From finite graphs to infinite; and beyond

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Mathematisches Seminar
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Many finite theorems fail for infinite graphs:
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- Hamilton cycle theorems
Hamilton cycle: A cycle containing all vertices.

Some examples:
Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
Things that go wrong in infinite graphs

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- Extremal graph theory
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- many others ...
Things that go wrong in infinite graphs

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⇒ need more general notions
Classical approach to ‘save’ Hamilton cycle theorems: accept double-rays as infinite cycles

\[ \cdots \rightarrow \cdots \]
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This approach only extends finite theorems in very restricted cases:
Classical approach to ‘save’ Hamilton cycle theorems: accept double-rays as infinite cycles

\[ \cdots \bullet \cdot \cdots \cdots \cdots \]

This approach only extends finite theorems in very restricted cases:

**Theorem (Tutte ’56)**

*Every finite 4-connected planar graph has a Hamilton cycle*
Classical approach: accept double-rays as infinite cycles

\[ \cdots \quad \bullet \quad \bullet \quad \bullet \quad \cdots \]

This approach only extends finite theorems in very restricted cases:

**Theorem (Yu ’05)**

*Every locally finite 4-connected planar graph has a spanning double ray ...*
Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles

\[ \cdots \cdot \cdot \cdot \cdots \]

This approach only extends finite theorems in very restricted cases:

**Theorem (Yu ’05)**

*Every locally finite 4-connected planar graph has a spanning double ray ... unless it is 3-divisible.*
Compactifying by Points at Infinity

A 3-divisible graph
A 3-divisible graph can have no spanning double ray.
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... but a Hamilton cycle?
end: equivalence class of rays
two rays are equivalent if no finite vertex set separates them
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two ends
**Ends**

**end**: equivalence class of rays

two rays are **equivalent** if no finite vertex set separates them

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**Two ends**

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**One end**
**Ends**

**end:** equivalence class of rays

two rays are **equivalent** if no finite vertex set separates them

![Diagram](image)

two ends

one end

uncountably many ends

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Infinite graphs
The End Compactification

Every ray converges to its end.
Every ray converges to its end
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Give each edge $e$ a length $\ell(e)$.
(Equivalent) definition of $|G|$

Give each edge $e$ a length $\ell(e)$

This naturally induces a metric $d_\ell$ on $G$
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Denote by $|G|_\ell$ the completion of $(G, d_\ell)$
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This naturally induces a metric $d_\ell$ on $G$

Denote by $|G|_\ell$ the completion of $(G, d_\ell)$

**Theorem (G ’06)**

If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell$ is homeomorphic to $|G|$. 
Circle:
A homeomorphic image of $S^1$ in $|G|$.
Infinite Cycles

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Hamilton circle:
a circle containing all vertices
Infinite Cycles

Circle:
A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices (and all ends?)
Infinite Cycles

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A homeomorphic image of $S^1$ in $|G|$.

Hamilton circle:
a circle containing all vertices, and thus also all ends.
Infinite Cycles

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the wild circle of Diestel & Kühn
Theorem (Fleischner ’74)

*The square of a finite 2-connected graph has a Hamilton cycle*
Fleischner’s Theorem

Theorem (Fleischner ‘74)

The square of a finite 2-connected graph has a Hamilton cycle

Theorem (Thomassen ‘78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle.
Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle.
Proof?
Proof?
Proof?
Proof?
Proof?
Proof?
Proof?
Proof?
Proof?
Proof?

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Infinite graphs
Proof?
Proof?
Proof?
Proof?
Hilbert’s space filling curve:

a sequence of injective curves with a non-injective limit
Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle
Theorem (G ’06)

The square of any locally finite 2-connected graph has a Hamilton circle

Corollary

Cayley graphs are “morally” hamiltonian.
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser ’59)

Does every finite connected Cayley graph have a Hamilton cycle?
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser ’59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1-ended Cayley graph have a Hamilton circle?
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser ’59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1-ended Cayley graph have a Hamilton circle?
Problem (Rapaport-Strasser ’59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1-ended Cayley graph have a Hamilton circle?

Problem

Prove that every connected Cayley graph of a finitely generated group $\Gamma$ has a Hamilton circle unless $\Gamma$ is the amalgamated product of more than $k$ groups over a subgroup of order $k$. 
Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory
- many others ...
Theorem (Mader ’72)

Any finite graph of minimum degree at least $4k$ has a $k$-connected subgraph.

$k$-connected means: you can delete any $k – 1$ vertices and the graph will still be connected.
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**Theorem (M. Stein ’05)**

Let $k \in \mathbb{N}$ and let $G$ be a locally finite graph such that every vertex has degree at least $6k^2 - 5k + 3$ and every end has degree at least $6k^2 - 9k + 4$. Then $G$ has a $k$-connected subgraph.
The **cycle space** \( C(G) \) of a finite graph:

- A vector space over \( \mathbb{Z}_2 \)
- Consists of all sums of cycles
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- Allows edge sets of infinite circles;
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The topological cycle space $\mathcal{C}(G)$ of a locally finite graph $G$ is defined similarly but:
- Allows edge sets of infinite circles;
- Allows infinite sums (whenever well-defined).
The topological Cycle Space

Known facts:
- A connected graph has an Euler tour iff every edge-cut is even (Euler)
- $G$ is planar iff $C(G)$ has a simple generating set (MacLane)
- If $G$ is 3-connected then its peripheral cycles generate $C(G)$ (Tutte)

Generalisations:
- Bruhn & Stein
- Bruhn & Stein
- Bruhn
Theorem (MacLane ’37)

A finite graph $G$ is planar iff $C(G)$ has a simple generating set.

**simple**: no edge appears in more than two generators.
MacLane’s Planarity Criterion

**Theorem (MacLane ’37)**

A finite graph $G$ is planar iff $\mathcal{C}(G)$ has a simple generating set.

**simple**: no edge appears in more than two generators.

**Theorem (Bruhn & Stein’05)**

... verbatim generalisation for locally finite $G$
There is a canonical homomorphism

\[ f : H_1(|G|) \rightarrow \mathcal{C}(G) \]
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**Theorem (Diestel & Sprüssel ’07)**
\[ f \text{ is surjective but not injective.} \]
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Theorem (Diestel & Sprüssel ’07)

\( f \) is surjective but not injective.
Problem

Modify $H_1$ to obtain a homology theory that captures $\mathcal{C}(G)$ when applied to $|G|$ and generalises graph-theoretical theorems to arbitrary continua.
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Modify $H_1$ to obtain a homology theory that captures $\mathcal{C}(G)$ when applied to $|G|$ and generalises graph-theoretical theorems to arbitrary continua.

In particular:

Problem

*Characterise the continua embeddable in the plane*
An **electrical network** is a graph $G$ with an assignment of resistances $r : E(G) \rightarrow \mathbb{R}^+$, and two special vertices (source – sink) pumping a flow of constant value $I$ into the network.
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**electrical flow**: A flow satisfying Kirchhoff’s second law (for finite cycles.)
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If $G$ is finite then the electrical flow is unique, if it is infinite then there might be several; but:
Electrical Networks

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**Theorem (G ’08)**

If $\sum_{e \in E} r(e) < \infty$ then there is a unique non-elusive electrical flow of finite energy.

energy $:= \sum_{e \in E} i^2(e)r(e)$. 

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