Uniqueness of currents in a network of finite total resistance

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Applications of electrical networks

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- in the study of Riemannian manifolds
- in Combinatorics

\[
\begin{bmatrix}
x | u, v \\
y | u, v \\
z | u, v
\end{bmatrix} = \begin{bmatrix} u \\
y \\
\log(\cos|u|\sec|v|) \end{bmatrix}
\]
The discrete Network Problem

The setup:

A graph $G = (V, E)$

A function $r : E \rightarrow \mathbb{R}^+$ (the resistances)

A source and a sink $p, q \in V$

A constant $I \in \mathbb{R}$ (the intensity of the current)

The problem:

Find a $p$-$q$ flow in $G$ with intensity $I$ that satisfies Kirchhoff's cycle law:

$$\sum_{\vec{e} \in \vec{E}} (C) v(\vec{e}) = 0$$

where $v(\vec{e}) := f(\vec{e}) r(e)$ (Ohm's law)
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Infinite Networks
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Not necessarily unique solution
Find a $p$-$q$ flow in $G$ with intensity $I$ that satisfies Kirchhoff’s cycle law:

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**Finite Networks**
- Unique solution

**Networks of finite total resistance**
- Unique solution

**Infinite Networks**
- Not necessarily unique solution
Non-elusive flows

The solution is not necessarily unique!

Non-elusive flow:
The net flow along any such cut must be zero:

\[ p - q = 0 \]
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The net flow along any such cut must be zero:
Theorem (G ’08)

In a network with \( \sum_{e \in E} r(e) < \infty \) there is a unique non-elusive flow with finite energy that satisfies Kirchhoff’s cycle law.
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In a network with \( \sum_{e \in E} r(e) < \infty \) there is a unique \textit{non-elusive} flow with finite energy that satisfies Kirchhoff’s cycle law.
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Energy of $f$: $\frac{1}{2} \sum_{e \in E} f^2(e)r(e)$
The Theorem

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Energy of \( f \): \( \frac{1}{2} \sum_{e \in E} f^2(e)r(e) \)
The proof

Finite case:

Assume there are two 'good' flows $f, g$ and consider $z := f - g$. 
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Finite case:

Infinite case:
A wild circle

Wild circles

[i.e. a homeomorphic image of $S^1$ in $|G|$ (discovered by Diestel & Kühn)]

Contains $\aleph_0$ double-rays arranged like the rational numbers.
The "gaps" between the double-rays are filled by a Cantor set of ends.
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Do wild circles satisfy Kirchhoff’s cycle law?

\[ \sum_{\vec{e} \in \vec{E}(C)} \vec{v}(\vec{e}) = 0 \]

\[ \vec{E}(F) = \sum_{\vec{e} \in \vec{E}(F)} \vec{v}(\vec{e}) = 0 \]

IT DEPENDS!

\[ \sum_{\vec{e} \in \vec{E}} \vec{v}(\vec{e}) = \sum_{F \text{ is a face boundary}} \sum_{\vec{e} \in \vec{E}(F)} \vec{v}(\vec{e}) = 0 \]

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Networks
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OK if \( \sum r(e) < \infty \)
The second tool

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**Theorem (G ’06 (easy))**

If \( \sum_{e \in E} \ell(e) < \infty \) then \( |G|_\ell \approx |G| \).
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Theorem (G ’06 (easy))

If $\sum_{e \in E} r(e) < \infty$ then $|G|_r \approx |G|$. 
Kirchhoff’s cycle law for wild circles

Theorem (Diestel & G)

The circles of an electrical network $N$ satisfy Kirchhoff’s cycle law if the sum of the resistances in $N$ is finite.
Theorem (G ’08)

In a network with \( \sum_{e \in E} r(e) < \infty \) there is a unique non-elusive flow with finite energy that satisfies Kirchhoff’s cycle law.
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The Dirichlet Problem

Continuous version

Let $X \subseteq \mathbb{R}^n$ be compact
Prescribe $\phi: \partial X \to \mathbb{R}$
Extend to $\phi': X \to \mathbb{R}$ that is harmonic inside $X$

$\nabla^2 \phi' = 0$

Discrete version

Let $G$ be a graph
Prescribe $\phi: \partial G \to \mathbb{R}$
Extend to $\phi': G \to \mathbb{R}$ that is harmonic in $G$

i.e. satisfies Kirchhoff's node law

Studied intensively (Woess, Kaimanovich, Benjamini & Schramm)
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Networks
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The Dirichlet Problem

Problem

For every assignment \( r : E \rightarrow \mathbb{R}_+ \) (such that \( |G|_r \) is compact) the Dirichlet problem is solvable for every continuous \( \phi : \partial |G|_r \rightarrow \mathbb{R} \).
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For every assignment $r : E \to \mathbb{R}_+$ (such that $|G|_r$ is compact) the Dirichlet problem is solvable for every continuous $\phi : \partial |G|_r \to \mathbb{R}$.

Interesting because:

Theorem (Gromov ’87 (indirect proof))

For every compact metric space $X$ there is a locally finite graph $G$ and $r : E \to \mathbb{R}_+$ such that $X = \partial |G|_r$. 

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Networks
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The converse works:

Theorem

If $f : \overrightarrow{E} \to \mathbb{R}$ is a flow of finite energy in $G$ satisfying Kirchhoff’s cycle law then it is possible to extend the corresponding potentials continuously to $\partial|G|_r$. 
Random Walks & Electrical networks

Every edge \( e \) has a weight \( c(e) \).

Go from \( x \) to \( y \) with probability
\[
P_{x \rightarrow y} := \frac{c(xy)}{c(x)}
\]

where
\[
c(x) := \sum_{xv \in E} c(xv)
\]

\( P_{pq}(x) \) is the probability that if you start in \( x \) you will hit \( p \) before \( q \).

\[r(e) < 1\]

Connect a source of voltage 1 to \( p \), \( q \).
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$\mathbb{P}_{pq}(x) :=$ the probability that if you start in $x$ you will hit $p$ before $q$.

$$c(e) \iff \frac{1}{r(e)}$$
Every edge $e$ has a weight $c(e)$

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$$P_{x \rightarrow y} := \frac{c(xy)}{c(x)}$$

where $c(x) := \sum_{xv \in E} c(xv)$

$$P_{pq}(x) := \text{the probability that if you start in } x \text{ you will hit } p \text{ before } q.$$  

$$c(e) \leftrightarrow \frac{1}{r(e)}$$

Connect a source of voltage 1 to $p, q$
Every edge $e$ has a weight $c(e)$

Go from $x$ to $y$ with probability

$$P_{x \rightarrow y} := \frac{c(xy)}{c(x)}$$

where $c(x) := \sum_{xv \in E} c(xv)$

$\mathbb{P}_{pq}(x) :=$ the probability that if you start in $x$ you will hit $p$ before $q$.

$$c(e) \iff \frac{1}{r(e)}$$

Connect a source of voltage 1 to $p, q$

$$\mathbb{P}_{pq}(x) = P(x)$$
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**Problem**

*Define brownian motion on } |G|_{\ell} *
Theorem (G ’08)

In a network with $\sum_{e \in E} r(e) < \infty$ there is a unique ‘good’ current.

Problem

Define brownian motion on $|G|_\ell$

Problem

Solve the Dirichlet Problem at the $|G|_\ell$ boundary