The planar cubic Cayley graphs

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Let $\Gamma$ be a group, and $S$ a generating set of $\Gamma$. Define the corresponding Cayley graph $G = \text{Cay}(\Gamma, S)$ by:

- $V(G) = \Gamma$,
- for every $g \in \Gamma$ and $s \in \{a, b, c, \ldots\}$, put in an edge: $g \cdot s \rightarrow gs$.

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Sabidussi’s Theorem

Theorem (Sabidussi’s Theorem)

A properly edge-coloured digraph is a Cayley graph iff for every \( x, y \in V(G) \) there is a colour-preserving automorphism mapping \( x \) to \( y \).

Properly edge-coloured := no vertex has two incoming or two outgoing edges with the same colour.

Let \( \Gamma \) be a group, and \( S \) a generating set of \( \Gamma \). Define the corresponding Cayley graph \( G = Cay(\Gamma, S) \) by:

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  \[
  g \quad s \quad gs
  \]
Charactisation of the finite planar groups

Theorem (Maschke 1886)

Every finite planar group is a group of isometries of $S^2$.

planar group := a group having at least 1 planar Cayley graph.
Let $\Gamma = \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ be a group presentation. Define the corresponding simplified Cayley complex $CC \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ by:
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- for every closed walk $C$ induced by a relator $R_i$, glue in a disc along $C$. 

Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries? Yes!
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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

Yes!
Proving Maschke’s Theorem

Given a finite plane Cayley graph $G$, consider the following group presentation:

Generators: the edge-colours of $G$;
Relators: the facial words starting at a fixed vertex.

This is indeed a presentation of $\Gamma(G)$. Let $X$ be the corresponding simplified Cayley complex. $X$ is homeomorphic to $S^2$. Since $\Gamma(G)$ acts on $X$, we have:

Theorem (Maschke 1886): Every finite planar group is a group of homeomorphisms of $S^2$. 

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**Theorem (Whitney ’32)**

*Let $G$ be a 3-connected plane graph. Then every automorphism of $G$ extends to a homeomorphism of the sphere.*
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The 1-ended planar groups

Theorem ((classic) Macbeath, Wilkie, ...)

Every 1-ended planar Cayley graph corresponds to a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$. 
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Planar Cayley graphs
Planar groups $\leftarrow$ fundamental groups of surfaces

Theorem (G '10)
A group has a planar simplified Cayley complex if and only if it has a VAP-free Cayley graph.
Planar groups $\prec \rightarrow$ fundamental groups of surfaces

... general classical theory, but only for groups with a planar simplified Cayley complex
Planar groups $\Leftarrow \Rightarrow$ fundamental groups of surfaces

... general classical theory, but only for groups with a planar simplified Cayley complex

*What about the other ones?*
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**Theorem (G ’10)**

*A group has a planar simplified Cayley complex if and only if it has a VAP-free Cayley graph.*
What about the non VAP-free ones?

Open Problems:

Problem (Mohar) How can you split a planar Cayley graph with > 1 ends into simpler Cayley graphs?

Problem (Droms et. al.) Is there an effective enumeration of the planar locally finite Cayley graphs?

Conjecture (Bonnington & Watkins / B. & Mohar) Every planar 3-connected locally finite transitive graph has at least one face bounded by a cycle.

Problem (G & Mohar) Is every planar 3-connected Cayley graph hamiltonian?

... and what about all the classical theory?
What about the non VAP-free ones?

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Theorem (G ’10)

Let $G$ be a planar cubic Cayley graph. Then $G$ is colour-isomorphic to precisely one element of the list.

Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.
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... and what about all the classical theory?
Examples

Planar Cayley graphs
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Planar Cayley graphs
Corollary (G ’10)

Every planar cubic Cayley graph has an almost planar Cayley complex.
Corollary (G & Hamann ’11)

Every planar Cayley graph has an almost planar Cayley complex.
Examples

Corollary (G & Hamann ’11)

Every planar Cayley graph has an almost planar Cayley complex...
maybe
Conjecture (Bonnington & Watkins)

Every planar 3-connected locally finite transitive graph has at least one face bounded by a cycle.
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Every planar 3-connected locally finite transitive graph has at least one face bounded by a cycle.

FALSE!
Cayley graphs without finite face boundaries
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Planar Cayley graphs
Spot the societies!
Spot the societies!
Theorem (Stallings ’71)

*Every group with \(>1\) ends can be written as an HNN-extension or an amalgamation product over a finite subgroup.*
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Conjecture

Let $G = \text{Cay}(\Gamma, S)$ be a Cayley graph with $> 1$ ends. Then there is a non-trivial splitting of $G$ as a union of subdivisions of Cayley graphs.

Corollary (G ’10)

True for planar cubic Cayley graphs.
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Corollary (G ’10)

True for planar cubic Cayley graphs.
1. $\langle a, b \mid b^2, a^n \rangle$, $n \in \{\infty, 2, 3, \ldots\}$
2. $\langle b, c, d \mid b^2, c^2, d^2, (bc)^n \rangle$, $n \in \{\infty, 1, 2, 3, \ldots\}$

$\kappa(G) = 1$

3. $G \cong \text{Cay} \langle a, b \mid b^2, (ab)^n \rangle$, $n \geq 2$
4. $G \cong \text{Cay} \langle a, b \mid b^2, (aba^{-1}b^{-1})^n \rangle$, $n \geq 1$
5. $G \cong \text{Cay} \langle a, b \mid b^2, a^4, (a^2b)^2 \rangle$
6. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (bc)(bd)^m \rangle$, $m \geq 2$
7. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (bcd)^m \rangle$, $n, m \geq 2$
8. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (bd)^m \rangle$, $n, m \geq 2$
9. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bd)^m \rangle$, $n \geq 1, m \geq 2$
10. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bcd)^m \rangle$, $m \geq 1$
11. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m \rangle$, $n \geq 1$

$\kappa(G) = 2$

12. $G \cong \text{Cay} \langle a, b \mid b^2, a^n, (ab)^m \rangle$, $n \geq 3, m \geq 2$
13. $G \cong \text{Cay} \langle a, b \mid b^2, a^n, (aba^{-1}b^{-1})^m \rangle$, $n \geq 3, m \geq 1$
14. $G \cong \text{Cay} \langle a, b \mid b^2, (ab)^m \rangle$, $m \geq 1$
15. $G \cong \text{Cay} \langle a, b \mid b^2, (ab^2)^m, m \geq 1 \rangle$

$\kappa(G) = 3$, G is 1-ended or finite, with two generators

16. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n \rangle$, $n \geq 1$
17. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bcd)^n \rangle$, $n \geq 1$
18. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bd)^n \rangle$, $n \geq 2, m \geq 1$
19. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (cd)^n, (cd)^m \rangle$, $n, m, p \geq 2$

$\kappa(G) = 3$, G is 1-ended or finite, with three generators

20. $G \cong \text{Cay} \langle a, b \mid b^2, (ab)^m, a^n \rangle$, $n \geq 3, m \geq 2$
21. $G \cong \text{Cay} \langle a, b \mid b^2, (aba^{-1}b^{-1})^m, a^n \rangle$, $n \geq 3, m \geq 1$
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25. $G \cong \text{Cay} \langle a, b \mid b^2, (ab)^m, (ab)^m \rangle$, $n \geq 2, m \geq 2$

$\kappa(G) = 3$, G is multi-ended, with two generators

Table 1: Classification of the cubic planar Cayley graphs. All presentations are planar in the sense of Section 1.4.