The planar Cayley graphs are effectively enumerable

Agelos Georgakopoulos

Neuchatel, 20/10/15
Groups need to act!
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Let them act on the plane
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Let them act on the plane
and be finitely generated
Planar discontinuous groups: ‘discrete’ groups of homeomorphisms of $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$.

**discrete:** orbits have no accumulation points

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discrete := orbits have no accumulation points

Examples:
Known facts

Planar discontinuous groups
- admit planar Cayley graphs
- are virtually surface groups
- admit one-relator presentations
- are effectively enumerable

see [Surfaces and Planar Discontinuous Groups, Zieschang, Vogt & Coldewey; Lecture Notes in Mathematics]
or [Lyndon & Schupp].
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Are Planar discontinuous groups exactly those having a planar Cayley graph? **NO!**
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Are Planar discontinuous groups exactly those having a planar Cayley graph? **NO!**

Definition: a group is **planar**, if it has a planar Cayley graph.
Charactisation of the finite planar groups

Definition: a group is planar, if it has a planar Cayley graph.

Theorem (Maschke 1886)

*Every finite planar group is a group of isometries of $S^2$.***

![Planar Cayley graph diagram](image-url)
The 1-ended planar groups

**Theorem ((classic) Macbeath, Wilkie, ...)**

Every 1-ended planar Cayley graph corresponds to a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$.

see [Surfaces and Planar Discontinuous Groups, Zieschang, Vogt & Coldewey; Lecture Notes in Mathematics]
Theorem (G ’12, Known?)

A group has a flat Cayley complex if and only if it has a accumulation-free Cayley graph.

(In which case it is a planar discontinuous group.)
Theorem (G ’12)

A Cayley graph admits an accumulation-free embedding if and only if it admits a facial presentation.
A facial presentation is a triple \((\mathcal{P} = \langle S \mid R \rangle, \sigma, \tau)\), where
- \(\sigma\) is a spin, i.e. a cyclic ordering on \(S\), and
- \(\tau : S \rightarrow \{T, F\}\) decides which generators are spin-preserving or spin-reversing, so that
- every relator is a facial word.
Facial presentations

**Theorem (G ’12)**

A Cayley graph admits an accumulation-free embedding if and only if it admits a facial presentation.

based on...

**Theorem (Whitney ’32)**

Let $G$ be a 3-connected plane graph. Then every automorphism of $G$ extends to a homeomorphism of the sphere.

... in other words, every automorphism of $G$ preserves facial paths.
A facial presentation is a triple \((\mathcal{P} = \langle S \mid R \rangle, \sigma, \tau)\), where
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Examples:

![Planar Cayley graphs with accumulation points](image)
Examples:
Examples:
Examples:

Planar Cayley graphs with accumulation points
Examples:
What we didn’t know

Open Problems:

Problem (Mohar)
How can you split a planar Cayley graph with \(\geq 1\) ends into simpler Cayley graphs?

Problem (Droms et. al.)
Is there an effective enumeration of the planar locally finite Cayley graphs?

Problem (Bonnington & Watkins (unpublished))
Does every planar 3-connected locally finite transitive graph have at least one face bounded by a cycle.

... and what about all the classical theory?
What we didn’t know

Open Problems:

Problem (Mohar)
How can you split a planar Cayley graph with > 1 ends into simpler Cayley graphs?

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... and what about all the classical theory?
Theorem (Dunwoody ’09)

If $\Gamma$ is a group and $G$ is a connected locally finite planar graph on which $\Gamma$ acts freely so that $\Gamma/G$ is finite, then $\Gamma$ or an index two subgroup of $\Gamma$ is the fundamental group of a graph of groups in which each vertex group is either a planar discontinuous group or a free product of finitely many cyclic groups and all edge groups are finite cyclic groups (possibly trivial).
Theorem (G ’10, to appear in Memoirs AMS)

Let $G$ be a planar cubic Cayley graph. Then $G$ is colour-isomorphic to precisely one element of the list.

Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.
Classification of the cubic planar Cayley graphs

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Presentations of planar Cayley graphs with accumulation points?

Recall that every accumulation-free Cayley graph has a facial presentation.

Recall that $G$ has a facial presentation $< = G$ has a flat Cayley complex.

How do we generalise?
Presentations of planar Cayley graphs with accumulation points?

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Presentations of planar Cayley graphs with accumulation points?

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Recall that $G$ has a facial presentation $\iff G$ has a flat Cayley complex

How do we generalise?
A presentation $\mathcal{P} = \langle S \mid R \rangle$ is planar, if it can be endowed with spin data $\sigma, \tau$ so that

- no two relator words cross
- every relator contains an even number of spin-reversing letters.

$\sigma$ is a spin, i.e. a cyclic ordering on $S$
$\tau : S \to \{ T, F \}$ decides which generators are spin-preserving or spin-reversing
Theorem (G & Hamann, ’14)

A Cayley graph $G$ is planar iff it admits a planar presentation.

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Cheat: this is a simplified definition, corresponding to the 3-connected case;
The general (2-connected) case is much harder to state and prove.
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- no two relator words cross
- every relator contains an even number of spin-reversing letters.

The proof of forward direction involves ramifications of Dunwoody cuts. The proof of the backward direction is elementary, and mainly graph-theoretic, but hard.
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Draw the corresponding tree $T_S := \text{Cay} \langle S \mid \emptyset \rangle$ accumulation-free in $\mathbb{R}^2$.
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Draw the corresponding tree $T_S := \text{Cay} \langle S \mid \emptyset \rangle$ accumulation-free in $\mathbb{R}^2$.

Let $D$ be a fundamental domain of $T_S$ w.r.t. $N(R)$. We can choose $D$ connected.
Proof ideas: backward direction

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Note that if $\partial D$ is nested, then $G$ is planar.
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It remains to prove that \( \partial D \) is nested.
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Assume $x, x'$ and $y, y'$ is a non-nested pair

Observe that in $\mathbb{R}^2$ every cycle has two sides; a non-nested pair would contradict this.
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We’ll reverse engineer: given a cycle $C$ in $G$, we want to define two ‘sides’ of $C$. 
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Two steps:
—Step 1: if $C$ comes from a relator $W$
—Step 2: for general $C$, write $C = \sum W_i$, and apply Step 1 to each $W_i$. 
Proof ideas: backward direction

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OK!
If $P = \langle S \mid R \rangle$ is a planar presentation, then its Cayley graph $G$ is planar.

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We are inclined to say ‘let the inside of $C$ be the union of insides of the $W_i$’... but we don’t know what’s inside/outside!
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Let’s still try:

\[
I_C := I_1 \triangle I_2 \triangle \ldots I_k
\]
\[
O_C := O_1 \triangle O_2 \triangle \ldots O_k
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Suppose it works;
Proof ideas: backward direction

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\[
I_C := I_1 \Delta O_2 \Delta \ldots I_k \\
O_C := O_1 \Delta I_2 \Delta \ldots O_k
\]

Suppose it works; then anything works!
Theorem (G & Hamann, ’14)

A Cayley graph $G$ is planar iff it admits a planar presentation.

Corollary

The planar groups are effectively enumerable.

(Answering Droms et. al.)
Generalise to include
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Does anybody know if the groups having a Cayley complex embeddable in $\mathbb{R}^3$ have been characterised?
Theorem (Stallings ’71)

Every group with \(>1\) ends can be written as an HNN-extension or an amalgamation product over a finite subgroup.

Can we generalise this to graphs?
Thank you!