A new homology for infinite graphs and metric continua

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Overview

- **Wild spaces** have a huge fundamental group $\pi_1$ and 1st homology group $(1\text{st Homology group } H_1 = \text{abelianization of } \pi_1)$

- We are going to *tame* $H_1$ by removing some ‘redundancy’
Wild spaces have a huge fundamental group $\pi_1$ and 1st homology group $H_1$ = abelianization of $\pi_1$.

We are going to *tame* $H_1$ by removing some ‘redundancy’.

... using experience from infinite graph theory.
Theorem (MacLane ’37)

A finite graph $G$ is planar iff $C(G)$ has a simple generating set.

$C(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology) = $\text{Abel}(\pi_1)$

simple: no edge appears in more than two generators.
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**simple**: no edge appears in more than two generators.

But using the right homology
(topological cycle space of Diestel & Kühn) ...:

**Theorem (Bruhn & Stein ’05)**

... *verbatim generalisation for locally finite $G$.*
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$$H'_1(X) := H_1(X) / d = 0$$

and, if you like, let $\hat{H}_1(X)$ be its completion.
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more precisely: $$d(a, b) := \inf_{\chi \text{ isom } \chi'} \text{ area}(\chi' \setminus \chi)$$

$\approx$ in $\chi'$

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\begin{align*}
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\text{more precisely: } d(a, b) &:= \inf_{\substack{X \text{ isom} \rightarrow X' \\\{a \approx b \text{ in } X'}} \text{ area}(X' \setminus X)
\end{align*}
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Let \[ H'_1(X) := H_1(X) / d=0 \]
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A wild space by Z. Virk & A. Zastrow.
Cycle decompositions
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\[ \rightarrow \]
Can you make a theorem out of this observation?
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Proposition

Every element of $C(G)$ can be written as a union of a set of edge-disjoint cycles.
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Every element of $\mathcal{C}(G)$ can be written as a union of a set of edge-disjoint cycles.

Theorem (G' 09)

For every compact metric space $X$ and $C \in \hat{\mathcal{H}}_1(X)$, there is a $\sigma$-representative $(z_i)_{i \in \mathbb{N}}$ of $C$ that minimizes the length $\sum_i \ell(z_i)$ among all representatives of $C$. 
Theorem (Diestel & Kühn)

Every element of the topological cycle space $C(G)$ of a locally finite graph $G$ can be written as a union of a set of edge-disjoint circles.
Theorem (Diestel & Kühn)

Every element of the topological cycle space \( C(G) \) of a locally finite graph \( G \) can be written as a union of a set of edge-disjoint circles.

One of many classical theorems recently extended to infinite graphs using our new homology, the topological cycle space \( C(G) \) in an ongoing series of >30 papers by Diestel, Kühn, Bruhn, Stein, G, Sprüssel, Richter, Vella, et. al.
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Proof sketch

**Theorem (G’ 09)**

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Specify a subset of well-behaved elements of $\hat{H}_1(X)$, called \textbf{primitive} elements;
Theorem (G’ 09)

For every compact metric space \( X \) and \( C \in \widehat{H}_1(X) \), there is a \( \sigma \)-representative \((z_i)_{i \in \mathbb{N}}\) of \( C \) that minimizes the length \( \sum_i \ell(z_i) \) among all representatives of \( C \).

- Specify a subset of well-behaved elements of \( \widehat{H}_1(X) \), called primitive elements;
- Prove the statement for primitive elements;
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- Specify a subset of well-behaved elements of $\hat{H}_1(X)$, called primitive elements;
- Prove the statement for primitive elements;
- Show that every other element can be expressed as a sum of primitive elements.
Proof sketch

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We say that $C \in \hat{H}_1(X)$ splits if there are $A, B \neq 0 \in \hat{H}_1(X)$ with

\[
C = A + B, \quad \text{and} \quad \ell(C) = \ell(A) + \ell(B).
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Specify a subset of well-behaved elements of $\widetilde{H}_1(X)$, called primitive elements;

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...where $\ell(C)$ is the minimal length of 1-simplices needed to represent $C$. 

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Proof sketch

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We say that $C \in \widehat{H}_1(X)$ splits if there are $A, B \neq 0 \in \widehat{H}_1(X)$ with

$$C = A + B, \text{ and}$$
$$\ell(C) = \ell(A) + \ell(B).$$

...where $\ell(C)$ is the minimal length of 1-simplices needed to represent $C$.

Then $C$ is \textit{primitive} if it doesn’t split.
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For every compact metric space $X$ and $C \in \hat{H}_1(X)$, there is a $\sigma$-representative $(z_i)_{i \in \mathbb{N}}$ of $C$ that minimizes the length $\sum_i \ell(z_i)$ among all representatives of $C$.

- Specify a subset of well-behaved elements of $\hat{H}_1(X)$, called primitive elements;
- Prove the statement for primitive elements;
- Show that every other element can be expressed as a sum of primitive elements.
Let \((\Gamma, +)\) be an abelian metrizable topological group, and suppose a function \(\ell : \Gamma \to \mathbb{R}^+\) is given satisfying the following properties

- \(\ell(a) = 0\) iff \(a = 0\);
- \(\ell(a + b) \leq \ell(a) + \ell(b)\) for every \(a, b \in \Gamma\);
- if \(b = \lim a_i\) then \(\ell(b) \leq \lim \inf \ell(a_i)\);
- Some “isoperimetric inequality” holds: e.g. \(d(a, 0) \leq U\ell^2(a)\)
  for some fixed \(U\) and for every \(a \in \Gamma\).

Then every element of \(\Gamma\) is a (possibly infinite) sum of primitive elements.
The Conjecture

Theorem (MacLane ’37)
A finite graph $G$ is planar iff $C(G)$ has a simple generating set.

Let $X$ be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then $X$ is planar iff there is a simple set $S$ of loops in $X$ and a metric $d$ inducing the topology of $X$ so that the set $U = \{ \chi \in \hat{H}_1(X) | \chi \in S \}$ 'spans' $\hat{H}_1(X)$. 

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**The Conjecture**

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**Theorem (MacLane ’37)**

A finite graph $G$ is planar iff $C(G)$ has a simple generating set.
\( \ell - \text{TOP} \)

Let \( G = (V, E) \) be any graph. Give each edge a length \( \ell(e) \). This induces a metric:

\[
d(v, w) := \inf \{ \ell(P) \mid P \text{ is a } v-w \text{ path} \}
\]

Let \( |G|_{\ell} \) be the completion of the corresponding metric space.

Theorem (Bourdon & Pajot, ...)

For every compact metric space \( X \) there is a locally finite graph \( G \) and \( \ell : E \to \mathbb{R}^+ \) such that the boundary of \( |G|_{\ell} \) is isometric to \( X \).
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**Theorem (G ’06)**

If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell \approx |G|$. ...and $\tilde{H}_1$ coincides with the topological cycle space and with $\tilde{H}_1(X)$. 
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**Problem**

*Does every compact metrizable space $X$ admit a metric such that $\tilde{H}_1(X) = \check{H}_1(X)$?*
let $G = (V, E)$ be any graph

give each edge a length $\ell(e)$

this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P$ is a $v$-$w$ path$\}$

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---

**Theorem (Bourdon & Pajot, ...)**

*For every compact metric space $X$ there is a locally finite graph $G$ and $\ell : E \to R_+$ such that the boundary of $|G|_\ell$ is isometric to $X$.***
Applications of $|G|_{\ell}$

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Applications of $\ell$-TOP

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All above authors “discovered” $|G|_{\ell}$ independently!
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- Generalise other graph-theoretical theorems to continua/fractals

Sources:

These slides are available online

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Sources:
AG: "Graph topologies induced by edge lengths" Discrete Math., 311, 1523–1542, 2011.
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Conjecture

Let \( X \) be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then \( X \) is planar iff there is a simple set \( S \) of loops in \( X \) and a metric \( d \) inducing the topology of \( X \) so that the set \( U := \{[\chi] \in \hat{H}_1(X) \mid \chi \in S\} \) ‘spans’ \( \hat{H}_1(X) \).