Group Walk Random Graphs

Agelos Georgakopoulos

Vancouver, 10.9.14
Random Graphs flashback

1269 papers on MathSciNet with "random graph" in their title
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... most of which on the Erdős-Renyi model $G(n, p)$:
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... most of which on the Erdős-Renyi model $G(n, p)$:

- $n$ vertices
- each pair joined with an edge, independently, with same probability $p = p(n)$.
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100. [Bollobás, B. Long paths in sparse random graphs. Combinatorica. 1982] shows that if \( p = \frac{c}{n} \), then almost every graph in \( G(n, p) \) contains a path of length at least \((1 - a(c))n\), where \( a(c) \) is an exponentially decreasing function of \( c \).
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=> shows that if \( p = \frac{c}{n} \), then almost every graph in \( G(n, p) \) contains a path of length at least \((1 - a(c))n\), where \( a(c) \) is an exponentially decreasing function of \( c \).

=> derives large deviation principles for the empirical neighbourhood measure of colored random graphs, defined as the number of vertices of a given colour with a given number of adjacent vertices of each colour. . . .
Random Graphs from trees
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Simulation on the binary tree by A. Janse van Rensburg.
A nice property
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Proposition

$$\mathbb{E}(\# \text{ edges } xy \text{ in } G_n(T) \text{ with } x \text{ in } X \text{ and } y \text{ in } Y)$$ converges.
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converges.

?
<table>
<thead>
<tr>
<th></th>
<th>One set</th>
<th>Two sets</th>
<th>Three sets</th>
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<td><img src="cycled_binary_tree_three_sets.png" alt="Image" /></td>
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</tbody>
</table>

Figure 4: Random graphs generated for various host graphs. Note the difference in number of components, isolated vertices, and component diameter.

Using the $k$-balls of the graphs in the preceding section (with boundaries as stated), we construct the random graphs $R_k(G, B)$. For all host graphs chosen, the probability that there was at least one isolated vertex (which implies that the graph was disconnected) tended to 1 as $k \to 1$. We examine several properties of the resulting random graphs:

- Number of isolated vertices
- Number of components
- Value: size of largest component / size of smallest component
- Diameter of largest connected component

For certain graphs, we also run random walks until the generated graph $R_k$ is connected. In what follows, 10,000 random graphs were generated for each $k$-value, and an average was taken.

Simulations by C. Midgley.
**Problem 1:** The (expected) number of connected components (or isolated vertices) is asymptotically proportional to $|B_n|$. 

**Problem 2:** The expected diameter of the largest component is asymptotically $c \log |B_n|$. 

Backed by simulations by C. Midgley.
**Metaproblem 1**: Which properties of the random graphs are determined by the group of the host graph $H$ and do not depend on the choice of a generating set?
What’s the point?

**Metaproblem 1:** Which properties of the random graphs are determined by the group of the host graph $H$ and do not depend on the choice of a generating set?

**Metaproblem 2:** Which group-theoretic properties of the host group are reflected in graph-theoretic properties of the random graphs?
The classical Douglas formula

\[ E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(\zeta, \eta) \, d\eta \, d\zeta \]

calculates the (Dirichlet) energy of a harmonic function \( h \) on \( \mathbb{D} \) from its boundary values \( \hat{h} \) on the circle \( \partial\mathbb{D} \).
Energy in finite electrical networks

\[ E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 C_{ab}, \]

Compare with Douglas:

\[ E(h) = \int_{\Omega} \int_{\Omega} \left( \hat{h}(\eta) - \hat{h}(\zeta) \right)^2 \Theta(\zeta, \eta) \, d\eta \, d\zeta \]

How can we generalise this to an arbitrary domain?

To an infinite graph?
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The classical Poisson formula

\[ h(z) = \int_0^1 \hat{h}(\theta) P(z, \theta) d\theta \]

where \( P(z, \theta) := \frac{1-|z|^2}{|e^{2\pi i \theta} - z|^2} \), recovers every continuous harmonic function \( h \) on \( \mathbb{D} \) from its boundary values \( \hat{h} \) on the circle \( \partial \mathbb{D} \).
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The Poisson-Furstenberg boundary

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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;

- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.
The Poisson-Furstenberg boundary

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups \([\textit{Annals of Math. '63}]\)
- Kaimanovich & Vershik give a general criterion using the entropy of random walk \([\textit{Annals of Probability '83}]\)
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria \([\textit{Annals of Math. '00}]\)
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[Doob '62] generalises this to Green spaces (or Riemannian manifolds) using their Martin boundary.
The energy of harmonic functions

Theorem (G & Kaimanovich ’14+)

For every locally finite network $G$, there is a measure $C$ on $\mathcal{P}^2(G)$ such that for every harmonic function $u$ the energy $E(u)$ equals

$$\int_{\mathcal{P}^2} \left( \hat{u}(\eta) - \hat{u}(\zeta) \right)^2 \, dC(\eta, \zeta).$$
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$$\int_{\mathcal{P}_2} (\hat{u}(\eta) - \hat{u}(\zeta))^2 \, dC(\eta, \zeta).$$
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$$\int_{\mathcal{P}_2} \left( \tilde{u}(\eta) - \tilde{u}(\zeta) \right)^2 dC(\eta, \zeta).$$

$C(X, Y) := \lim_n \mathbb{E}(\# \text{ edges } xy \in G_n(H) \text{ with } x \text{ ‘close to’ } X, \text{ and } y \text{ ‘close to’ } Y)$
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for $h$ fine-continuous quasi-everywhere [Doob ’63].
The Naim Kernel

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where the Naim Kernel \( \Theta \) is defined as

\[
\Theta(\zeta, \eta) := \lim_{z_n \to \zeta, y_n \to \eta} \frac{F(z_n, y_n)}{G(o, o)} \frac{F(z_n, o)}{F(o, y_n)}
\]
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... in the fine topology [Naim ’57].
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**Remark:**

\[ \frac{1}{\Theta(z, y)} = G(o, o) \Pr(o < y | y), \]

where \( \Pr_z(o < y | y) \) is the conditional probability to visit \( o \) before \( y \) subject to visiting \( y \).
Convergence of the Naim Kernel

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\Theta(\zeta, \eta) := \frac{1}{G(o, o)} \lim_{z_n \to \zeta, y_n \to \eta} \frac{F(z_n, y_n)}{F(z_n, o)F(o, y_n)}
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Problem: Let \((z_i)_{i \in \mathbb{N}}\) and \((w_i)_{i \in \mathbb{N}}\) be independent simple random walks from \(o\). Then \(\lim_{n, m \to \infty} \Theta(z_n, w_m)\) exists almost surely.
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**Proposition**

*For every measurable \( X, Y \subseteq P(G)\)*

\[
C_n(X, Y) = E(\Theta^n(x_n, y_n)1_{XY}).
\]

*Therefore, \( C(X, Y) = \lim_n E(\Theta^n(x_n, y_n)1_{XY}). \)*
Random Interlacements $\mathcal{I}$ [Sznitman]:

- A Poisson point process whose 'points' are 2-way infinite trajectories
- Applied to study the vacant set on the discrete 3D-torus
- Governed by a certain $\sigma$-finite measure $\nu$

Claim: $C(X, Y) = \nu(1_{XY}^\infty)$. 

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Claim: $C(X, Y) = \nu(1_{XY} \mathcal{W}^*)$. 
The effective conductance measure $C$, The Naim kernel $\Theta$, Random Interlacements $\mathcal{I}$, and Group Walk Random Graphs $G_n(H)$
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Can we use the one to study the other?
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Can we use them to study groups?