Discrete Riemann mapping and the Poisson boundary

Agelos Georgakopoulos

31/1/14
Theorem (Riemann? ’1851, Carathéodory 1912)

For every simply connected open set $\Omega \subset \subset \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from $\Omega$ onto the open unit disk.
The Riemann mapping theorem

Theorem (Riemann? ’1851, Carathéodory 1912)
For every simply connected open set $\Omega \subset \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from $\Omega$ onto the open unit disk.

Theorem (Koebe 1908)
For every open set $\Omega \subset \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from $\Omega$ onto a circle domain.
The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph $G$, there is a circle packing in the plane (or $S^2$) with nerve $G$. The packing is unique (up to Möbius transformations) if $G$ is a triangulation of $S^2$. 

![Diagram showing a planar graph and its corresponding circle packing]
The Koebe-Andreev-Thurston circle packing theorem

Figure 3: An circle packing approximation of an triple connected domain and its nerve and its completion to an triangulation of a triangulation and an combinatorially equivalent circle packing and a from Oded's thesis. Thanks to Andrei Mishchenko for creating a.
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Figure 3:n An circlen packingn approximationn ofn an ... circlen packing;n vax-vcxn aren fromn Odedusn
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[S. Rohde: "Oded Schramm: From Circle Packing to SLE", '10]
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Circle Packing $\leftrightarrow$ Conformal map

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Figure 3: An circle packing approximation of an triangulated domain and its completion to an triangulation of an and an combinatorially equivalent circle packing; are from Oded's thesis; thanks to Andrei Mishchenko for creating

[109x4] Agelos Georgakopoulos

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The Koebe-Andreev-Thurston circle packing theorem

Circle Packing <=> Conformal map

Figure 3: An circle packing approximation of an triangulated domain and its nerve. From Oded Schramm's thesis, thanks to Andrei Mishchenko for creating the diagrams.

[Source: S. Rohde: "Oded Schramm: From Circle Packing to SLE", 2010]
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Figure 3: An circle packing approximation of an triply connected domain $\Rightarrow$ a nerve $\Rightarrow$ its completion $\Rightarrow$ a triangulation of 52 faces.$\Rightarrow$ a combinatorial equivalent circle packing.$\Rightarrow$ are from Oded's thesis.$\Rightarrow$ thanks to Andrei Mishchenko for creating the figure.
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Circle Packing $\Rightarrow$ Conformal map

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Circle Packing $\leftrightarrow$ Conformal map

Circle Packing $\Rightarrow$ Conformal map

[S. Rohde: "Oded Schramm: From Circle Packing to SLE", ’10]
Theorem (Brooks, Smith, Stone & Tutte ’40)

... for every finite planar graph $G$, there is a square tiling with incidence graph $G$ ...
Properties of square tilings

- Every edge is mapped to a square;
- Vertices correspond to horizontal segments tangent with their edges;
- There is no overlap of squares, and no 'empty' space left;
- The square tiling of the dual of $G$ can be obtained from that of $G$ by a 90° rotation.

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\[ \text{Agelos Georgakopoulos} \]
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Think of the graph as an electrical network; impose an electrical current from $p$ to $q$; let the square corresponding to edge $e$ have side length the flow $i(e)$; place each vertex $x$ at height equal to the potential $h(x)$; use a duality argument to determine the width coordinates.
The construction of square tilings

Think of the graph as an electrical network; impose an electrical current from \( p \) to \( q \); let the square corresponding to edge \( e \) have side length \( i(e) \); place each vertex \( x \) at height equal to the potential \( h(x) \); use a duality argument to determine the width coordinates.

Agléos Georgakópoulos
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“... Riemann, like Klein in the passage quoted from Poincare, may have considered the quadrilateral as a metallic conducting plate with battery terminals connected to its ‘top’ and ‘bottom’. “The current must pass” as Klein is supposed to have said. The current flow lines, connecting top to bottom, would have filled the quadrilateral from side to side one line through each point of the quadrilateral. Equipotential lines, connecting side to side, would likewise have filled the quadrilateral from top to bottom. The pair of families would meet one another orthogonally and give rectilinear flat coordinates for the quadrilateral.”
Theorem (Benjamini & Schramm ’96)

Every (transient) graph $G$ of bounded degree that admits a uniquely absorbing embedding in the plane admits a square tiling.

\[ C \]
Theorem (Benjamini & Schramm ’96)

Every (transient) graph $G$ of bounded degree that admits a uniquely absorbing embedding in the plane admits a square tiling. Moreover, random walk on $G$ converges a. s. to a point in $C$. 

$C$
The boundary of the square tiling coincides with the Poisson boundary

Question (Benjamini & Schramm ’96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?
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*Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?*

Theorem (G ’12)

*Yes.*
This is not about groups
Theorem (G '12)

For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with $C$. 

Angel, Barlow, Gurel-Gurevich & Nachmias recently identified the Poisson & Martin boundary of any bounded degree, transient, 1-ended triangulation of the plane with the boundary of its circle packing.
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Theorem (G ’12)

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Sharp functions can be combined by elementary operations:

- **‘Union’**:
  $$\bigcup_i f_i(x) := \mathbb{P}\{\exists i, f_i(X_n) \to 1 \text{ for random walk } X_n \text{ starting at } x\}$$

- **‘Intersection’**:
  $$\bigcap_i f_i(x) := \mathbb{P}\{\forall i, f_i(X_n) \to 1 \text{ for random walk } X_n \text{ starting at } x\}$$
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Thus they satisfy the $\sigma$-algebra axioms, except that there is no ground set.
The criterion

**Theorem (G '12)**

*(Informal statement)* Let $M$ be a Markov chain. Any measurable space that can be used as the ground set of the \(\sigma\)-algebra* of sharp harmonic functions on $M$ is a realisation of the Poisson boundary of $M$. 

*Agelos Georgakopoulos*
Conjecture (Northshield ’93)

Let $G$ be an accumulation-free plane, non-amenable graph with bounded vertex degrees. Then the Northshield circle of $G$ is a realisation of its Poisson boundary.
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Theorem (G ’13)

Indeed.
Let $G$ be an infinite, Gromov-hyperbolic, non-amenable, 1-ended, plane graph with bounded degrees and no infinite faces. Then the following five boundaries of $G$ are canonically homeomorphic to each other:

- the hyperbolic boundary
- the Martin boundary \cite{Ancona '88}
- the boundary of the square tiling
- the Northshield circle, and
- the boundary $\partial_{\infty}(G)$. 

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Conjecture (G)

Let $M$ be a complete, simply connected Riemannian surface with Gaussian curvatures bounded between two negative constants. Let $f : M \to \mathbb{D}$ be a conformal map. Then for every 1-way infinite geodesic $\gamma$ in $M$, the image $f(\gamma)$ converges to a point in the boundary $\mathbb{S}^1$ of $\mathbb{D}$, and this convergence determines a homeomorphism from the sphere at infinity of $M$ to $\mathbb{S}^1$. 
You can do more with the Poisson boundary...
The classical Douglas formula

\[ E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta \]

calculates the (Dirichlet) energy of a harmonic function \( h \) on \( \mathbb{D} \) from its boundary values \( \hat{h} \) on the circle \( \partial \mathbb{D} \).
Energy in finite electrical networks

\[ E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 C_{ab}, \]

where

\[ C_{ab} = d(a) P_a(b). \]
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Compare with Douglas: \( E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta \)
Theorem (G & V. Kaimanovich ’14+)

For every locally finite network $G$, there is a measure $C$ on $P^2(G)$ such that for every harmonic function $u$ the energy $E(u)$ equals

$$\int_{P^2} \left( \hat{u}(\eta) - \hat{u}(\zeta) \right)^2 dC(\eta, \zeta).$$
The energy of harmonic functions

Theorem (G & V. Kaimanovich ’14+)

For every locally finite network $G$, there is a measure $C$ on $\mathcal{P}^2(G)$ such that for every harmonic function $u$ the energy $E(u)$ equals

$$
\int_{\mathcal{P}^2} \left( \hat{u}(\eta) - \hat{u}(\zeta) \right)^2 dC(\eta, \zeta).
$$

This is a discrete version of a result of [Doob ’62] on Green spaces (or Riemannian manifolds), which generalises Douglas’ formula

$$
E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta
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Agelos Georgakopoulos
Energy in finite electrical networks

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Plans to generalise Sznitman’s random interlacements ...
Summary