Infinite cycles in graphs

Agelos Georgakopoulos

Mathematisches Seminar
Universität Hamburg

Marburg, 2.5.2008
Many finite theorems involving paths or cycles fail for infinite graphs:
Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
Things that go wrong in infinite graphs

Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
- MacLane’s planarity criterion
Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
- MacLane’s planarity criterion
- the Tutte/Nash-Williams tree packing theorem
Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
- MacLane’s planarity criterion
- the Tutte/Nash-Williams tree packing theorem
- ...

Things that go wrong in infinite graphs
Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
- MacLane’s planarity criterion
- the Tutte/Nash-Williams tree packing theorem
- ...
- all hamilton-cycle theorems
Many finite theorems involving paths or cycles fail for infinite graphs:

- Euler’s theorem
- MacLane’s planarity criterion
- the Tutte/Nash-Williams tree packing theorem
- ...
- all hamilton-cycle theorems

\[ \Rightarrow \text{ need more general notions of paths and cycles}\]
Classical approach: accept double-rays as infinite cycles
Classical approach: accept double-rays as infinite cycles

\[ \cdots \quad \bullet \quad \bullet \quad \bullet \quad \cdots \]

This approach only extends finite theorems in very restricted cases:
Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles

\[ \cdots \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \cdots \]

This approach only extends finite theorems in very restricted cases:

**Theorem (Tutte ’56)**

*Every finite 4-connected planar graph has a Hamilton cycle*
Classical approach: accept double-rays as infinite cycles

This approach only extends finite theorems in very restricted cases:

Theorem (Yu ’05)

Every locally finite 4-connected planar graph has a spanning double ray...
Classical approach: accept double-rays as infinite cycles

\[ \cdots \rightarrow \cdots \leftarrow \cdots \]

This approach only extends finite theorems in very restricted cases:

**Theorem (Yu ’05)**

*Every locally finite 4-connected planar graph has a spanning double ray ... unless it is 3-divisible.*
Compactifying by Points at Infinity

A 3-divisible graph
A 3-divisible graph can have no spanning double ray
A 3-divisible graph can have no spanning double ray
A 3-divisible graph can have no spanning double ray

... but a Hamilton cycle?
end: equivalence class of rays
two rays are equivalent if no finite vertex set separates them
end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them
end: equivalence class of rays

two rays are **equivalent** if no finite vertex set separates them

---

**two ends**

---

**one end**
end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

... two ends

one end

uncountably many ends
Every ray converges to its end
The End Compactification

Every ray converges to its end.
Every ray converges to its end
Every ray converges to its end
Circle:
A homeomorphic image of $S^1$ in $|G|$.
Circle:
A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices
Circle:
A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices (and all ends?)
Circle:
A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices, and thus also all ends.
Circle:
A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices, and thus also all ends.
Circle:
A homeomorphic image of $S^1$ in $|G|$. 
Circle:
A homeomorphic image of $S^1$ in $|G|$. 

the wild circle of Diestel & Kühn
Fleischner’s Theorem

Theorem (Fleischner ’74)

The square of a finite 2-connected graph has a Hamilton cycle
Fleischner’s Theorem

Theorem (Fleischner ’74)

The square of a finite 2-connected graph has a Hamilton cycle

Theorem (Thomassen ’78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle.
Theorem (G ’06)

*The square of any locally finite 2-connected graph has a Hamilton circle*
Introduction
Topological cycles
Fleischner’s Theorem

Proof?
Proof?

Infinite cycles in graphs
Proof?
Proof?

Infinite cycles in graphs
Introduction
Topological cycles
Fleischner’s Theorem

Proof?
Proof?
Proof?
Introduction

Topological cycles

Fleischner’s Theorem

Proof?

Agelos Georgakopoulos

Infinite cycles in graphs
Proof?
Proof?

Introduction  Topological cycles  Fleischner's Theorem

Agelos Georgakopoulos  Infinite cycles in graphs
Introduction

Topological cycles

Fleischner’s Theorem

Proof?

Agelos Georgakopoulos

Infinite cycles in graphs
Proof?

Infinite cycles in graphs
Proof?
Proof?

Fleischner’s Theorem: Infinite cycles in graphs
Hilbert’s space filling curve:

a sequence of injective curves with a non-injective limit
Structure of the Finite Proof

Theorem (G ’06)

*The square of any locally finite 2-connected graph has a Hamilton circle*
Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
Structure of the Finite Proof

Theorem (G ’06)

*The square of any locally finite 2-connected graph has a Hamilton circle*

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle
Structure of the Finite Proof
Structure of the Finite Proof

Infinite cycles in graphs
Structure of the Finite Proof
Structure of the Finite Proof
Structure of the Infinite Proof

Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle
**Structure of the Infinite Proof**

**Theorem (G ’06)**

*The square of any locally finite 2-connected graph has a Hamilton circle*

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

It will not work if we have too many crossings
Structure of the Infinite Proof

Theorem (G ’06)

*The square of any locally finite 2-connected graph has a Hamilton circle*

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

It will not work if we have too many crossings

Extra problems for infinite graphs:

- you need a topological Euler tour
**Theorem (G ’06)**

The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

*It will not work if we have too many crossings*

**Extra problems for infinite graphs:**

- you need a topological Euler tour, thus you have to guarantee even degree at ends too
Theorem (G ’06)

*The square of any locally finite 2-connected graph has a Hamilton circle*

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

*It will not work if we have too many crossings*

**Extra problems for infinite graphs:**

- you need a topological Euler tour, thus you have to guarantee even degree at ends too
- the (topological) Euler tour has to be injective at ends
Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

It will not work if we have too many crossings

Extra problems for infinite graphs:

- you need a topological Euler tour, thus you have to guarantee even degree at ends too
- the (topological) Euler tour has to be injective at ends
- deleting edges may change the end topology
Structure of the Infinite Proof

Infinite cycles in graphs
Structure of the Infinite Proof

Infinite cycles in graphs
Structure of the Infinite Proof
Structure of the Infinite Proof

Infinite cycles in graphs
Structure of the Infinite Proof

Fleischner's Theorem

Agelos Georgakopoulos

Infinite cycles in graphs
The square of any locally finite 2-connected graph has a Hamilton circle

- make all vertex degrees even by deleting some edges and doubling some others
- pick an Euler tour
- bridge crossings to turn the Euler tour into a Hamilton cycle

It will not work if we have too many crossings

Extra problems for infinite graphs:

- you need a topological Euler tour, thus you have to guarantee even degree at ends too
- the (topological) Euler tour has to be injective at ends
- deleting edges may change the end topology
Problem (Rapaport-Strasser ’59)

*Does every finite connected Cayley graph have a Hamilton cycle?*
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser ’59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1-ended Cayley graph have a Hamilton circle?
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser ’59)
Does every finite connected Cayley graph have a Hamilton cycle?

Problem
Does every connected 1-ended Cayley graph have a Hamilton circle?

Problem
Prove that every connected Cayley graph of a finitely generated group $\Gamma$ has a Hamilton circle unless $\Gamma$ is the amalgamated product of more than $k$ groups over a subgroup of order $k$. 