Decay of correlations for flows with unbounded roof function, including the infinite horizon planar periodic Lorentz gas

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Abstract

We introduce a technique for studying nonuniformly hyperbolic flows with unbounded roof functions. In particular, we establish the decay of correlation rate $1/t$ for all infinite horizon planar periodic Lorentz gases. (Previously this result was proved only in some special cases.)

Our method is useful for analysing the statistical properties of other classes of flows with unbounded roof functions. For geometric Lorenz attractors (including the classical Lorenz attractor), consequences include a greatly simplified proof of the central limit theorem and almost sure invariance principle.

1 Introduction

For the planar periodic Lorentz gas with infinite horizon, it has been widely anticipated [7] that correlations in continuous time decay at the rate $1/t$. Melbourne [12] proved that this rate holds for sufficiently regular observables in certain special cases, including the classical one where the obstacles are circular disks. However, in general the results in [12] yield only the decay rate $O(1/t^{1-\epsilon})$ where $\epsilon > 0$ is arbitrarily small.

Roughly speaking, slowness of mixing of nonuniformly hyperbolic flows can arise from (i) slowness of mixing of certain Poincaré maps, and/or (ii) unboundedness of the return time for the flow to the Poincaré cross-section (from now on called the roof function). The method in [12] deals well with mechanism (i) and less well with mechanism (ii), and infinite horizon Lorentz gases fall under the second category. A related issue arises in work of [9] which gives the correct statistical limit laws for

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geometric Lorenz attractors, but with a convoluted proof due to unbounded roof functions.

In this paper, we present a different method for dealing with nonuniformly hyperbolic flows with unbounded roof function. For geometric Lorenz attractors, we obtain greatly simplified proofs. For Lorentz gases we obtain optimal results (again with less effort):

Theorem 1.1 *Suppose that ϕ_t is the flow corresponding to an infinite horizon planar periodic Lorentz gas (with convex scatters of nonvanishing curvature and C^3 boundaries). Let ν denote Liouville measure. Then for sufficiently regular observables v, w, the correlation function $\rho(t) = \int v\, w \circ \phi_t \, d\nu - \int v \, d\nu \int w \, d\nu$ satisfies $|\rho(t)| \leq C_{v,w} t^{-1}$ for all $t > 0$.*

Remark 1.2 As in Dolgopyat [5], *sufficiently regular* observables are those having sufficiently many uniformly Hölder derivatives in the flow direction.

The new method in this paper is *rearrangement* of certain Young tower models [20, 21, 3] so that the roof function becomes bounded without altering statistical properties of the flow. The change from unbounded to bounded roof function is compensated for by a deterioration in the tails of the Young tower. As discussed above, techniques for statistical properties of flows deal better with poor tails than unbounded roof functions, and so rearrangement of the tower is advantageous.

In particular, the situation for the general infinite horizon Lorentz gas is as follows. In [12], the correct result was obtained up to a logarithmic factor for the associated semiflow (quotiented along stable manifolds) and up to a factor t^ϵ for the flow. As verified in Section 4 of the current paper, a result of Szász & Varjú [18] can be used to remove logarithmic factors for the semiflow (hence yielding the optimal result for the semiflow). However, this method cannot deal with the factor t^ϵ for the flow. Our approach in this paper combined with [12] yields a logarithmic factor for both the semiflow and the flow. Applying [18] removes the logarithmic factors yielding Theorem 1.1.

The remainder of this paper is organised as follows. In Section 2, we recall some background material about Young towers and suspension flows. In Section 3, we introduce the idea of rearranging towers and consider applications to Lorentz gases and Lorenz attractors. In Section 4, we apply [18] to complete the proof of Theorem 1.1.

2 Suspension flows, towers, and induced roof functions

Suspension flows Given a map $f : X \to X$ with ergodic invariant probability measure μ and an integrable roof function $h : X \to \mathbb{R}^+$, we define the *suspension* $X^h = \{(x, u) \in X \times \mathbb{R} : 0 \leq u \leq h(x)\}/\sim$ where $(x, h(x)) \sim (fx, 0)$. The *suspension* *flow* $\phi_t : X^h \to X^h$ is given by $\phi_t(x,u) = (x, u+t)$ computed modulo $\sim$. An ergodic flow-invariant probability measure is given by $\mu^h = \mu \times \text{Lebesgue}/\bar{h}$ where $\bar{h} = \int_X h\,d\mu$.

Young towers Young [20, 21] introduced a class of nonuniformly hyperbolic maps $T : M \to M$ modelled by Young towers. In particular, there is a set $Y \subset M$ possessing an appropriate hyperbolic structure. Furthermore, there exists a countable measurable partition $\{Y_j\}$ of Y and a return time function $r : Y \to \mathbb{Z}^+$ constant on partition elements, such that the induced map $F = T^r : Y \to Y$ is smooth and uniformly hyperbolic with physical (SRB) measure μ_Y.

The corresponding *Young tower* Δ is defined to be $\Delta = \{(y,\ell) \in Y \times \mathbb{Z} : 0 \le \ell < r(y)\}$ with *tower map* $f : \Delta \to \Delta$ given by $f(y,\ell) = (y, \ell+1)$ for $\ell \le r(y) - 2$ and $f(y, r(y)-1) = (Fy, 0)$. The projection $\pi : \Delta \to M$, $\pi(y,\ell) = T^\ell y$, is a semiconjugacy. We note that r need not be the first return time to Y for $T : M \to M$ but is the first return for $f : \Delta \to \Delta$, so f is a extension of T. Provided $r \in L^1(\mu_Y)$, we obtain an f-invariant ergodic probability measure μ on Δ (and correspondingly $\pi^*\mu$ on M) given by $\mu = \mu_Y \times \text{counting}/\bar{r}$ where $\bar{r} = \int_Y r\,d\mu_Y$.

Given $\theta \in (0,1)$, we define the symbolic metric d_θ on Y by setting $d_\theta(x,y) = \theta^{s(x,y)}$ where $s(x,y)$ is the least integer $n \ge 0$ such that $F^n x$ and $F^n y$ lie in distinct partition elements Y_j. This extends to a metric on Δ with $d_\theta((x,\ell),(y,m)) = d_\theta(x,y)$ if $\ell = m$ and 1 otherwise.

Let $\Delta_{j,\ell} = Y_j \times \{\ell\}$ denote the corresponding partition elements of Δ. An observable $v : \Delta \to \mathbb{R}$ is said to be *locally Lipschitz* if $v|\Delta_{j,\ell}$ is Lipschitz wrt d_θ for each j,ℓ, with Lipschitz constant $|v1_{\Delta_{j,\ell}}|_\theta$ and Lipschitz norm $\|v1_{\Delta_{j,\ell}}\|_\theta = |v1_{\Delta_{j,\ell}}|_\infty + |v1_{\Delta_{j,\ell}}|_\theta$. If $\sup_{j,\ell} \|v1_{\Delta_{j,\ell}}\|_\theta < \infty$, then v is *uniformly locally Lipschitz*. Similarly, we define the notion of (uniformly) locally Lipschitz observable on Y.

Induced roof functions Suppose that $T : X \to X$ is a Poincaré map for a flow with roof function $h_0 : X \to \mathbb{R}$. Suppose further that $T : X \to X$ is modelled by a Young tower $f : \Delta \to \Delta$ with return time function $r : Y \to \mathbb{Z}^+$. Then the underlying flow is modelled by the suspension flow Δ^h where $h = h_0 \circ \pi$ and $\pi : \Delta \to X$ is projection. As above, we suppose that r and h are integrable and refer to ergodic invariant probability measures μ_Y, μ, μ^h on Y, Δ, and Δ^h respectively.

An alternative model is obtained from Δ^h by taking Y as the cross-section to the flow with *induced roof function* $H(y) = \sum_{\ell=0}^{r(y)-1} h(y,\ell)$. The flow on Y^H is identical to the flow on Δ^h, so the suspension flow Y^H is an extension of the underlying flow.

For clarity of exposition we introduce some more notation. Given $y \in Y$ let $h_1(y) = \max_{0 \le k < r(y)} h(T^k y)$, and $\ell_1(y) = \min\{0 \le k < r(y) : h(T^k y) = h_1(y)\}$, that is, the first iterate for which the maximum value h_1 is attained. Let $r(j) = r|_{Y_j}$ and set $Y_{j,i} = \{y \in Y_j : \ell_1(y) = i\}$, $0 \le i < r(j)$. This defines a refinement of the original partition $\{Y_j\}$ of Y. Correspondingly, $\Delta_{j,i,\ell} = Y_{j,i} \times \{\ell\}$ defines a refinement of the original partition $\{\Delta_{j,\ell}\}$ of Δ. Below we use the fact that f^ℓ defines a measure preserving isomorphism between $\Delta_{j,i,0}$ and $\Delta_{j,i,\ell}$ (equipped with the measure μ) for all $j \ge 1$, $0 \le i, \ell < r(j)$.

Proposition 2.1 *$\mu_Y(H > n) \le \mu_Y(r > k) + \bar{r}\mu(h > n/k)$ for all $k \ge 1$, $n \ge 1$.*

Proof If $r(y) \le k$ and $H(y) > n$, then $h \circ f^\ell > n/k$ for at least one $\ell \in \{0, \dots, k-1\}$, in particular, $h_1(y) = h(f^{\ell_1(y)}y) > n/k$. Restricting to $y \in Y_{j,i}$ for $i < r(j) \le k$, we have

$$\begin{aligned}\mu_Y(y \in Y_{j,i} : H(y) > n) &\le \mu_Y(y \in Y_{j,i} : h(f^i(y,0)) > n/k) \\ &= \bar{r}\mu(x \in \Delta_{j,i,0} : h(f^i x) > n/k) = \bar{r}\mu(x \in \Delta_{j,i,i} : h(x) > n/k).\end{aligned}$$

Summing up over $0 \le i < r(j) \le k$ we obtain

$$\mu_Y(y \in Y : r(y) \le k, H(y) > n) \le \bar{r}\mu(x \in \Delta : h(x) > n/k),$$

and so $\mu_Y(H > n) \le \mu_Y(r > k) + \mu_Y(r \le k, H > n) \le \mu_Y(r > k) + \bar{r}\mu(h > n/k)$ as required. ∎

For future reference, we list some special cases of Proposition 2.1.

Corollary 2.2 *(a) If $\mu_Y(r > n)$ decays exponentially and $\mu(h > n) = O(n^{-(\beta+1)})$, then the optimal choice is $k = C \log n$ for some large enough constant C, which implies $\mu_Y(H > n) = O((\log n)^{\beta+1} n^{-(\beta+1)})$.*

(b) If $\mu_Y(r > n) = O(e^{-cn})$ and $\mu(h > n) = O(e^{-dn})$, then the optimal choice is $k = \sqrt{dn/c}$, which implies $\mu_Y(H > n) = O(e^{-(cdn)^{\frac{1}{2}}})$.

(c) If $\mu_Y(r > n) = O(n^{-(\beta_1+1)})$ and $\mu(h > n) = O(n^{-(\beta_2+1)})$, then the optimal choice is $k = n^\alpha$ with $\alpha = \frac{\beta_2+1}{\beta_1+\beta_2+2}$, which implies $\mu_Y(H > n) = O(n^{-(\beta+1)})$ where $\beta = \frac{\beta_1\beta_2-1}{\beta_1+\beta_2+2}$. ∎

3 Rearrangements of towers

Suppose as in Section 2 that we have a flow (ϕ_t, Λ, ν) modelled by a suspension flow Δ^h over a tower Δ with return map $F : Y \to Y$, return time function $r : Y \to \mathbb{Z}^+$, roof function $h : \Delta \to \mathbb{R}^+$, and induced roof function $H(y) = \sum_{\ell=0}^{r(y)-1} h(y, \ell)$. Note that the extension flow on Δ^h is determined by (F, Y, μ_Y), r and h.

Now let (F, Y, μ_Y) remain unchanged but replace r and h by a modified return time function $\tilde{r}$ and a modified roof function $\tilde{h}$. Then we can define the corresponding modified Young tower $\tilde{\Delta}$ and modified suspension $\tilde{\Delta}^{\tilde{h}}$.

If the induced roof function is unchanged: $H(y) = \sum_{\ell=0}^{\tilde{r}(y)-1} \tilde{h}(y, \ell)$, then the suspension flow on $\tilde{\Delta}^{\tilde{h}}$ is identical to the suspension flow on Y^H and hence is identical to the suspension flow on Δ^h. Thus $\tilde{\Delta}^{\tilde{h}}$ is an alternative extension of the underlying flow. The aim is to carry out this modification in such a way that $\tilde{h}$ is bounded above and below.

Lemma 3.1 *Suppose that H is locally Lipschitz and bounded below. Suppose further that there exists $\delta > 0$ such that $\inf_{Y_j} H \ge \delta |H1_{Y_j}|_\theta$ for all j. Then we can choose $\tilde{r} : Y \to \mathbb{Z}^+$ and $\tilde{h} : \tilde{\Delta} \to \mathbb{R}^+$ so that*

(a) $\tilde{h}$ is uniformly locally Lipschitz and bounded below.

(b) $H(y) = \sum_{\ell=0}^{\tilde{r}(y)-1} \tilde{h}(y, \ell)$.

Proof Define $\tilde{r}1_{Y_j} = \big[\, \|H1_{Y_j}\|_\theta \,\big] + 1$ and $\tilde{h}(y, \ell) = H(y)/\tilde{r}(y)$ for all $y \in Y_j$, $0 \le \ell < \tilde{r}(y)$. Part (b) is immediate. Also, $\|\tilde{h}1_{\Delta_{j,\ell}}\|_\theta = \|H1_{Y_j}\|_\theta/(\tilde{r}1_{Y_j}) \le 1$ so $\tilde{h}$ is uniformly locally Lipschitz.

It remains to verify that $\tilde{h}$ is bounded below. Let $y \in Y_j$, $0 \le \ell < r(y)$. Since H is bounded below, there is a constant $\epsilon > 0$ such that

$$\tilde{h}(y, \ell) = \frac{H(y)}{\big[\, \|H1_{Y_j}\|_\theta \,\big] + 1} \ge \epsilon \frac{H(y)}{\|H1_{Y_j}\|_\theta}.$$

Moreover,

$$\begin{aligned} \frac{H(y)}{\|H1_{Y_j}\|_\theta} &\ge \frac{\inf_{Y_j} H}{|H1_{Y_j}|_\infty + |H1_{Y_j}|_\theta} = \frac{\inf_{Y_j} H}{\inf_{Y_j} H + \sup_{Y_j} H - \inf_{Y_j} H + |H1_{Y_j}|_\theta} \\ &\ge \frac{\inf_{Y_j} H}{\inf_{Y_j} H + 2|H1_{Y_j}|_\theta} \ge \frac{\inf_{Y_j} H}{\inf_{Y_j} H + 2\delta^{-1} \inf_{Y_j} H} = (1 + 2\delta^{-1})^{-1}. \end{aligned}$$

Hence $\tilde{h}(y, \ell) \ge \epsilon(1 + 2\delta^{-1})^{-1} > 0$ as required. ∎

Remark 3.2 Since $\tilde{h}$ is bounded above and below, it is immediate that $\mu_Y(H > n) \approx \mu_Y(\tilde{r} > n)$. By Proposition 2.1, we have an estimate for $\mu_Y(\tilde{r} > n)$.

Corollary 3.3 *Suppose that the hypotheses of Lemma 3.1 are satisfied and that $\mu_Y(H > n) = O((\log n)^\gamma n^{-(\beta+1)})$, where $\beta > 0$, $\gamma \ge 0$. Then typically correlations decay at the rate $O((\log t)^\gamma t^{-\beta})$ for sufficiently regular observables, and the word "typically" can be removed if there is a contact structure as in the case of a Lorentz gas.*

Proof By Remark 3.2, $\mu_Y(\tilde{r} > n) = O((\log n)^\gamma n^{-(\beta+1)})$. Since $\tilde{h}$ is bounded above and below, the result follows from [12, Theorem 2.6] and [12, Subsection 5.3]. ∎

Corollary 3.4 *Suppose that the hypotheses of Lemma 3.1 are satisfied and that $\mu_Y(H > n) = O(n^{-(\beta+1)})$. Let $v : \Lambda \to \mathbb{R}$ be a mean zero uniformly Hölder observable of the flow.*

(a) If $\beta > 0$, then we have the large deviations estimate

$$\nu\Big\{t^{-1}\Big|\int_0^t v\circ\phi_s\,ds\Big| > \epsilon\Big\} \le C(\epsilon, \|v\|)t^{-\beta} \quad \textit{for all } t > 0.$$

(b) If $\beta > 1$, then the (vector-valued) almost sure invariance principle holds. In particular, $t^{-1/2}\int_0^t v\circ\phi_s\,ds \to_d G$ as $t \to \infty$, where G is normally distributed with mean zero and variance $\sigma^2 \ge 0$.

(c) If $\beta > 1$, then qth moments converge for all $q < 2\beta$. That is, $\lim_{t\to\infty}\int_\Lambda |t^{-1/2}\int_0^t v\circ\phi_s\,ds|^q\,d\nu = E|G|^q$.

Proof Part (a) follows from [14] and [13]. Part (b) follows from [15]. Part (c) follows from [17]. ∎

Remark 3.5 As usual, statistical limit laws of the type discussed in Corollary 3.4 are valid much more generally than decay of correlations. In contrast to Corollary 3.3, it suffices that the observables are Hölder. Furthermore, no "typicality" condition is required on the flow.

Now we consider various situations where this method applies.

Example 3.6 (Lorenz attractors) If H is locally Lipschitz and bounded below, and $\sup_j |H1_{Y_j}|_\theta < \infty$, then Lemma 3.1 is applicable. This situation is likely to arise when the roof function h has a logarithmic singularity, since Young towers are built to achieve bounded distortion, counteracting the distortion of log-Jacobian. In doing so, a byproduct is that $|H1_{Y_j}|_\theta$ is bounded.

A specific example is the classical Lorenz attractor [10, 19] (or more generally, the class of geometric Lorenz attractors [1, 8]). It standard to construct the Young tower so that $\mu_Y(r > n)$ decays exponentially and $|H1_{Y_j}|_\theta$ is bounded. Since the singularity of the roof function is logarithmic, $\mu(h > n)$ also decays exponentially. By Corollary 2.2(b), $\mu_Y(H > n)$ decays at least stretched exponentially. By Corollary 3.4(b), we obtain the vector-valued almost sure invariance principle (and hence the functional CLT and functional LIL), recovering the result of [9]. Note that [9] required a complicated inducing scheme to take account of the joint unboundedness of r and h, whereas the argument here requires a much simpler inducing scheme independent of h.

The result described in this paper has been used in [17] to obtain convergence of moments of all orders (Corollary 3.4(c)). Further, by Corollary 3.4(a), we obtain superpolynomial large deviations estimates. This is exactly the kind of estimate required in [16] to obtain convergence of fast-slow skew product flows to stochastic limits when the fast dynamics is governed by the Lorenz attractor. (In fact, it is known [2] that large deviations decay exponentially for all continuous observables v, however, here we establish that the constant $C = C(\epsilon, \|v\|)$ depends on v only via $\|v\|$, which is crucial for [16].)

For completeness, we recall that little is known about decay of correlations for the Lorenz attractor. By [11], geometric Lorenz attractors are always mixing, and it is a consequence of [12] that typical geometric Lorenz attractors have superpolynomial decay of correlations (for sufficiently regular observables). It follows from [6, 12] that superpolynomial decay is open and dense for geometric Lorenz attractors, but the specific case of the Lorenz attractor is still not known. The results in this paper yield nothing new about decay of correlations for geometric Lorenz attractors.

Example 3.7 (Lorentz gases) A second class of examples is provided by situations where there is a constant $C > 0$ such that, for all j, $\sup_{Y_j} H \le C \inf_{Y_j} H$ and $|H1_{Y_j}|_\theta \le C \sup_{Y_j} H$.

For planar periodic Lorentz gases, partition elements Y_j are refined at each grazing collision occurring before the return to Y, so it is evident that the roof function $h : \Delta \to \mathbb{R}^+$ satisfies $\sup_{\Delta_{j,\ell}} h \le C \inf_{\Delta_{j,\ell}} h$ and $|h1_{\Delta_{j,\ell}}|_\theta \le C \sup_{\Delta_{j,\ell}} h$ for all j, ℓ with $0 \le \ell < r1_{Y_j}$. Hence the required estimates are satisfied by H.

Moreover, $\mu_Y(r > n)$ decays exponentially [3] and it is well-known that $\mu([h] = n) = O(n^{-3})$, so that $\mu(h > n) = O(n^{-2})$ in the infinite horizon case, hence it follows from Corollary 2.2(a) that $\mu_Y(H > n) = O((\log n)^2 n^{-2})$. By Corollary 3.3, we deduce that correlations for the Lorentz flow decay at the rate $O((\log t)^2 t^{-1})$. This recovers the result in [12] for the semiflow and is already an improvement for the flow.

As shown in Section 4, the logarithmic factor in the estimate for $\mu_Y(H > n)$ can be removed and by Corollary 3.3, we obtain the optimal decay rate $1/t$.

Example 3.8 We end with a hypothetical class of examples, where r and h both have polynomial tails, $\mu_Y(r > n) = O(n^{-(\beta_1+1)})$ and $\mu(h > n) = O(n^{-(\beta_2+1)})$. By Corollary 2.2(c), we obtain $\mu_Y(H > n) = O(n^{-(\beta+1)})$ where $\beta = \frac{\beta_1\beta_2 - 1}{\beta_1+\beta_2+2}$.

If $\beta_1\beta_2 > 1$ (so that $\beta > 0$), then typically correlations decay at the rate $O(t^{-\beta})$ by Corollary 3.3, and we obtain the large deviations estimate $O(t^{-\beta})$ by Corollary 3.4(a). If $\beta > 1$, then we can apply Corollary 3.4(b) and (c).

4 Decay rate for the infinite horizon Lorentz gas

In this section, we return to the infinite horizon planar periodic Lorentz gas and complete the proof of Theorem 1.1. As described in Example 3.7, it suffices by Corollary 3.3 to remove the logarithmic factor in the tail estimate for the induced roof function H.

Lemma 4.1 *For infinite horizon planar periodic Lorentz gases, $\mu_Y(H > n) = O(n^{-2})$.*

Recall that

$$\mu_Y(y \in Y : r(y) > n) = O(e^{-cn}) \text{ for some } c > 0. \tag{4.1}$$

$$\mu(x \in X : h(x) > n) = O(n^{-2}). \tag{4.2}$$

By Corollary 2.2(a), we have $\mu_Y(H > n) = O((\log n)^2 n^{-2})$. The crucial ingredient for proving Lemma 4.1 is due to Szász & Varjú [18].

Lemma 4.2 ([18, Lemma 16], [4, Lemma 5.1]) *There are constants $p, q > 0$ with the following property. For any b sufficiently large there is a constant $C = C(b) > 0$ such that*

$$\mu\big\{x \in X : [h(x)] = m \ \text{and}\ h(T^j x) > m^{1-q} \ \text{for some}\ j \in \{1, \dots, b \log m\}\big\}$$
$$\le Cm^{-p}\mu(x \in X : [h(x)] = m),$$

for all $m \ge 1$. ∎

For $b > 0$, define

$$Y_b(n) = \{y \in Y : r(y) \le b \log n \text{ and } \max_{0 \le \ell < r(y)} h(T^\ell y) \le \tfrac{1}{2} n \text{ and } H(y) \ge n\}.$$

Corollary 4.3 *For b sufficiently large, $\mu_Y(Y_b(n)) = o(n^{-2})$.*

Proof Let $y \in Y_b(n)$. We will use some of the notations, in particular $h_1(= h_1(y))$ and $\ell_1(= \ell_1(y))$ from Section 2. Further, define $h_2 = \max_{0 \le \ell < r(y), \ell \ne \ell_1}\{h(T^\ell y)\}$, and $m_i = [h_i]$, $i = 1, 2$. Then we have

$$\frac{n}{2b \log n} - 1 \le m_2 \le m_1 \le \frac{n}{2}. \tag{4.3}$$

(Proof: It is clear that $m_2 \le m_1 \le n/2$, and $n \le H \le h_1 + (b \log n - 1)h_2 \le n/2 + (b \log n)h_2$.)

The inequalities (4.3) imply that there are two free flights $h \circ T^\ell$ of comparable lengths h_1, h_2, during the iterates $\ell = 0, \dots, b \log n$. Choosing b sufficiently large, it follows from Lemma 4.2 that

$$\mu_Y(Y_b(n)) \le Cm_2^{-p}\mu(x \in X : [h(x)] = m_2).$$

By (4.2),

$$\mu_Y(Y_b(n)) \ll m_2^{-(2+p)} \ll (n^{-1} \log n)^{2+p} = o(n^{-2})$$

as required. ∎

Proof of Lemma 4.1 We continue to use the notation of Section 2. As in the proof of Proposition 2.1, we have

$$\mu_Y\{y \in Y : \max_{0 \le \ell < r(y)} h(T^\ell y) > n/2\} = \bar{r}\mu\{y \in Y : h(T^{\ell_1(y)} y) > n/2\}$$
$$= \bar{r}\mu\{T^{\ell_1(y)} y : h(T^{\ell_1(y)} y) > n/2\} \le \bar{r}\mu\{x \in \Delta : h(x) > n/2\},$$

and so $\mu_Y\{y \in Y : \max_{0 \le \ell < r(y)} h(T^\ell y) > n/2\} = O(n^{-2})$ by (4.2). Hence it follows from Corollary 4.3 that

$$\mu_Y\{y \in Y : r(y) \le b \log n \;\text{ and }\; H(y) \ge n\} = O(n^{-2}).$$

Finally, by (4.1), $\mu_Y(r > b \log n)) = O(n^{-bc}) = o(n^{-2})$ for any $b > 2/c$ and so $\mu_Y(H \ge n) = O(n^{-2})$ as required. ∎

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