mathbb{R}_+$ such that $\ell\text{-}TOP(\Gamma)$ is the hyperbolic compactification of $\Gamma$.

(because the *Floyd boundary* is a special case of $\ell\text{-}TOP$.

**Problem**

*Are there other important spaces that are a special case of $\ell\text{-}TOP$?*
Generality of $\ell$-TOP

Theorem (Gromov ’87)

Every compact metric space is isometric to the hyperbolic boundary of some hyperbolic graph

(separable: has a countable dense subset)
Generality of $\ell$-TOP

Theorem (Gromov ’87)

Every compact metric space is isometric to the hyperbolic boundary of some hyperbolic graph

Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.

(separable: has a countable dense subset)
Generality of $\ell$-TOP

**Theorem (G ’08)**

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Generality of $\ell$-TOP

Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Generality of $\ell$-TOP

Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Generality of $\ell$-TOP

Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Theorem (G ’08)

A metric space $X$ is isometric to the $\ell$-TOP boundary of some connected locally finite graph iff $X$ is complete and separable.
Random walks

Classical Problem: Given a graph $\Gamma$, is random walk on $\Gamma$ transient or recurrent?
Random walks

Classical Problem: Given a graph $\Gamma$, is random walk on $\Gamma$ transient or recurrent?

New Problem: Can random walk on $\Gamma$ go to infinity and come back?

Why not: model random walk by brownian motion, and...

Problem: Define brownian motion on $\ell$-TOP of a (Cayley) graph.
Random walks

Classical Problem: Given a graph $\Gamma$, is random walk on $\Gamma$ transient or recurrent?

New Problem: Can random walk on $\Gamma$ go to infinity and come back?
**Random walks**

*Classical Problem:* Given a graph $\Gamma$, is random walk on $\Gamma$ transient or recurrent?

*New Problem:* Can random walk on $\Gamma$ go to infinity and come back?

*Why not:* model random walk by brownian motion, and ...
Random walks

Classical Problem: Given a graph $\Gamma$, is random walk on $\Gamma$ transient or recurrent?

New Problem: Can random walk on $\Gamma$ go to infinity and come back?

Why not: model random walk by brownian motion, and ...

Problem

Define brownian motion on $\ell$-TOP of a (Cayley) graph.
The topological cycle space $\mathcal{C}(\Gamma)$ of a locally finite graph $\Gamma$:
The topological cycle space $\mathcal{C}(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$

Known facts:

- MacLane: $\Gamma$ is planar iff $C(\Gamma)$ has a simple generating set
- Tutte: If $\Gamma$ is 3-connected then its peripheral circuits generate $C(\Gamma)$

Easy: The fundamental circuits of a spanning tree generate $C(\Gamma)$

Generalisations:

- Bruhn & Stein
- Diestel & Kühn
- Agelos Georgakopoulos
The topological cycle space $\mathcal{C}(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$
- Allows infinite *thin* sums
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$  
- Allows infinite \textit{thin} sums

Known facts:

- \textbf{MacLane}: $\Gamma$ is planar iff $C(\Gamma)$ has a simple generating set

Generalisations:

- Bruhn & Stein

- Diestel & Kühn

- Agelos Georgakopoulos
The Cycle Space of an Infinite Graph

The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$  
- Allows infinite *thin* sums

Known facts:

- **MacLane:** $\Gamma$ is planar iff $C(\Gamma)$ has a simple generating set
- **Tutte:** If $\Gamma$ is 3-connected then its peripheral circuits generate $C(\Gamma)$

Generalisations:

- Bruhn & Stein
- Bruhn
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$.
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$.
- Allows infinite *thin* sums.

Known facts:

- **MacLane**: $\Gamma$ is planar iff $C(\Gamma)$ has a simple generating set.
- **Tutte**: If $\Gamma$ is 3-connected then its peripheral circuits generate $C(\Gamma)$.
- **easy**: The fundamental circuits of a spanning tree generate $C(\Gamma)$.

Generalisations:

- Bruhn & Stein
- Bruhn
- Diestel & Kühn
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$
- Allows infinite *thin* sums

Diagram:

```
  x  \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \\
  |   \

y
```
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$
- Allows infinite *thin* sums
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$
- Allows infinite *thin* sums
The topological cycle space $C(\Gamma)$ of a locally finite graph $\Gamma$:

- A vector space over $\mathbb{Z}_2$
- Consists of sums of edge sets of (finite or infinite) circles in $|\Gamma|$.
- Allows infinite thin sums.

\[ x \quad y \]
\[ \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \]

*Idea:* allow only sums of families of circuits of finite total length.
A new Homology

Work in progress:

We aim at a homology that

- generalises the topological cycle space
- is defined for any metric space
- allows generalisations of theorems from graphs to other spaces
A monster