A Short Proof of Fleischner’s Theorem

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Abstract
We give a short proof of the fact that the square of a 2-connected finite graph is Hamiltonian.

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1 Introduction
The square $G^2$ of a graph $G$ is the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most 2 in $G$. In 1974, Fleischner [3, 4] proved that the square of every 2-connected finite graph has a Hamilton cycle. Thomassen [7] extended this fact to locally finite 1-ended graphs, where a Hamilton cycle is taken to be an infinite path containing all vertices. Using Thomassen’s method, Říha (see [8] or [2]) produced a shorter proof of Fleischner’s Theorem. History repeated itself, and once again the study of infinite graphs led to a new proof of Fleischner’s Theorem: a proof is presented here that uses an idea developed for the recent extension of Fleischner’s Theorem to locally finite graphs with any number of ends\(^1\) to shorten Říha’s proof.

In [5] the present proof is adapted to give a short proof of another theorem of Fleischner [3], stating that the total graph of every finite 2-edge-connected graph has a Hamilton cycle.

2 Definitions
We will be using the terminology of [2]. Let $G$ be a multigraph, and $J$ a walk in $G$. A pass of $J$ through a vertex $x$ is a subwalk of $J$ of the form $uexfv$, where $e$ and $f$ are edges. By lifting this pass we mean replacing it in $J$ by the walk\(^2\).

\(^1\)Settling a problem of Diestel [1], it is shown in [6] that the square of every locally finite 2-connected graph contains a Hamilton circle, a homeomorphic image of the real unit circle $S^1$ in the topological space $|G|$ formed by $G$ and all its ends.

\(^2\)
ugu, where g is a u-v edge, if u ≠ v, or by the trivial walk u if u = v (in fact, the latter case will never occur).

A double edge is a pair of parallel edges, and a multipath is a multigraph obtained from a path by replacing some of its edges by double edges. If C ⊆ G are multigraphs, then a C-trail in G is either a path having precisely its endvertices (but no edge) in common with C, or a cycle having precisely one vertex in common with C. A vertex y on some cycle C is called C-bound if all neighbours of y lie on C.

3 The proof

We will use the following lemma of Ríha [8]. For the convenience of the reader the proof is repeated here.

Lemma 1. If G is a 2-connected finite graph and x ∈ V(G), then there is a cycle C ⊆ G that contains x as well as a C-bound vertex y ≠ x.

Proof. As G is 2-connected, it contains a cycle C′ that contains x. If C′ is a Hamilton cycle there is nothing more to show, so let D be a component of G − C′. Assume that C′ and D are chosen so that |D| is minimal. Easily, C′ contains a path P between two distinct neighbours u, v of D whose interior P′ does not contain x and has no neighbour in D. Replacing P′ in C′ by a u-v-path through D, we obtain a cycle C that contains x and a vertex y ∈ D. By the minimality of |D| and the choice of P′, y has no neighbour in G − C, so C satisfies the assertion of the lemma.

We will prove Fleischner's Theorem in the following stronger form, which is similar to the assertion proved by Ríha [8].

Theorem 1. If G is a 2-connected finite graph and x ∈ V(G), then G^2 has a Hamilton cycle whose edges at x lie in E(G).

Proof. We perform induction on |G|. For |G| = 3 the assertion is trivial. For |G| > 3, let C be a cycle as provided by Lemma 1. Our first aim is to define, for every component D of G − C, a set of C-trails in G^2 + E′, where E′ will be a set of additional edges parallel to edges of G. Every vertex of D will lie in exactly one such trail, and for every such trail T and every edge e of T incident with a vertex of C, e will lie in E(G) or in E′.

If D consists of a single vertex u, we pick any C-trail in G containing u, and let E_D be the set of its two edges. If |D| > 1, let D be the (2-connected) graph obtained from G by contracting G − D to a vertex x. Applying the induction hypothesis to D, we obtain a Hamilton cycle H of D^2 whose edges at x lie in E(D). Write E for the set of those edges of H that are not edges of G^2. Replacing these by edges of G or new edges e' ∈ E′, we shall turn E(H) into the edge set of a union of C-trails. Consider an edge u v ∈ E, with u ∈ D. Then either v = x, or u, v have distance at most 2 in D but not in G, and are hence neighbours of x in D. In either case, G contains a u-C edge. Let E_D be obtained from E(H) \ E by adding at every vertex u ∈ D as many u-C edges as u has incident edges in E; if u has two incident edges in E but sends only one edge e to C, we add both e and a new edge e' parallel to e. Then every vertex of D has the same degree (two) in (V(G), E_D) as in H, so E_D is the edge set
of a union of C-trails. Let \( G' := (V(G), E(C) \cup \bigcup_D E_D) \) be the union of \( C \) and all those trails, the union taken over the set of all components \( D \) of \( G - C \).

Let \( y \) be a \( C \)-bound vertex of \( C \) and pick a vertex \( z \) and edges \( d_1, d_2, g_1, g_2 \) of \( C \), so that \( C = xg_1z \ldots d_1yd_2 \ldots g_2x \) (the vertices and edges named here need not be distinct). We will add parallel edges to some edges of \( C - g_1 \), to turn \( G' \) into an eulerian multigraph \( G'_y \) — i.e. a connected multigraph in which every vertex has even degree (and which therefore has an Euler tour [2]). Every vertex in \( G' - C \) already has degree 2. In order to obtain even degrees at the vertices in \( C \) we consider these vertices in reverse order, starting with \( x \) and ending with \( z \). Let \( u \) be the vertex currently considered, and let \( v \) be the vertex to be considered next. Add a new edge parallel to \( uv \) if and only if \( u \) has odd degree in the multigraph obtained from \( G' \) so far. When finally \( u = z \) is considered, every other vertex has even degree, so by the “hand-shaking lemma” \( z \) must have even degree too and no edge parallel to \( g_1 \) will be added. Let \( G'_y \) be the resulting multigraph, and let \( C'_0 = G'_y[V(C)] \).

If \( g_2 \) has a parallel edge \( g_2' \) in \( G'_y \), then delete both \( g_2, g_2' \). If \( g_2 \) has no parallel edge, and \( d_2 \) has a parallel edge \( d_2' \), then delete both \( d_2 \) and \( d_2' \). Let \( G'_y \) be the resulting (eulerian) multigraph. If \( g_2 \) has been deleted, then let \( P_3 \) be the multipath \( C'_y - \{g_2, g_2'\} \). If not, let \( P_1 \) be the maximal multipath in \( C'_y \) with endvertices \( x, y \) containing \( g_1 \), and let \( P_2 \) be the multipath containing all edges in \( E(C'_y \cap G'_y) - E(P_1) \) (Figure 1).

Figure 1: The paths \( P_i \) (three cases). The bold edges are known to be single.

Our plan is to find an Euler tour \( J' \) of \( G'_y \) that can be transformed into a Hamilton cycle of \( G^2 \). In order to endow \( J' \) with the required properties we will derive it from an Euler tour of an auxiliary multigraph, which we define next.

For every \( i \) such that \( P_i \) has been defined, do the following. Write \( P_i = x_0^i x_1^i \ldots x_l^i \) with \( x_0^i = x \), and \( e_j^i \) or just \( e_j \) for the \( x_{j-1}^i \ldots x_j^i \) edge of \( P_i \) in \( E(C) \). Its parallel edge, if it exists, will again be denoted by \( e_j^i \) (when \( i \) is fixed). Now for \( j = 1, \ldots, l_i \), if \( e_j^i \) exists, replace \( e_j \) and \( e_{j+1} \) by a new edge \( f_j \) joining \( x_{j-1}^i \) to \( x_{j+1}^i \); we say that \( f_j \) represents the walk \( x_{j-1}^i e_j x_{j+1}^i \) (Figure 2). Note that every such replacement leaves the current multigraph connected, and it preserves the parity of all degrees. Hence, the multigraph \( G^2 \) finally obtained by all these replacements is eulerian, so pick an Euler tour \( J \) of \( G^2 \). Transform \( J \) into an Euler tour \( J' \) of \( G'_y \) by replacing every edge in \( E(J) - E(G'_y) \) by the walk it represents.

Our next aim is to perform some lifts in \( J' \) to transform it into a Hamilton cycle. To this end, we will now mark some passes for later lifting. Start by marking all passes of \( J' \) through \( x \) except for one arbitrarily chosen pass. We
want to mark some more passes, so that for any vertex \( v \in V(C) - x \) the following assertion holds:

for any \( i, j \), if \( v = x_i^j \) then all passes of \( J' \) through \( v \) are marked except for the pass containing \( e_i^j \).

This is easy to satisfy for \( v \neq y \), as there is precisely one pair \( i, j \) so that \( v = x_i^j \) in that case. A difficulty can only arise if \( v = y = x_1^1 = x_2^2 \), in case both \( P_1 \) and \( P_2 \) contain \( y \). By the definition of the \( P_i \), this case only materialises if there are no edges \( g_{2}', f_{2}' \) in \( G_\beta \), and as \( y \) is \( C \)-bound, it has degree at most 3 and hence degree 2 in \( G_\beta \) in that case. But then, there is only one pass of \( J' \) through \( v \), which consists of \( e_1^1, e_2^2 \), and leaving it unmarked satisfies (1).

So we assume that (1) holds, and now we claim that

for every edge \( e = uv \) in \( J' \), at most one of the two passes of \( J' \) that contain \( e \) is marked, and moreover if \( u = x \), then the pass of \( J' \) through \( v \) containing \( e \) is unmarked.

This is clear for edges in \( E(G_\beta) - E(C_0) \), so pick an \( e \in P_i \). If \( e = e_j \) for some \( j \), then by (1) the pass of \( J' \) through \( x_i^j \) containing \( e \) is unmarked; in particular, if \( e \) is incident with \( x = x_i^0 \), then \( j = 1 \) and the pass of \( J' \) through \( x_i^1 \) containing \( e \) is unmarked. If \( e = e_j' \), then \( e \) is not incident with \( x \) by the construction of \( G_\beta \), and an edge \( f_{j-1} \) was defined to represent the walk \( x_{j-2}e_{j-1}x_{j-1}e_j'x_j \). Since \( J \) contained \( f_{j-1} \), this walk is a pass in \( J' \). This pass is unmarked by (1), because it is a pass through \( x_{j-1} \) containing \( e_{j-1} \).

So we proved our claim, which implies that no two marked passes share an edge. Thus we can now lift each marked pass of \( J' \) to an edge of \( G^2 \), to obtain a new closed walk \( H' \) in \( G^2 + E' \). Every vertex of \( G \) is traversed precisely once by \( H' \), since by (1) we marked, and eventually lifted, for each vertex \( v \) of \( G \) all passes of \( J' \) through \( v \) except precisely one pass. (This is trivially true for a vertex \( u \) in \( G - C \), as there is only one pass of \( J' \) through \( u \) and this pass was not marked.) In particular, \( H' \) cannot contain any pair of parallel edges, so we can replace every edge \( e' \) in \( H' \) that is parallel to an edge \( e \) of \( G \) by \( e \) to obtain a Hamilton cycle \( H \) of \( G^2 \). Since by the second part of (2) no edge incident with \( x \) was lifted at its other end, both edges of \( H \) at \( x \) lie in \( G \) as desired.
4 Total graphs

The subdivision graph $S(G)$ of a graph $G$ is the bipartite graph with partition classes $V(G), E(G)$ where $x \in V(G)$ and $e \in E(G)$ are joined by an edge if $x$ is incident with $e$ in $G$. The total graph $T(G)$ of $G$ is the square of $S(G)$; equivalently, $T(G)$ is the graph on $V(G) \cup E(G)$ where two vertices are adjacent if the respective objects are adjacent or incident in $G$. Fleischner [3] proved that:

**Theorem 2.** If $G$ is a finite, 2-edge-connected graph then $T(G)$ has a Hamilton cycle.

In [5] the proof of Section 3 was adapted to give a short proof of Theorem 2, exploiting the fact that $T(G)$ is the square of a graph. We do not repeat that proof here, but we will point out the main differences to the proof in Section 3.

Instead of looking for a cycle $C$ with a $C$-bound vertex, we just pick any cycle $C$ in $G$; the reason is that later we will consider the subdivision graph $C'$ of $C$, and then any of the vertices of degree 2 that will arise after subdividing an edge will be $C'$-bound. Again we use induction, and apply the induction hypothesis to all components of $S(G) - S(C')$ to obtain a set of $C'$-trails covering all vertices in $S(G) - S(C')$ (this step is more complicated though). After constructing the $C'$-trails we have a very similar situation to that in the proof of Section 3, and we can proceed in the same way; the fact that we have a big choice of $C'$-bound vertices only simplifies the proof.

References


