From mafia expansion to analytic functions in percolation theory

Agelos Georgakopoulos

Joint work with John Haslegrave, and with Christoforos Panagiotis

These slides are on-line
A “social” network evolves in (continuous or discrete) time according to the following rules.
When a (Poisson) clock ticks, vertices split into two;
A model for Mafia growth

- When a (Poisson) clock ticks, vertices split into two;
- When a vertex splits, each of its edges gets randomly inherited by one of its offspring (with probability 1/2);

Theorem (G & Haslegrave (thanks to G. Ray), 18+)
As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).

How does the expected size depend on $\lambda$?

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Does the limit \(M(\lambda)\) depend on the starting network?
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As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).

Does the limit $M(\lambda)$ depend on the starting network? No! In other words,

**Theorem**

There is a unique random graph $M(\lambda)$ invariant under the above operation.
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Theorem (G & Haslegrave (thanks to G. Ray), 18+)

As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).

Is $M(\lambda)$ finite or infinite?
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As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).

Is $M(\lambda)$ finite or infinite?
It is finite almost surely
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Is its expected size finite or infinite?
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Is its expected size finite or infinite?

- **finite** in the synchronous case,
- **we don’t know** in the asynchronous case
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*As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).*

How does the expected size depend on $\lambda$?
Simulations by C. Moniz (Warwick).

Further examples. Here are some more random graphs $R_{k,n}$ outputted by my algorithm for different values of $k$ and $n$:

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47 48
49 50 51
52 53
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Figure 3: $R_{k,n}$ obtained from binary trees, for different values of $k$ and $n$. From top-left to bottom-right, the values are $k = 3$ and $n = 6$, $k = 3$ and $n = 8$, $k = 6$ and $n = 10$, $k = 7$ and $n = 10$, respectively. Drawn using Python igraph [20].

Notice that while the graph is connected for $k = 3$ walks on a tree of depth $n = 6$, it is disconnected for $k = 3$ walks on a tree of depth $n = 8$. The example of $R_{3,8}$ illustrated in figure 3 has 15 connected components.

One may be interested in how often the random graph obtained from this construction is connected, for different values of $k$ and $n$. In the case of $n = 10$, I observed (informally) that more often than not, a graph with $k = 6$ would be disconnected and a graph with $k = 7$ walkers would be connected. Some examples of this are illustrated in figure 3.
The expected size of $M(\lambda)$

Let $\chi(\lambda) := \mathbb{E}(|M(\lambda)|)$

**Theorem (G & Haslegrave ’18+)
**

$$e^{c\lambda} \leq \chi(\lambda) \leq e^{e^{c\lambda}}$$
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$$e^{c\lambda} \leq \chi(\lambda) \leq e^{e^{c\lambda}}$$

Conjecture:

$$\chi(\lambda) \sim \lambda^{c\lambda}$$

(backed by simulations)
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Is $\chi(\lambda)$ continuous in $\lambda$?
Bernoulli bond percolation on an infinite graph, i.e.

Each edge
- present with probability $p$,
and
- absent with probability $1 - p$

independently of other edges.

Percolation threshold:

$$p_c := \sup\{p \mid \mathbb{P}_p(\text{component of } o \text{ is infinite}) = 0\}$$
Classical era:
Introduced by physicists Broadbent & Hammersley ’57

\[ p_c(\text{square grid}) = 1/2 \] (Harris ’59 + Kesten ’80)

Many results and questions on phase transitions, continuity, smoothness etc. in the ’80s:
Aizenman, Barsky, Chayes, Grimmett, Hara, Kesten, Marstrand, Newman, Schulman, Slade, Zhang ... (apologies to many!)
Historical remarks on percolation theory

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Thought of as part of statistical mechanics
Historical remarks on percolation theory

Modern era:
Benjamini & Schramm ’96 popularised percolation on groups ‘beyond $\mathbb{Z}^d$’

See the textbooks [Lyons & Peres ‘15], [Pete ‘18+] for more.
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Modern era:
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... for example, percolation can characterise amenability:

Theorem ($\Leftarrow$ Aizenman, Kesten & Newman ’87,
$\Rightarrow$ Pak & Smirnova-Nagnibeda ’00)

A finitely generated group is non-amenable iff it has a Cayley graph with $p_c < p_u$. 

See the textbooks [Lyons & Peres ’15], [Pete ’18+] for more.
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Modern era:
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Theorem (Kesten ’59)

A finitely generated Cayley graph is non-amenable iff spectral radius of Laplacian < 1 iff n-step return probability of random walk decays exponentially in n.
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Historical remarks on percolation theory

Post-modern era:
Scaling limits of critical percolation in the plane
Conformal invariance thereof
SLE

Lawler-Schramm-Werner, Smirnov ... (apologies to many!)
Not covered in this talk.
\( \chi(p) := \mathbb{E}_p(|C(o)|) \),
i.e. the expected size of the component of the origin \( o \).

**Theorem (Kesten ’82)**

\( \chi(p) \) is an analytic function of \( p \) for \( p \in [0, p_c) \) when \( G \) is a lattice in \( \mathbb{R}^d \).
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**Theorem (Kesten ’82)**

\( \chi(p) \) is an analytic function of \( p \) for \( p \in [0, p_c) \) when \( G \) is a lattice in \( \mathbb{R}^d \).

‘Trying to think of negative probabilities gave me cultural shock at first...’


Let’s just extend \( p \) to the complex numbers...

—Harry Kesten ’81; blatantly paraphrased
Some complex analysis basics

**Theorem (Weierstrass):** Let $f = \sum f_n$ be a series of analytic functions which converges uniformly on each compact subset of a domain $\Omega \subset \mathbb{C}$. Then $f$ is analytic on $\Omega$.

**Weierstrass M-test:** Let $(f_n)$ be a sequence of functions such that there is a sequence of ‘upper bounds’ $M_n$ satisfying

$$|f_n(z)| \leq M_n, \forall x \in \Omega \quad \text{and} \quad \sum M_n < \infty.$$

Then the series $\sum f_n(x)$ converges uniformly on $\Omega$. 

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**Theorem (Aizenman & Barsky ’87)**

In every vertex-transitive percolation model,

\[
\mathbb{P}_p(|C| \geq n) \leq c_p^{-n},
\]

for every \( p < p_c \) and some \( c_p > 1 \).
Conjectures on the percolation probability

\[ \theta(p) := \mathbb{P}_p(|C| = \infty), \]

i.e. the percolation probability.

Fig. 1.1. It is generally believed that the percolation probability \( \theta(p) \) behaves roughly as indicated here. It is known, for example, that \( \theta \) is infinitely differentiable except at the critical point \( p_c \). The possibility of a jump discontinuity at \( p_c \) has not been ruled out when \( d \geq 3 \) but \( d \) is not too large.

1.2 Some Possible Questions

Here are some apparently reasonable questions, some of which turn out to be feasible.

• What is the value of \( p_c \)?
• What are the structures of the subcritical and supercritical phases?
• What happens when \( p \) is near to \( p_c \)?
• Are there other points of phase transition?
• What are the properties of other 'macroscopic' quantities, such as the mean size of the open cluster containing the origin?
• What is the relevance of the choice of dimension or lattice?
• In what ways are the large-scale properties different if the states of nearby edges are allowed to be dependent rather than independent?

There is a variety of reasons for the explosion of interest in the percolation model, and we mention next a few of these.

• The problems are simple and elegant to state, and apparently hard to solve.
• Their solutions require a mixture of new ideas, from analysis, geometry, and discrete mathematics.
• Physical intuition has provided a bunch of beautiful conjectures.
• Techniques developed for percolation have applications to other more complicated spatial random processes, such as epidemic models.
• Percolation gives insight and method for understanding other physical models of spatial interaction, such as Ising and Potts models.
• Percolation provides a 'simple' model for porous bodies and other 'transport' problems.
Open problem:

Is $\theta(p)$ analytic for $p > p_c$?

Appearing (for $G = \mathbb{Z}^d$) in the textbooks

*Kesten ’82, Grimmett ’96, Grimmett ’99.*
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\textit{‘it is a well-known problem of debatable interest...’}
–Grimmett ’99
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‘it is a well-known problem of debatable interest...’

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‘...this is not just an academic matter. For instance, there are examples of disordered systems in statistical mechanics that develop a Griffiths singularity, i.e., systems that have a phase transition point even though their free energy is a $C^\infty$ function.’

–Braga, Proccaci & Sanchis ’02
$\theta(p) := \mathbb{P}_p(|C| = \infty)$,  
i.e. the percolation probability.

For percolation on the \textit{d-regular tree}, we have

$$\theta(p) = 1 - (1 - p\theta_0(p))^d$$

where $\theta_0$ solves $1 - \theta_0 = (1 - p\theta_0)^{d-1}$. 
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\textbf{Proposition} \(\theta\) is analytic for \(p > p_c\) on any regular tree (G & Panagiotis ’18+).
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\textbf{Proposition} \( \theta \) is analytic for \( p > p_c \) on any regular tree (G & Panagiotis ’18+).

Trivial for binary tree, but what about higher degrees?
**Proposition** \( \theta \) is analytic for \( p > p_c \) on any regular tree (G & Panagiotis ’18+).

We deduce this from

**Theorem (G & Panagiotis ’18+)**

\[
\theta \text{ is analytic for } p > \frac{1}{1+h} \text{ on any bounded-degree graph with Cheeger constant } h.
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Which builds upon

**Theorem (Benjamini & Schramm ’96)**

\[
p_c \leq \frac{1}{1+h} \text{ on any such graph.}
\]
Theorem (G & Panagiotis ’18+)

\( \theta(p) \) is analytic for \( p > p_c \) on any planar lattice.

Fig. 1.1. It is generally believed that the percolation probability \( \theta(p) \) behaves roughly as indicated here. It is known, for example, that \( \theta \) is infinitely differentiable except at the critical point \( p_c \). The possibility of a jump discontinuity at \( p_c \) has not been ruled out when \( d \geq 3 \) but \( d \) is not too large.
Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer $n$ is of order

$$\exp(\sqrt{n}).$$

Elementary proof: [P. Erdös, Annals of Mathematics '42]
Conjecture (Benjamini & Schramm ’96):
$p_c < 1$ for every finitely generated Cayley graph.
Finitely presented Cayley graphs

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\[ p_c < 1 \] for every finitely generated Cayley graph.

Theorem (Babson & Benjamini ’99):
\[ p_c < 1 \] for every finitely presented Cayley graph.

Theorem (GP ’18+):
\[ \theta(p) \] is analytic for \( p \) near 1 for every finitely presented Cayley graph.

Theorem (Häggström ’00):
Every bounded degree graph exhibits a phase transition in all or none of the following models: bond/site percolation, Ising, Widom-Rowlinson, beach model.
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–Similar arguments, but we had to generalise interfaces to all graphs.

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Theorem (Duminil-Copin, Goswami, Raoufi, Severo & Yadin ’18+)
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Proof involves the Gaussian Free Field.

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$\theta(p)$ is analytic for $p > 1 - p_c$ for site percolation on any ‘triangulated’ lattice in $\mathbb{Z}^d$, $d \geq 2$. 

Proof based on the notion of interfaces from before, and an exponential decay thereof...
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Is $\theta(p)$ analytic at $1 - p_c$?
Continuous at $p_c$?
Further reading: [H. Duminil-Copin, Sixty years of percolation]

Further examples. Here are some more random graphs $R_{k,n}$ outputted by my algorithm for different values of $k$ and $n$:

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Notice that while the graph is connected for $k = 3$ walks on a tree of depth $n = 6$, it is disconnected for $k = 3$ walks on a tree of depth $n = 8$. The example of $R_{3,8}$ illustrated in figure 3 has 15 connected components.

One may be interested in how often the random graph obtained from this construction is connected, for different values of $k$ and $n$. In the case of $n = 10$, I observed (informally) that more often than not, a graph with $k = 6$ would be disconnected and a graph with $k = 7$ walkers would be connected. Some examples of this are illustrated in figure 3.

Further reading: [G. & Panagiotis, Analyticity results in Bernoulli Percolation]

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