Discrete Riemann mapping and the Poisson boundary

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The Riemann mapping theorem

Theorem (Riemann? 1851, Carathéodory 1912)

For every simply connected open set $\Omega \subseteq \mathbb{C}$, $\Omega \neq \emptyset$, there is a bijective conformal map from $\Omega$ onto the open unit disk.
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Theorem (Koebe 1920)

For every open set $\Omega \subset \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from $\Omega$ onto a circle domain.
The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph $G$, there is a circle packing in the plane (or $S^2$) with nerve $G$. The packing is unique (up to Möbius transformations) if $G$ is a triangulation of $S^2$. 
The Koebe-Andreev-Thurston circle packing theorem
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Circle Packing $\leftrightarrow$ Conformal map

Figure 3: An circle packing approximation of an triangulated domain to an triangulation of a combinatorially equivalent circle packing; are from Oded’s thesis; thanks to Andrei Mishchenko for creating

[S. Rohde: “Oded Schramm: From Circle Packing to SLE”, ’10]
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Figure 3: An circle packing approximation of an triangulated domain and its nerve completion to an triangulation of its combinatorially equivalent circle packing. From Oded's thesis; thanks to Andrey Mishchenko for creating the diagrams.
The Koebe-Andreev-Thurston circle packing theorem

Circle Packing $\iff$ Conformal map

Figure 3: An circle packing approximation of an triply connected domain and its nerve and completion to an triangulation of 52 and an combinatorially equivalent circle packing; $\{u-pn_\text{vax-vcxn}\} \text{aren from Odedu's thesis; thanks to Andreyn Mishchenkon for creating vdxn 6}

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Figure 3: An circle packing approximation of a triangle domain and its nerve completion to an triangulation of an circle packing; $\alpha - \omega$ are from Oded’s thesis; thanks to Andrei Mishchenko for creating $\omega$.
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Circle Packing $\Rightarrow$ Conformal map

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Theorem (Brooks, Smith, Stone & Tutte ’40)

... for every finite planar graph $G$, there is a square tiling with incidence graph $G$ ...
Properties of square tilings

- Every edge is mapped to a square;
- Vertices correspond to horizontal segments tangent with their edges;
- There is no overlap of squares, and no 'empty' space left;
- The square tiling of the dual of $G$ can be obtained from that of $G$ by a $90^\circ$ rotation.

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The construction of square tilings

Think of the graph as an electrical network;

- Place each vertex $x$ at height equal to the potential $h(x)$;
- Use a duality argument to determine the width coordinates.
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The construction of square tilings

The square tilings of Benjamini & Schramm

**Theorem (Benjamini & Schramm ’96)**

Every transient (infinite) graph $G$ of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling.
The square tilings of Benjamini & Schramm

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Every transient (infinite) graph $G$ of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling. Moreover, random walk on $G$ converges a.s. to a point in $C$. 

---

$C$
The classical Poisson formula

\[ h(z) = \int_{0}^{2\pi} \hat{h}(\theta) P(z, \theta) d\theta \]

where \( P(z, \theta) := \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \),

recovers every continuous harmonic function \( h \) on \( \mathbb{D} \) from its boundary values \( \hat{h} \) on the circle \( \partial \mathbb{D} \).
The classical Poisson formula

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The Poisson integral representation formula

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Can the bounded harmonic functions on a plane graph $G$ be expressed as a Poisson-like integral using $C$?
The boundary of the square tiling coincides with the Poisson boundary

Can the bounded harmonic functions on a plane graph $G$ be expressed as a Poisson-like integral using $C$?

A function $h : V(G) \to \mathbb{R}$, is harmonic, if $h(x) = \sum_{y \sim x} h(y)/d(x)$. 
The boundary of the square tiling coincides with the Poisson boundary.

**Question (Benjamini & Schramm ’96)**

*Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?*
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Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?

Theorem (G '12)

Yes!
The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph $G$ consists of
- a measurable space $(\mathcal{P}_G, \Sigma)$, and
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- every bounded harmonic function $h$ can be obtained by

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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function
  $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$. 
Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups  
  [Annals of Math. '63]

- Kaimanovich & Vershik give a general criterion using the entropy of random walk  
  [Annals of Probability '83]

- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria  
  [Annals of Math. '00]

General survey:

- Erschler: Poisson-Furstenberg Boundaries, Large-scale Geometry and Growth of Groups  
  [Proceedings of ICM 2010]

Textbooks:

- Woess: Random Walks on Infinite Graphs and Groups
- Lyons & Peres: Probability on Trees and Networks
Theorem (G ’12)

For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with $C$. 
Probabilistic interpretation of the tiling

**Lemma (G ’12)**

Let $C$ be a ‘horizontal’ circle in the tiling $T$ of $G$, and let $B$ the set of points of $G$ at which $C$ ‘dissects’ $T$. Then the widths of the points of $B$ in $T$ coincide with the probability distribution of the first visit to $B$ by brownian motion on $G$ starting at $o$. 

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Lemma (G ’12)

Let $C$ be a ‘horizontal’ circle in the tiling $T$ of $G$, and let $B$ the set of points of $G$ at which $C$ ‘dissects’ $T$. Then the widths of the points of $B$ in $T$ coincide with the probability distribution of the first visit to $B$ by brownian motion on $G$ starting at $o$. 
Lemma

For every ‘meridian’ $M$ in $T$, the probability that brownian motion on $G$ starting at $o$ will ‘cross’ $M$ clockwise equals the probability to cross $M$ counter-clockwise.
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Conjecture (Northshield ’93)

Let $G$ be an accumulation-free plane, non-amenable graph with bounded vertex degrees. Then the Northshield circle of $G$ is a realisation of its Poisson boundary.
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Theorem (G ’13)

Indeed.
Theorem (G ’13)

Let $G$ be an infinite, Gromov-hyperbolic, non-amenable, 1-ended, plane graph with bounded degrees and no infinite faces. Then the following 5 boundaries of $G$ (and the corresponding compactifications of $G$) are canonically homeomorphic to each other:

- the hyperbolic boundary
- the Martin boundary [Ancona]
- the boundary of the square tiling
- the Northshield circle $\partial_\infty(G)$ and
- the transience boundary $\partial_\prec(G)$ [Northshield].
Conjecture (G)

Let $M$ be a complete, simply connected Riemannian surface with sectional curvatures bounded between two negative constants. Let $f : M \to \mathbb{D}$ be a conformal map. Then for every 1-way infinite geodesic $\gamma$ in $M$, the image $f(\gamma)$ converges to a point in the boundary $\mathbb{S}^1$ of $\mathbb{D}$, and this convergence determines a homeomorphism from the sphere at infinity of $M$ to $\mathbb{S}^1$. 
Open problems

Problem

Is every planar graph with the Liouville property amenable?

–For Cayley graphs this is true even without planarity [Kaimanovich & Vershik];
–for general graphs it is false even assuming bounded degrees [e.g. Benjamini & Kozma].
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Is every planar graph with the Liouville property amenable?

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Problem

Is there a planar, Gromov-hyperbolic graph with bounded degrees, no infinite faces, and the Liouville property?
Here come some ‘geometric’ random graphs
The classical Douglas formula

\[ E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta \]

calculates the (Dirichlet) energy of a harmonic function \( h \) on \( \mathbb{D} \) from its boundary values \( \hat{h} \) on the circle \( \partial \mathbb{D} \).
Energy in finite electrical networks

\[ E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 C_{ab}, \]

where \( C_{ab} \) is the capacitance between nodes \( a \) and \( b \).

Compare with Douglas:

\[ E(h) = \int_{2\pi}^{0} \int_{2\pi}^{0} \left( \hat{h}(\eta) - \hat{h}(\zeta) \right)^2 \Theta(z, \eta) \, d\eta \]
Energy in finite electrical networks

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Theorem (G & V. Kaimanovich ’14+)

For every locally finite network $G$, there is a measure $C$ on $\mathcal{P}^2(G)$ such that for every harmonic function $u$ the energy $E(u)$ equals

$$\int_{\mathcal{P}^2} \left( \hat{u}(\eta) - \hat{u}(\zeta) \right)^2 \, dC(\eta, \zeta).$$
The energy of harmonic functions

Theorem (G & V. Kaimanovich ’14+)

For every locally finite network $G$, there is a measure $C$ on $P^2(G)$ such that for every harmonic function $u$ the energy $E(u)$ equals

$$\int_{P^2} (\widehat{u}(\eta) - \widehat{u}(\zeta))^2 dC(\eta, \zeta).$$

This is a discrete version of a result of [Doob ’62] on Green spaces (or Riemannian manifolds), which generalises Douglas’ formula $E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta)d\eta$.
Energy in finite electrical networks

\[ E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 \ C^{ab} \]
Summary