From mafia expansion to analytic functions in percolation theory

Agelos Georgakopoulos

Joint work with John Haslegrave, and with Christofoforos Panagiotis
A “social” network evolves in (continuous or discrete) time according to the following rules.
When a (Poisson) clock ticks, vertices split into two;
A model for Mafia growth

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- When a vertex splits, each of its edges gets randomly inherited by one of its offspring (with probability 1/2);

Theorem (G & Haslegrave (thanks to G. Ray), 18+)
As time goes to infinity, the distribution of the component of a designated vertex converges (to a random graph $M(\lambda)$).

How does the expected size depend on $\lambda$?

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No! In other words,

Theorem

There is a unique random graph $M(\lambda)$ invariant under the above operation.

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It is finite almost surely.

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finite in the synchronous case,
we don’t know in the asynchronous case.

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Let $\chi(\lambda) := \mathbb{E}(|M(\lambda)|)$

**Theorem (G & Haslegrave ’18+)**

$$e^{c_1 \lambda} \leq \chi(\lambda) \leq e^{e^{c_1 \lambda}}$$
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Conjecture:

$\chi(\lambda) \sim \lambda^{c\lambda}$

(backed by simulations)
The expected size of $M(\lambda)$

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Is $\chi(\lambda)$ continuous in $\lambda$?
Percolation model

Bernoulli bond percolation on an infinite graph, i.e.

Each edge
- present with probability $p$,
and
- absent with probability $1 - p$

independently of other edges.

Percolation threshold:

$$p_c := \sup\{p \mid \mathbb{P}_p(\text{component of } o \text{ is infinite}) = 0\}$$

E.g. $p_c(\text{square grid}) = 1/2$ (Harris ’59 + Kesten ’80)
$\chi(p) := \mathbb{E}_p(|C(o)|)$,
i.e. the expected size of the component of the origin $o$.

**Theorem (Kesten ’82)**

$\chi(p)$ is an analytic function of $p$ for $p \in [0, p_c)$ when $G$ is a lattice in $\mathbb{R}^d$. 

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Proved by extending $p$ and $\chi(p)$ to the complex numbers, and using classical complex analysis (Weierstrass).
Theorem (Weierstrass): Let \( f = \sum f_n \) be a series of analytic functions which converges uniformly on each compact subset of a domain \( \Omega \subset \mathbb{C} \). Then \( f \) is analytic on \( \Omega \).

Weierstrass M-test: Let \((f_n)\) be a sequence of functions such that there is a sequence of ‘upper bounds’ \( M_n \) satisfying

\[
|f_n(z)| \leq M_n, \forall x \in \Omega \quad \text{and} \quad \sum M_n < \infty.
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Then the series \( \sum f_n(x) \) converges uniformly on \( \Omega \).
Some complex analysis basics

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**Theorem (Aizenman & Barsky ’87)**

In every vertex-transitive percolation model,

\[
\mathbb{P}_p(|C| > n) \leq c_p^{-n},
\]

for every \( p < p_c \) and some \( c_p > 1 \).

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Conjectures on the percolation probability

\[
\theta(p) := \mathbb{P}_p(|C| = \infty),
\]
i.e. the percolation probability.

Fig. 1.1. It is generally believed that the percolation probability \(\theta(p)\) behaves roughly as indicated here. It is known, for example, that \(\theta\) is infinitely differentiable except at the critical point \(p_c\). The possibility of a jump discontinuity at \(p_c\) has not been ruled out when \(d \geq 3\) but \(d\) is not too large.
Open problem:

Is \( \theta(p) \) analytic for \( p > p_c \)?

Appearing in the textbooks Kesten ’82, Grimmett ’96, Grimmett ’99.
Our results (G & Panagiotis ’18+)

- $p_c = p_\mathbb{C}$ for all regular trees.
  - trivial for binary tree, but what about higher degrees?

- $p_c = p_{\mathbb{C}}$ for all planar lattices.
  - previously open for all graphs; $C_\infty$ known for $\mathbb{Z}^d$.
  - $\tau, \tau_f$ analytic for $p > p_c$ on all planar lattices.
  - Braga et al. ’04 prove analyticity near $p = 1$ for $\mathbb{Z}^d$.

- $p_c = p_\mathbb{C}$ for continuum percolation in $\mathbb{R}^2$.
  - asked by Last et al. ’16; $C_\infty$ known for all finitely presented Cayley graphs.
  - proved for $\mathbb{Z}^d$ by Braga et al. ’02.
  - $p_c < 1$ for all non-amenable graphs.

- $p_c \leq 1/2$ for certain families of triangulations.

- Progress on questions of Benjamini & Schramm ’96, and Benjamini ’16.
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‘...this is not just an academic matter. For instance, there are examples of disordered systems in statistical mechanics that develop a Griffiths singularity, i.e., systems that have a phase transition point even though their free energy is a $C^\infty$ function.’
–Braga, Proccaci & Sanchis ’02
Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer $n$ is of order

$$\exp(\sqrt{n}).$$

Question:

Does the expected number of separating ‘surfaces’ of $\mathbb{Z}^3$ of size $n$ surrounding $o$ decay exponentially in $n$ for all $p \neq p_c$?
Outlook

- Is the expected size of the asynchronous mafia finite?
- Find other mafia-type rules
- Prove $p_C = p_c$ in higher dimensions

Further reading:

[A. Georgakopoulos and J. Haslegrave, Percolation on an infinitely generated group]

[H. Duminil-Copin, Sixty years of percolation]

[H. Duminil-Copin & V. Tassion, A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$]

These slides are on-line

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