# The Structure of Symmetric Attractors *

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**Abstract**

We consider discrete equivariant dynamical systems and obtain results about the structure of attractors for such systems. We show, for example, that the symmetry of an attractor cannot, in general, be an arbitrary subgroup of the group of symmetries. In addition, there are group-theoretic restrictions on the symmetry of connected components of a symmetric attractor.

Our methods are topological in nature and exploit connectedness properties of the ambient space. In particular, we prove a general lemma about connected components of the complement of preimage sets and how they are permuted by the mapping.

These methods do not themselves depend on equivariance. For example, we use them to prove that the presence of periodic points in the dynamics limits the number of connected components of an attractor, and, for one-dimensional mappings, to prove results on sensitive dependence and the density of periodic points.

## 1 Introduction

Our goal in this paper is to describe mathematical properties of symmetric attractors that have been observed in the numerical simulations of equivariant

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discrete dynamical systems in [3],[5],[9]. These properties include connectedness, sensitive dependence on initial conditions and, indeed, the actual symmetry of the attractor. While pursuing this goal we have used techniques that are equally valid for systems possessing no symmetry, and these techniques lead to interesting results for asymmetric systems as well.

In contrast with much of the literature on discrete dynamical systems we make no assumptions on the mapping other than continuity, but our definition of attractor includes the requirement that the basin be open (on the other hand, we do not require that there is a dense orbit). We will prove results of the following type.

(a) If an attractor contains a point of period $k$, then it has at most $k$ connected components (see Theorem 2.9). Symmetry often forces the origin to be fixed. So when an attractor for such a system contains the origin, it must be connected. (Such attractors also have full symmetry (see Proposition 4.8).) In addition, topologically mixing attractors are connected (see Theorem 2.6).

(b) For continuous maps on the line, attractors (and, more generally, $\omega$-limit sets) are contained in the closure of the set of periodic points (see Theorem 3.1). This 'closing lemma' holds without the usual assumptions of genericity and differentiability. In addition, nonminimal attractors of continuous maps on the line display sensitive dependence on initial conditions (see Theorem 3.2).

(c) There are representation-theoretic restrictions on the symmetry of attractors (see Theorem 4.10).

(d) There are group theoretic restrictions on the symmetry of connected components of symmetric attractors (see Theorem 4.6).

(e) Mappings of the plane having $\mathbf{D}_m$-symmetry (where $\mathbf{D}_m$ is the *dihedral* group of symmetries of the regular $m$-sided polygon in the plane) *cannot* have attractors with symmetry $\mathbf{D}_k$ where $2 < k < m$ (see Theorem 5.2). Also, if a $\mathbf{D}_m$-equivariant mapping has an attractor with symmetry $\mathbf{D}_m$, then each of its components must have $\mathbf{D}_m$-symmetry (see Theorem 5.7).

We will also show that the proofs of all of these theorems rely in part on a single topological lemma (see Lemma 2.1) which uses connected components of the complement of a preimage set to cover invariant sets. This and related results are described in Section 2. Results concerning one-dimensional mappings are given in Section 3. In Section 4 we discuss how symmetry is brought into the study of attractors of mappings and prove some general results; our main theorems about symmetric attractors for mappings with dihedral symmetry are proved in Section 5.

It was observed in [3] using computer experimentation that symmetry-increasing bifurcations seem to be the rule in the discrete dynamics of maps in the plane with dihedral $\mathbf{D}_m$ symmetry. These bifurcations occur through the collision of conjugate attractors. In [4] we use the results about preimage sets, in particular Theorems 3.5 and 5.8, as the basis for a numerical algorithm for the computation of certain types of symmetry increasing bifurcations.

An important observation in the theory of equivariant steady-state bifurcations states that there are restrictions on the possible symmetry of bifurcating equilibria. Our results in Sections 4 and 5 indicate that a similar remark holds for general attractors.

# 2 Topological dynamics using preimage sets

In this section we introduce the main topological results that we use. The main observation (see Lemma 2.1) states that $\omega$-limit sets are either contained in the closure of a preimage set or are covered by a finite number of connected components of the complement of that preimage set. This result along with the definition of preimage sets is presented in Subsection (a). This observation has a number of applications which appear throughout this paper. In Subsections (b) and (c) we show how Lemma 2.1 can be used to prove the connectedness theorems promised in the Introduction and a general result concerning sensitive dependence.

## (a) Preimage sets

Let $X$ be a finite-dimensional Euclidean space and suppose that $f : X \to X$ is a continuous mapping. (In fact, we need only assume that $X$ is a locally connected metric space for our results to be valid.) Recall that if $x \in X$, the *$\omega$-limit set* of $x$ is defined to be the set $\omega(x)$ consisting of points $y \in X$ for which there is an increasing sequence $\{n_k\}$ of positive integers such that $f^{n_k}(x) \to y$. Basic properties of $\omega(x)$ include

1. $\omega(x)$ is closed,

2. $\omega(f(x)) = \omega(x)$,

3. $f(\omega(x)) \subset \omega(x)$ with equality if $\omega(x)$ is compact.

We call a set $A$ *topologically transitive* if $A = \omega(x)$ for some $x \in A$. Equivalently, $A$ has a dense orbit. The set $A$ is *topologically mixing* if for any open subsets $U, V \in A$ there exists a positive integer $N$ such that $f^{-n}(U) \cap V \neq \emptyset$ for all $n \geq N$. If $A$ is topologically mixing, then $A$ is topologically transitive under $f^k$ for all $k \geq 1$.

Let $S$ be a subset of $X$. We define

$$\mathcal{P}_S(f) = \bigcup_{n=0}^{\infty} (f^n)^{-1}(S)$$

to be the set consisting of $S$ and all of the preimages of $S$ under $f$. Usually, when the context is clear we write $\mathcal{P}_S(f) = \mathcal{P}_S$. Observe that $f^{-1}(\mathcal{P}_S) \subset \mathcal{P}_S$. It follows that $f$ induces a mapping

$$f : X - \mathcal{P}_S \to X - \mathcal{P}_S.$$

Since $f$ is continuous, connected components of $X - \mathcal{P}_S$ are mapped into connected components.

The following topological lemma is used repeatedly throughout the paper.

**Lemma 2.1** *Let $x \in X$ and let $S$ be a subset of $X$. Then either $\omega(x) \subset \overline{\mathcal{P}_S}$ or the following are valid.*

*(a) $\omega(x) - \mathcal{P}_S$ is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $X - \mathcal{P}_S$.*

*(b) These components can be ordered so that $f(C_i) \subset C_{i+1 \bmod r}$.*

*(c) $\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$.*

**Proof:** We assume that $\omega(x) \not\subset \overline{\mathcal{P}_S}$. Choose $y \in \omega(x) - \overline{\mathcal{P}_S}$ and $\epsilon > 0$ such that $B_\epsilon(y) \subset X - \overline{\mathcal{P}_S}$ where $B_\epsilon(y)$ is the open ball of radius $\epsilon$ centered at $y$. Since $B_\epsilon(y)$ is connected it lies inside a connected component $C_0$ of $X - \mathcal{P}_S$. Since $y \in \omega(x)$, there exist a smallest integer $k \geq 0$ such that $f^k(x) \in B_\epsilon(y)$.

Also, there is a smallest integer $\ell > k$ such that $f^\ell(x) \in B_\varepsilon(y)$ If $r = \ell - k$, then $f^r(B_\epsilon(y)) \cap B_\epsilon(y) \neq \emptyset$ and it follows by continuity that $f^r(C_0) \subset C_0$.

Write $x' = f^k(x)$ and let $C_i$ be the connected component of $X - \mathcal{P}_S$ containing $f^i(x')$ for $i = 0, \ldots, r-1$. It follows by continuity that $f(C_i) \subset C_{i+1 \bmod r}$, and so

$$f^i(x') \in C_0 \cup \cdots \cup C_{r-1}, \quad i \geq 0.$$

Hence,

$$\omega(x) = \omega(x') \subset \overline{C_0 \cup \cdots \cup C_{r-1}} = \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}.$$

In addition, since there are only finitely many connected components, they have no limit points lying in another connected component of $X - \mathcal{P}_S$. Hence

$$\omega(x) \subset C_0 \cup \cdots \cup C_{r-1} \cup \mathcal{P}_S$$

from which (a) follows. ∎

An $f$-invariant set $A$ is *stable* if for any neighborhood $U$ of $A$ there is a smaller neighborhood $V$ such that $f^n(V) \subset U$ for all $n \geq 0$. $A$ is called *asymptotically* stable if it is stable and upon iteration all points in $V$ converge to $A$.

**Definition 2.2** An *attractor* is an asymptotically stable $\omega$-limit set.

We note that there are more general definitions of attractor where the open basin requirement is relaxed and many of our results hold under these more general definitions.

**Proposition 2.3** *Let $S$ and $A$ be closed sets and suppose that $A$ is a stable $f$-invariant set. Then the following statements are equivalent,*

*(a)* $A \cap S = \emptyset$,

*(b)* $A \cap \overline{\mathcal{P}_S} = \emptyset$.

**Proof:** Since $S \subset \overline{\mathcal{P}_S}$, it is clear that (b) implies (a). Now suppose that (a) is valid. Since $A$ and $S$ are closed, there is an open set $U$ containing $A$ with $U \cap S = \emptyset$. Let $V$ be a smaller neighborhood of $A$ such that $f^n(V) \subset U$ for $n = 0, 1, 2, \ldots$. It follows that $V \cap \mathcal{P}_S = \emptyset$ and so $A \cap \overline{\mathcal{P}_S} = \emptyset$ as required. ∎

**Corollary 2.4** *Suppose that $A$ is an attractor, $S$ is closed and $A \cap S = \emptyset$. Then*

$$A \subset C_0 \cup \cdots \cup C_{r-1}.$$

**Lemma 2.5** *Let $M$ and $S$ be closed subsets. Assume:*

*(a) $A$ is an attractor and $A \cap S = \emptyset$.*

*(b) $C$ is a connected component of $X - \mathcal{P}_S$ and $A \cap C \neq \emptyset$.*

*(c) $M$ is $f$-invariant and $A \cap M \neq \emptyset$.*

*Then $M$ intersects $C$.*

**Proof:** By (b) $A$ intersects $C$; hence $C$ must be one of the connected components guaranteed by Lemma 2.1. These connected components are permuted cyclically by $f$; by (a) and Corollary 2.4 they cover the whole of $A$. By (c), $M$ intersects at least one connected component; invariance implies that $M$ intersects all the connected components. In particular, $M$ intersects $C$. ∎

## (b) Connectedness results

We can now prove two rather strong connectedness results for attractors.

**Theorem 2.6** *Let $A$ be a topologically mixing attractor for a continuous mapping $f$. Then $A$ is connected.*

**Proof:** Suppose that $A$ is not connected. Then we may write $A$ as the disjoint union of two closed sets $A_1$ and $A_2$. Let $S$ be a closed subset of $X$ such that $A_1$ and $A_2$ lie inside distinct connected components of $X - S$. By Corollary 2.4, $A \subset C_0 \cup \cdots C_{r-1}$, where $C_0, \ldots, C_{r-1}$ are connected components of $X - \mathcal{P}_S$ and are permuted cyclically by $f$. Also $r \geq 2$ by construction. In particular $A \cap C_0$ and $A \cap C_1$ are invariant under $f^r$, so that $f^r$ has no attractor containing both of these subsets of $A$. Hence $A$ is not an attractor for $f^r$ which contradicts the assumption that $A$ is topologically mixing. ∎

**Corollary 2.7** *If $A$ is a topologically mixing attractor topologically conjugate to a subshift then $A$ is a fixed point.*

**Proof:** It is well-known that spaces topologically conjugate to a subshift are totally disconnected, see for example Proposition 11.9 in Mañé [10]. Combining this with Theorem 2.6, we see that $A$ is both connected and totally disconnected. ∎

**Remark 2.8** The standard examples of nontrivial topologically mixing spaces conjugate to subshifts are not attracting by any definition. On the other hand, it is not difficult to construct examples of nontrivial topologically mixing attractors that are semiconjugate to subshifts and even conjugate to subshifts off a negligible subset.

**Theorem 2.9** *Let $A$ be an attractor for a continuous mapping $f$. Suppose $A$ contains a periodic point of period $k$. Then $A$ has at most $k$ connected components.*

**Proof:** Suppose we can write $A$ as a disjoint union of closed sets

$$A = A_1 \cup \cdots \cup A_{k+1}.$$

Choose $S$ to be a closed set that separates the $A_j$s and such that $S \cap A = \emptyset$. By Corollary 2.4

$$A \subset C_0 \cup \cdots \cup C_{r-1}$$

where $C_0, \ldots, C_{r-1}$ are connected components of $X - \mathcal{P}_S$. Since $S$ separates the $A_j$s at most one $A_j$ can intersect a given $C_i$. It follows that

$$k + 1 \leq r.$$

Now we let $M$ be the periodic trajectory consisting of $k$ points that is assumed to exist in $A$. Since $M$ is $f$-invariant, we may apply Lemma 2.5 to conclude that $M \cap C_j \neq \emptyset$ for each $C_j$. If follows that

$$k \geq r.$$

This contradicts the assumption that we can write $A$ as a disjoint union of $k+1$ closed subsets. ∎

There are a number of consequences of Theorem 2.9 — we mention two.

**Corollary 2.10** *(a) If an attractor contains a fixed point, then it is connected.*

*(b) If an attractor contains a periodic point, then it cannot be a Cantor set.*

## (c) Sensitive dependence

We begin by recalling the notion of sensitive dependence on initial condition. The following definition is essentially due to Guckenheimer [6] (the only difference being that we speak of sensitive dependence of $A$ rather than of $f$).

**Definition 2.11** An invariant set $A$ has *sensitive dependence* if there is a set $Y \supset A$ of positive (Lebesgue) measure and an $\varepsilon > 0$ such that for every $x \in Y$ and every $\delta > 0$ there is a point $y$ that is $\delta$-close to $x$ and an integer $m > 0$ such that

$$|f^m(x) - f^m(y)| > \varepsilon.$$

We also introduce a weaker notion than sensitive dependence that is equivalent to Definition 2.11 for invariant sets of positive measure.

**Definition 2.12** An invariant set $A$ has *weak dependence* if there is an $\varepsilon > 0$ such that for every $x \in A$ and every $\delta > 0$ there is a point $y$ that is $\delta$-close to $x$ and an integer $m > 0$ such that

$$|f^m(x) - f^m(y)| > \varepsilon.$$

**Proposition 2.13** *Let $f : X \to X$ be continuous, and $x \in X$. Suppose that $S \subset X$ satisfies the following conditions:*

*(a) $f(S) \subset S$,*

*(b) $\omega(x) \subset \overline{\mathcal{P}_S}$, and*

*(c) $\omega(x) \not\subset \overline{S}$.*

*Then $\omega(x)$ has weak dependence. If, in addition, $\omega(x)$ has positive measure, then $\omega(x)$ has sensitive dependence.*

**Proof:** Using (c) choose a point $p$ in $\omega(x) - \overline{S}$. Let $d$ be the distance from $p$ to $\overline{S}$ and choose $\varepsilon$ to be less than $d$. Let $y \in \omega(x)$ and let $\delta > 0$. In the $\delta$-neighborhood of $y$ there exists a point $x' = f^k(x)$, some $k$, and a point $z$ that iterates to $S$ (by (b)) — say in $\ell$ iterates. Since $p \in \omega(x)$ there exists an $m \geq \ell$ such that $f^m(x')$ is $(d - \varepsilon)$-close to $p$. By (a) $f^m(z) \in S$ and hence is distance at least $\varepsilon$ away from $f^m(x')$. ∎

## 3 One-dimensional maps

In this section we prove results about maps on the line which illustrate the methods of the previous section, in particular Lemma 2.1 and Proposition 2.13. Our main results deal with the two issues of density of periodic orbits and sensitive dependence for $\omega$-limit sets of these one-dimensional maps. Many authors have considered these issues but usually when assuming stronger hypotheses on the mappings. We focus on results that can be obtained for a general class of mappings using the topological methods of Section 2. We note that the results in this section are *not* required in subsequent sections.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let Per(f) denote the periodic points of $f$. Then $\mathcal{P}_{\mathrm{Per}(f)}$ is the set of eventually periodic points of $f$. An $\omega$-limit set is *minimal* if it contains no proper closed invariant subsets.

We now state our two main theorems.

**Theorem 3.1** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map, and $x \in \mathbb{R}$. Then*

*(a)* $\omega(x) \subset \overline{\mathcal{P}_{\mathrm{Per}(f)}}$.

*(b) The limit points of $\omega(x)$ lie in $\overline{\mathrm{Per}(f)}$. In particular, if $\omega(x)$ is topologically transitive, then $\omega(x) \subset \overline{\mathrm{Per}(f)}$.*

**Theorem 3.2** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and $x \in \mathbb{R}$. If $\omega(x)$ is compact and is not minimal then $\omega(x)$ has weak dependence. If, in addition, $\omega(x)$ has positive measure, then $\omega(x)$ has sensitive dependence.*

**Corollary 3.3** *Suppose that $\omega(x) \subset \mathbb{R}$ consists of a finite union of closed intervals. Then $\omega(x)$ is topologically transitive, periodic points are dense in $\omega(x)$ and there is sensitive dependence.*

**Proof:** Topological transitivity is clear since $\omega(x)$ has interior in $\mathbb{R}$. Hence by Theorem 3.1(b), $\omega(x) \subset \overline{\mathrm{Per}(f)}$. Again using the fact that $\omega(x)$ has interior, the dense set of periodic orbits can be taken to lie in $\omega(x)$.

Suppose that $\omega(x)$ consists of $k$ intervals. Then $f^k$ maps a single interval into itself and has a fixed point. This implies that $\omega(x)$ contains a period $k$ point and is not minimal. In addition, $\omega(x)$ has positive measure and sensitive dependence follows from Theorem 3.2. ∎

**Remark 3.4** (a) Theorem 3.2 holds also for continuous circle maps. In addition, Theorem 3.1 is valid for mappings of the circle provided the set of periodic points is nonempty. The proofs are completely analogous to those for mappings on the line.

(b) Theorem 3.1 is reminiscent of Pugh's closing lemma [11], [12]. Note however that no genericity or differentiability assumptions on $f$ are required, in contrast with mappings of the circle or higher-dimensional manifolds. However, even in $\mathbb{R}$, it is true only generically that the *nonwandering set* $\Omega(f)$ is equal to $\overline{\mathrm{Per}(f)}$, see Young [13]. Part (a) of Theorem 3.1 was proved previously by Block [2] using similar methods. We present a proof here to focus on the way Lemma 2.1 is used in the proof.

(c) If $A$ is an attractor in the sense of Definition 2.2 then Theorem 3.1 implies that $A$ must contain a periodic point. But then by Theorem 2.9 $A$ has finitely many connected components. In particular, an attractor for a mapping of the line must be a periodic orbit or a finite union of closed intervals, and cannot be a Cantor set. Of course there are more general definitions of attractor that do not exclude Cantor sets.

(d) There are examples of minimal Cantor sets both with zero and positive measure for continuous mappings on an interval. Moreover these Cantor sets attract almost every point in the interval and do not display weak dependence. See, for example, [7] and [8].

We end with a result that is useful in the computation of symmetry-increasing bifurcations, see [4]. Let $p$ be a periodic point for $f$ and let $S$ denote the corresponding periodic orbit. We call $p$ (or $S$) *unstable* if there exists a neighborhood $U$ of $S$ such that $\mathrm{dist}(f(x), S) \geq \mathrm{dist}(x, S)$ for all $x \in U$. Note that if $p$ has period $k$, $f$ is differentiable and $|(f^k)'(p)| > 1$, then $p$ is unstable.

**Theorem 3.5** *Suppose that $x \in \mathbb{R}$ and that the orbit of $x$ under $f$ is bounded. If $p \in A$ is an unstable periodic point, then*

$$\omega(x) \subset \overline{\mathcal{P}_p}.$$

**Corollary 3.6** *Suppose that an odd continuous mapping $f : \mathbb{R} \to \mathbb{R}$ has a compact attractor $A$ containing the unstable fixed point* 0. *Then* *(a)* $A \subset \overline{\mathcal{P}_0}$.

*(b)* *A is connected.*

*(c)* *A has sensitive dependence.*

*(d)* *Periodic points are dense in A.*

**Proof:** Statement (a) follows from Theorem 3.5, (b) follows from Theorem 2.9, (c) follows from Theorem 3.2 and (d) follows from Theorem 3.1 ∎

We now turn to the proofs of Theorems 3.1, 3.2 and 3.5.

**Proof of Theorem 3.1:** (a) Setting $S = \mathrm{Per}(f)$ in Lemma 2.1 implies that either $\omega(x) \subset \overline{\mathcal{P}_{\mathrm{Per}(f)}}$ or $\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$ where the $C_j$ are connected components of $\mathbb{R} - \mathcal{P}_{\mathrm{Per}(f)}$ and are cyclically permuted by $f$. We show that the second alternative implies that $\omega(x) \subset \mathcal{P}_{\mathrm{Per}(f)}$ which proves part (a).

Let $\omega_j = \omega(x) \cap \overline{C_j}$. Then $\omega_j$ is an $\omega$-limit set for $f^r$. By construction, $f$ has no periodic points in $C_j$. Therefore $f^r$ has no fixed points in $C_j$ and either $f^r(x) > x$ for each $x \in C_j$ or $f^r(x) < x$ for each $x \in C_j$.

Since $\omega_j$ is an $\omega$-limit set for $f^r$, $\omega_j$ lies in the boundary of $C_j$. It follows that $\omega(x)$ is finite (possibly empty). If $y \in \omega(x)$ then the orbit of $y$ under $f$ consists of finitely many points and so $y \in \mathcal{P}_{\mathrm{Per}(f)}$ as required.
(b) Let $(a, b)$ be an open interval containing $y$. We show that $(a, b) \cap \mathrm{Per}(f) \neq \emptyset$ thus proving the first statement in part (b). By part (a), there is an eventually periodic point $z \in (a, b)$. Suppose that $z$ iterates to a periodic orbit of period $r$. Let $g = f^r$. Then $z$ iterates under $g$ to a fixed point $p$, $g^k(z) = p$ for some $k$. If $p = z$ then we are finished, so we may assume that $p > z$.

Let $\omega_j(x)$ denote the $\omega$-limit set of $f^j(x)$ under $g$. Then $y$ is a limit point of $\omega_j(x)$ for at least one $j$, $1 \leq j \leq r$. Hence, there are points $y_1 < y_2$ contained in $\omega_j(x) \cap (a, b)$. Let $\varepsilon = (y_2 - y_1)/2$. Since $g$-transitive points are dense in $\omega_j(x)$, there is a transitive point $x' \in (a, b)$ within distance $\varepsilon$ of $y_2$. Now $g^k(x')$ is also transitive and so $g^\ell(x')$ is within distance $\varepsilon$ of $y_1$ for some $\ell \geq k$. Therefore, we have

$$g^\ell(x') < x', \quad g^\ell(z) = p > z.$$

It follows from the intermediate value theorem that $g^\ell$ has a fixed point between $x'$ and $z$ and hence in $(a, b)$. This is the required periodic point for $f$.

Suppose now that $\omega(x)$ is topologically transitive. To complete the proof of part (b) it is sufficient to show that $\omega(x)$ is either a periodic orbit or a perfect set. That is, either every point is periodic or every point is a limit point. Since $\omega(x)$ is topologically transitive we can assume without loss of generality that $x \in \omega(x)$. If $f^{n_1}(x) = f^{n_2}(x)$ for positive integers $n_1 \neq n_2$, then $\omega(x)$ is a periodic orbit. Otherwise the orbit $\{f^n(x); n = 1, 2, \dots\}$ consists of distinct points. Let $y \in \omega(x)$. Then there is an increasing sequence $n_k$ such that $f^{n_k}(x) \to y$. The points $f^{n_k}(x)$ lie in $\omega(x)$ and are distinct so that $y$ is a limit point of $\omega(x)$ as required. ∎

Next we turn to the proof of Theorem 3.2. We require two preliminary results. The first of these, Lemma 3.7, is used also in the proof of Theorem 3.5. The second result, Lemma 3.8, contains the technical part of the proof of Theorem 3.2.

**Lemma 3.7** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $x \in \mathbb{R}$ and $\omega(x)$ is compact. If $z \in \omega(x) - \mathrm{Per}(f)$, then $\omega(x) \subset \overline{\mathcal{P}_z}$.*

**Proof:** Since $f(\omega(x)) = \omega(x)$, there exists a sequence $z_n \in \omega(x)$, $n \geq 0$ such that $f(z_n) = z_{n-1}$. We have assumed that $z$ is not periodic, so the points in the sequence are distinct. Hence $\omega(x) \cap \mathcal{P}_z$ is an infinite set. Suppose that $\omega(x) \not\subset \overline{\mathcal{P}_z}$. By Lemma 2.1, $\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$ where each $C_j$ is a connected component of $\mathbb{R} - \mathcal{P}_z$. But then $\omega(x) \cap \mathcal{P}_z$ consists at most of the union of the end points of the intervals $\overline{C_j}$ and hence has at most $2r$ points. This is a contradiction. ∎

**Lemma 3.8** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x \in \mathbb{R}$. Suppose further that $\omega(x) \subset \overline{\mathrm{Per}(f)}$ and $\omega(x) \cap \mathrm{Per}(f) \neq \emptyset$. Then either $\omega(x)$ has sensitive dependence or $\omega(x)$ is a periodic orbit.*

**Proof:** The proof divides into two cases depending on whether or not the following property (†) is satisfied.

(†) For any positive integer $\ell$, there is a periodic orbit $S \subset \mathbb{R}$ such that at least $\ell$ connected components of $\mathbb{R} - S$ intersect $\omega(x)$.

Suppose first that property (†) is valid and let $y$ be a periodic point in $\omega(x)$ with period $k$. We may choose a periodic orbit $S \subset \mathbb{R}$ so that at least $k + 1$ components of $\mathbb{R} - S$ intersect $\omega(x)$. Suppose that $\omega(x) \not\subset \overline{\mathcal{P}_S}$.

Then there are at least $k+1$, but by Lemma 2.1 finitely many, components of $\mathbb{R} - \mathcal{P}_S$ that cover $\omega(x)$ and that are cyclically permuted by $f$. One of these components contains $y$ and there is a contradiction. Hence $\omega(x) \subset \overline{\mathcal{P}_S}$. Either $\omega(x) = S$ in which case $\omega(x)$ is a periodic orbit or $S$ satisfies the hypotheses of Proposition 2.13, so that $\omega(x)$ has weak dependence.

It remains to consider the case when property (†) is not valid. Since $\omega(x) \subset \overline{\mathrm{Per}(f)}$ it follows that $\omega(x) \subset \overline{\mathrm{Fix}(f^k)}$ for some $k$. By continuity, $\omega(x) \subset \mathrm{Fix}(f^k)$ and consists entirely of periodic orbits. We show that if $\omega(x)$ contains more than one periodic orbit, then there is sensitive dependence.

It is sufficient to show that $\omega(x)$ has sensitive dependence under $g = f^k$. Suppose that $P_1$ and $P_2$ are distinct periodic orbits in $\omega(x)$ and define

$$\varepsilon = \frac{1}{4} \min_{p_1 \in P_1, p_2 \in P_2} |p_1 - p_2|.$$

Suppose that $y \in \omega(x)$ and $\delta > 0$. Choose an iterate $z = f^\ell(x)$ that is $\delta$-close to $y$. There are integers $m_1, m_2 \geq 0$ such that $g^{m_j}(z)$ is within distance $\varepsilon$ of points $p_j \in P_j$, $j = 1, 2$. On the other hand, $y$ is fixed by $g$ so that $g^{m_j}(y) = y$. Hence $|g^{m_j}(y) - g^{m_j}(z)| = |y - g^{m_j}(y)| > \varepsilon$ for $j = 1$ or $j = 2$, thus proving sensitive dependence. ∎

**Proof of Theorem 3.2:** We shall prove weak dependence. Strong dependence then follows from the additional assumption that $\omega(x)$ has positive measure. The strategy of our proof is to reduce, under the assumption that $\omega(x)$ does not have weak dependence, to the situation where the hypotheses of Lemma 3.8 are valid in which case the theorem follows.

The proof varies depending on whether or not $\omega(x)$ is topologically transitive. Suppose first that $\omega(x)$ is topologically transitive. Since $\omega(x)$ is not minimal, there is an invariant closed subset $S$ contained properly in $\omega(x)$. If $z \in S$ is not periodic, then by Lemma 3.7, $\omega(x) \subset \overline{\mathcal{P}_z} \subset \overline{\mathcal{P}_S}$ and weak dependence follows from Proposition 2.13. So we may assume that $S$, and hence $\omega(x)$, contains periodic points. In addition, $\omega(x) \subset \overline{\mathrm{Per}(f)}$ by Theorem 3.1(b) so that the hypotheses of Lemma 3.8 are satisfied as required.

Next suppose that $\omega(x)$ is not topologically transitive. We claim that either $\omega(x) \subset \mathrm{Per}(f)$ or $\omega(x)$ has sensitive dependence. If the first possibility holds, the hypotheses of Lemma 3.8 are satisfied.

We prove the claim by assuming that $\omega(x) \not\subset \mathrm{Per}(f)$ and showing that $\omega(x)$ has sensitive dependence. Choose $z \in \omega(x) - \mathrm{Per}(f)$ and let $S = \overline{\{f^n(z);\, n \geq 0\}}$. By Lemma 3.7, $\omega(x) \subset \overline{\mathcal{P}_z} \subset \overline{\mathcal{P}_S}$. Note that if $S$ is a proper subset of $\omega(x)$ then we are finished by Proposition 2.13.

Choose $y \in \omega(x)$ such that $f(y) = z$. By the above discussion we may assume that $y \in S$. So either $z$ iterates onto $y$ under $f$ or $z \in \omega(z)$. Since $z \notin \mathrm{Per}(f)$ it is the second possibility that is valid: $z \in \omega(z)$. Hence we may again apply Lemma 3.7 to deduce that $\omega(x) \subset \overline{\mathcal{P}_{\omega(z)}}$. Since $\omega(x)$ is not topologically transitive, $\omega(z)$ is a proper subset of $\omega(x)$ and sensitive dependence follows from Proposition 2.13 ∎

**Proof of Theorem 3.5:** Let $S$ be the periodic orbit corresponding to $p$. We show that there are points in $\omega(x)$ that iterate to $S$ and do not lie in $S$. Such a point $q$ is not periodic and hence $\omega(x) \subset \overline{\mathcal{P}_q} \subset \overline{\mathcal{P}_p}$ by Lemma 3.7.

Since the orbit of $x$ under $f$ is bounded, $\omega(x)$ is compact. Hence, $\omega(x)$ lies in the interior of a closed interval $I$. Let $U$ be the neighborhood of $S$ in the definition of unstable periodic point. We may assume that $U \subset \mathbb{R}$. Let $V \subset U$ be a smaller neighborhood of $S$ and let $W \subset \mathbb{R}$ be a neighborhood of $\omega(x)$.

We claim that there must be a point $q \in W - U$ such that $f(q) \in V$. Choose a sequence of neighborhoods $V_j$ converging to $S$ and a sequence of neighborhoods $W_j$ converging to $\omega(x)$. By the claim we obtain a sequence of points $q_j \in W_j - U$, such that $f(q_j) \in V_j$. The sequence $q_j$ lies in $I - U$ which is compact, so passing to a convergent subsequence, we have that $q_j \to q$. Moreover it follows from the construction of the sequence that $q \in \omega(x) - S$ and $f(q) \in S$ as required.

It remains to verify the claim. First observe that there is an integer $K \geq 0$ such that $f^k(x) \in W$ for all $k \geq K$. For otherwise $f^j(x) \in \mathbb{R} - W$ for infinitely many integers $j$, and since this set of points is bounded there would ba an $\omega$ limit point in $\mathbb{R} - W$. This contradicts the fact that $\omega(x) \subset W$.

So without loss of generality, we may assume that $f^j(x) \in W$ for all $j \geq 0$. Since $S \subset \omega(x)$, $x$ eventually iterates into $V$. Let $k$ be the least integer satisfying $f^k(x) \in V$. Then $q = f^{k-1}(x) \in W - U$. For if $q \in U$, then $q \in U - V$ and must iterate out of $U$ before entering $V$. ∎

# 4 Symmetry of an attractor

Suppose that $\Gamma$ is a finite group acting linearly on $X$, and that $f$ is $\Gamma$-equivariant, that is

$$f(\gamma x) = \gamma f(x).$$

If $x \in X$, we define the *isotropy subgroup* of $x$ to be the subgroup

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma x = x\}.$$

If $\Sigma$ is a subgroup, then it has a *fixed-point subspace*

$$\mathrm{Fix}(\Sigma) = \{x \in X \ : \ \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

We can now define the symmetry of a nonempty set $A \subset X$. The subgroup of $\Gamma$ that fixes each point in $A$ is denoted by

$$T_A = \bigcap_{x \in A} \Sigma_x,$$

and the *isotropy subgroup* of $A$ consists of the group elements that preserve the set $A$, and is denoted by

$$\Sigma_A = \{\gamma \in \Gamma \ : \ \gamma A = A\}.$$

**Proposition 4.1** *Let $x \in X$, and $A = \omega(x) \neq \emptyset$. Then*

*(a)* $A \subset \mathrm{Fix}(T_A)$.

*(b)* $T_A$ *is a normal subgroup of* $\Sigma_A$.

**Proof:** These statements follow easily from the definitions of $T_A$ and $\Sigma_A$. ∎

By Proposition 4.1, $\Sigma_A$ is contained in the normalizer $N(T_A)$ of $T_A$. The *symmetry group* $S_A$ of $A$ is defined to be the quotient group

$$S_A = \Sigma_A / T_A \subset N(T_A)/T_A.$$

Note that if $A$ consists of a single point, then $S_A$ is the trivial group.

**Remark 4.2** It is the group $S_A$ that plays the most significant role in the theoretical issues discussed in this paper. However it is interesting to compare the meaning of the three groups $T_A$, $\Sigma_A$ and $S_A$ in applications, particularly to nonequilibrium solutions of PDEs. The group $T_A$ refers to the symmetries of a solution at each instant in time while $\Sigma_A$ refers to symmetries of the time-average of that solution. The important observation for applications is that $\Sigma_A$ can be larger than $T_A$, [4]. In this sense, $S_A = \Sigma_A/T_A$ are the new symmetries that appear in solutions by taking time-averages.

If $T_A = 1$ then $\Sigma_A$ can be identified with $S_A$ and we say that $A$ is *$\Sigma$-symmetric*.

**Proposition 4.3** *Assume:*

*(a)* $Y$ *is a finite set with* $|Y| = r$.

*(b) The finite group* $\mathbb{Z}_r$ *acts transitively on* $Y$.

*(c)* $\Sigma$ *is a group acting fixed-point free on* $Y$.

*(d) The actions of* $\Sigma$ *and* $\mathbb{Z}_r$ *commute.*

*Then* $\Sigma$ *is isomorphic to a subgroup of* $\mathbb{Z}_r$, *that is* $\Sigma \cong \mathbb{Z}_k$ *where* $k$ *divides* $r$.

**Proof:** Let $a$ be a generator of $\mathbb{Z}_r$, fix $y \in Y$, and let $\sigma \in \Sigma$. Since $\sigma y \in Y$, there is a unique $a^p \in \mathbb{Z}_r$ such that $\sigma y = a^p y$, using (a) and (b). Define $\chi : \Sigma \to \mathbb{Z}_r$ by $\chi(\sigma) = a^p$. We show that $\chi$ is a monomorphism. The proposition follows with $k = r/p$.

To see that $\chi$ is a homomorphism, suppose that $\chi(\sigma_j) = a^{p_j}$ for $j = 1, 2$. Then $\sigma_j y = a^{p_j} y$ and

$$\sigma_2 \sigma_1 y = \sigma_2 a^{p_1} y = a^{p_1} \sigma_2 y$$

since by (d) the actions of $\Sigma$ and $\mathbb{Z}_r$ commute. Hence

$$\sigma_2 \sigma_1 y = a^{p_1} a^{p_2} y = a^{p_2} a^{p_1} y$$

since $\mathbb{Z}_r$ is commutative. It follows that

$$\chi(\sigma_2 \sigma_1) = \chi(\sigma_2)\chi(\sigma_1)$$

We now show that $\chi$ is injective. Suppose that $\chi(\sigma) = 1$. Then $\sigma y = y$ and $\sigma = 1$ since by (c) $\Sigma$ acts fixed-point free. ∎

**Corollary 4.4** *Suppose that $f : X \to X$ is $\Gamma$-equivariant and $A$ is a periodic orbit of period $r$. Then $S_A \cong \mathbb{Z}_k$ where $k$ divides $r$.*

**Proof:** Let $Y = A = \{x, f(x), \ldots, f^{r-1}(x)\}$. Note that the action of $f$ on $Y$ is a transitive $\mathbb{Z}_r$ action. Moreover $\Sigma_{f^j(x)} = T_A$ for each $j$ from which it follows that $S_A$ acts fixed-point free on $Y$. Since $\Gamma$-equivariance means that the actions of $S_A$ and $\mathbb{Z}_r$ commute, the result follows from Proposition 4.3. ∎

**Remark 4.5** Often we shall discuss properties of $\Gamma$-symmetric attractors $A$ for $\Gamma$-equivariant mappings $f$ so that $S_A = \Gamma$. This assumption can be verified in two distinct ways. First, when $A$ contains a point with trivial isotropy then $T_A = 1$ and we can identify $S_A$ with $\Sigma_A \subset \Gamma$. If $\Sigma_A \neq \Gamma$ discard the elements of $\Gamma$ that are not in $\Sigma_A$ and redefine $\Gamma = \Sigma_A$. Then $f$ is still $\Gamma$-equivariant and $A$ is $\Gamma$-symmetric.

Second, even when $A$ does not contain points with trivial isotropy this hypothesis can be satisfied – if in addition we restrict $f$. Suppose that $T_A \neq 1$. Note that $A \subset \mathrm{Fix}(T_A)$ by Proposition 4.1(a) and $\mathrm{Fix}(T_A)$ is an $f$-invariant subspace. Let $g = f|_{\mathrm{Fix}(T_A)}$. Then $g$ is a $\Delta$-equivariant mapping where $\Delta = N(T_A)/T_A$. Inside $\Delta$, $S_A = \Sigma_A$ and we are back in the first case.

**Theorem 4.6** *Let $f : X \to X$ be a $\Gamma$-equivariant map with an attractor $A$, $S_A = \Gamma$. Suppose that $A$ is the disjoint union of two compact sets $A_1$ and $A_2$. Then $S_{A_1}$ is a normal subgroup of $\Gamma$ and the quotient group $\Gamma/S_{A_1}$ is cyclic.*

**Proof:** Choose $S_0$ to be a closed set separating $A_1$ and $A_2$ such that $A \cap S_0 = \emptyset$. Let $S = \bigcup_{\gamma\in\Gamma} \gamma S_0$. Since $A$ is $\Gamma$-symmetric we have that $A \cap S = \emptyset$. By Corollary 2.4, we have

$$A \subset C_0 \cup \cdots \cup C_{r-1}$$

where the $C_j$ are connected components of $X - \mathcal{P}_S$. Moreover the connected components are permuted cyclically by $f$ and permuted by elements of $\Gamma$.

It is easy to check that $S_{A_1}$ is a normal subgroup of $\Gamma$. Suppose that $\gamma C_j = C_j$ for some $\gamma \in \Gamma$ and some connected component $C_j$. We claim that $\gamma \in S_{A_1}$. It follows that $\Gamma/S_{A_1}$ acts fixed-point free on $\{C_0, \ldots, C_{r-1}\}$ and hence is cyclic by Proposition 4.3.

It remains to verify the claim. First observe that by equivariance of $f$, the element $\gamma$ fixes some $C_j$ if and only if $\gamma$ fixes all the components $C_0, \ldots, C_{r-1}$. But $S_A = \Gamma$ so that $\gamma(A \cap C_j) = A \cap C_j$ for each $j$. The choice of $S_0$ guarantees that if a $C_j$ intersects $A_1$ then $C_j \cap A_2 = \emptyset$ so that $\gamma(A_1 \cap C_j) = A_1 \cap C_j$. Thus $A_1$ is made up of $\gamma$-symmetric pieces and is itself $\gamma$-symmetric. ∎

**Remark 4.7** (a) Suppose in Theorem 4.6 that $\Gamma$ is a simple noncyclic group and $S_A = \Gamma$. Then $S_{A_1} = \Gamma$. (Recall that a group $G$ is *simple* if the only normal subgroups are $G$ and 1.) An example of such a group $\Gamma$ is given by the symmetry group of the icosahedron which is isomorphic to the alternating group $A_5$.

(b) Suppose that $\Gamma = \mathbf{D}_m$ in Theorem 4.6. Then $S_{A_1} = \mathbf{D}_m$ or $\mathbb{Z}_m$ if $m$ is odd, and $S_{A_1} = \mathbf{D}_m, \mathbb{Z}_m$ or $\mathbf{D}_{m/2}$ if $m$ is even. It is easily verified that these subgroups obey the hypotheses of the theorem. The remaining normal subgroups $\mathbb{Z}_k$, where $k$ divides $m$, may be ruled out thanks to the isomorphism $\mathbf{D}_m/\mathbb{Z}_k \cong \mathbf{D}_{m/k}$.

(c) For certain representations of a group $\Gamma$, there may be further restrictions on the symmetry of disjoint parts of the attractor. For example we show in Theorem 5.7 that if $\mathbf{D}_m$ is acting faithfully on $\mathbb{R}^2$, $m \geq 3$, then $S_{A_1} = \mathbf{D}_m$.

We have seen that there are goup-theoretic restrictions on the symmetry groups of periodic orbits (Corollary 4.4) and on the symmetry of connected components of symmetric attractors (Theorem 4.6). We now show that there are restrictions on the symmetry groups of attractors. In contrast to the previous results, these restrictions are not purely group-theoretic, but depend on the representation of the group. Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group and let $\Sigma$ be a subgroup of $\Gamma$. Recall that $\tau \in \Gamma$ is a reflection if $\mathrm{Fix}(\tau)$ is a hyperplane in $\mathbb{R}^n$. Let $K_\Sigma$ be the set of reflections in $\Gamma - \Sigma$ and define

$$L_\Sigma = \bigcup_{\tau \in K_\Sigma} \mathrm{Fix}(\tau).$$

The connected components of $\mathbb{R}^n - L_\Sigma$ are permuted by elements of $\Sigma$. We will use the fact that a $\Sigma$-symmetric attractor for a $\Gamma$-equivariant map cannot intersect $L_\Sigma$. This is a consequence of the following result proved in [3], Proposition 1.1.

**Proposition 4.8** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and commute with a matrix $\rho$. Let $A \subset \mathbb{R}^n$ be an attractor for $f$. If*

$$A \cap \rho(A) \neq \emptyset,$$

*then*

$$\rho(A) = A.$$

**Proposition 4.9** *Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group with subgroup $\Sigma$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous $\Gamma$-equivariant mapping with a $\Sigma$-symmetric topologically mixing attractor. Then there is a connected component of $\mathbb{R}^n - L_\Sigma$ that is preserved by $\Sigma$.*

**Proof:** By Proposition 4.8, any $\Sigma$-symmetric attractor $A$ must satisfy $A \cap L_\Sigma = \emptyset$. In addition, $A$ is connected by Theorem 2.6. Hence $A$ lies inside a single connected component $C$ of $\mathbb{R}^n - L_\Sigma$. But $\Sigma$ fixes $A$ and hence $C$. ∎

If we drop the topological mixing assumption in Proposition 4.9 then the situation is more complicated, but there is still a representation-theoretic restriction on $\Sigma$.

**Theorem 4.10** *Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group with subgroup $\Delta$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous $\Gamma$-equivariant mapping with a $\Delta$-symmetric attractor. Then there is a subgroup $\Sigma$ such that*

*(a) $\Sigma$ is a normal subgroup of $\Delta$,*

*(b) $\Delta/\Sigma$ is cyclic, and*

*(c) $\Sigma$ fixes a connected component of $\mathbb{R}^n - L_\Sigma$.*

**Proof:** Let $A$ be a $\Delta$-symmetric attractor for a $\Gamma$-equivariant continuous mapping $f$. Let $L$ be the union of the reflection hyperplanes that are not intersected by $A$. The connected components of $\mathbb{R}^n - L$ that intersect $A$ are permuted by the elements of $\Delta$.

By Corollary 4.4, $A$ is covered by finitely many connected components $C_1, \ldots, C_r$ of $\mathbb{R}^n - \mathcal{P}_L$ and these connected components are permuted cyclically by $f$. Let $\Sigma_i$ denote the subgroup of $\Delta$ that fixes the connected component containing $C_i$. It is an easy argument using equivariance of $f$ to show that $\Sigma = \Sigma_i$ is independent of $i$ and that $\Sigma$ is normal in $\Delta$.

Now suppose that $\tau \in \Gamma$ is a reflection and that $\mathrm{Fix}(\tau) \not\subset L$. Then $A \cap \mathrm{Fix}(\tau) \neq \emptyset$ and by Proposition 4.8, $\tau \in \Delta$. Moreover there is a connected component, $C_1$ say, of $\mathbb{R}^n - \mathcal{P}_L$ intersecting $\mathrm{Fix}(\tau)$ and hence $\tau \in \Sigma_1 = \Sigma$, so that $\mathrm{Fix}(\tau) \not\subset L_\Sigma$. It follows that $L_\Sigma \subset L$ and so $\Sigma$ fixes a connected component of $\mathbb{R}^n - L_\Sigma$.

It remains to show that $\Delta/\Sigma$ is cyclic. But $\Delta/\Sigma$ acts fixed-point-free on the set of connected components $\{C_1, \ldots, C_r\}$ (by definition of $\Sigma$). Since $f$ commutes with this action and permutes the $C_i$ cyclically, the result follows from Proposition 4.3. ∎

**Remark 4.11** (a) In Section 5 we apply Theorem 4.10 when $\Gamma = \mathbf{D}_m$ acting on $\mathbb{R}^2$ and show that not all subgroups of $\mathbf{D}_m$ can be the symmetry group of an attractor for a $\mathbf{D}_m$-equivariant mapping.

(b) Suppose that $\Gamma$ is finite. Then the representation-theoretic restriction obtained in Theorem 4.10 is necessary and sufficient, see Ashwin and Melbourne [1]. In particular, there are no restrictions on cyclic subgroups of $\Gamma$ nor on subgroups of $\Gamma$ that contain all the reflections in $\Gamma$. (For the case of cyclic subgroups, see also King and Stewart [9]). Proposition 4.9 is also optimal except when $\Gamma$ is a cyclic subgroup of $\mathbf{O}(2)$ ([1]).

**Definition 4.12** Let $\mathcal{D}$ be a finite collection of closed subsets of $X$. The collection $\mathcal{D}$ is a *fundamental decomposition* for the action of $\Gamma$ if

(a) $X = \bigcup_{B\in\mathcal{D}} B$.

(b) $\overline{\mathrm{int}(B)} = B$, for each $B \in \mathcal{D}$.

(c) The sets $\mathrm{int}(B)$ are pairwise disjoint.

(d) The group $\Gamma$ acts on $\mathcal{D}$; that is, $\gamma(B) \in \mathcal{D}$ for all $B \in \mathcal{D}$ and $\gamma \in \Gamma$.

(e) If $\gamma(B) = B$ for some $B \in \mathcal{D}$ and nontrivial $\gamma \in \Gamma$, then there is an element $\delta \in \Gamma$ such that $\gamma\delta B \neq \delta B$.

**Remark 4.13** (a) Definition 4.12(e) states that $\Gamma$ acts fixed-point free on group orbits in $\mathcal{D}$.

(b) This definition is similar to that of a fundamental domain. However, we allow the possibility that $\gamma B = B$ for some $B \in \mathcal{D}$ and some nontrivial $\gamma \in \Gamma$.

(c) A natural way to produce fundamental decompositions is to choose a hyperplane in $X$ passing through the origin. Let $S_0$ denote a half-plane inside this hyperplane, and let $S = \bigcup_{\gamma\in\Gamma} \gamma S_0$. Let $\mathcal{D}$ be the collection of closures of connected components of $X - S$. It is clear that $\mathcal{D}$ satisfies Definition 4.12 a–d. Condition (e) must be verified in each case.

Recall that $\mathcal{P}_S$ is defined to be the set of preimages of a set $S$ under $f$.

**Proposition 4.14** *Suppose that $\Gamma$ is not cyclic and $A$ is an $\omega$-limit set with $S_A = \Gamma$. Let $\mathcal{D}$ be a fundamental decomposition for $\Gamma$ and let $S$ be the closed set $\bigcup_{B \in \mathcal{D}} \partial B$. Then*

*(a) $A \subset \overline{\mathcal{P}_S}$.*

*(b) If $A$ is an attractor, then $A \cap S \neq \emptyset$.*

*(c) If $A$ is an attractor and $\mathcal{D}$ is constructed as in Remark 4.13(c), then $A$ intersects $\gamma S_0$ for each $\gamma \in \Gamma$.*

**Proof:** (a) Suppose that $A \not\subset \overline{\mathcal{P}_S}$. Then by Lemma 2.1, $A - \mathcal{P}_S$ is covered by connected components $C_0, \dots, C_{r-1}$ of $X - \mathcal{P}_S$, and these connected components are permuted cyclically by $f$. We claim that $\Gamma$ acts fixed-point free on the connected components. Then it follows from Proposition 4.3 that $\Gamma$ is cyclic which we had assumed not to be the case. When applying that proposition, set $\Sigma = \Gamma$ and $Y = \{C_0, \dots, C_{r-1}\}$.

It remains to verify the claim. Suppose that $\gamma C_0 \subset C_0$. Then the $\Gamma$-equivariance of $f$ implies that $\gamma C_j \subset C_j$ for each $j$. Let $B_j$ denote the unique subset of $\mathcal{D}$ that contains $C_j$ — note that uniqueness follows from Definition 4.12(c). Then $\gamma B_j = B_j$ for every $j$. Since $A$ is $\Gamma$-symmetric, the collection of subsets $\{B_j\}$ consists of a collection of group orbits of $\Gamma$ by Definition 4.12(d), each of which is fixed by $\gamma$. But Definition 4.12(e) states that $\Gamma$ acts fixed-point free on group orbits, so $\gamma = 1$ as required.

(b) follows from Proposition 2.3.

(c) If $S = \bigcup_{\gamma \in \Gamma} \gamma S_0$ then $A$ intersects $\gamma S_0$ for some $\gamma \in \Gamma$, and hence for all $\gamma \in \Gamma$ since $S_A = \Gamma$. ∎

# 5 Planar maps with dihedral symmetry

The dihedral group $\mathbf{D}_m$ consists of the symmetries of the regular $m$-sided polygon and is generated by a rotation $\theta$ through $2\pi/m$ and a reflection $\kappa$. The irreducible representations of $\mathbf{D}_m$ are one- or two-dimensional and the faithful representations are given on $\mathbb{C}$ by

$$\theta \cdot z = e^{2\ell\pi i/m} z, \quad \kappa \cdot z = \bar{z}$$

where $\ell$ and $m$ are coprime. We consider here only the standard two-dimensional representation $\ell = 1$; the results for the other two-dimensional irreducible representations are identical.

The subgroups of $\mathbf{D}_m$ are $\mathbf{D}_k$ and $\mathbb{Z}_k$, $k \geq 1$, where $k$ divides $m$. There are $m$ axes of symmetry for $\mathbf{D}_m$ which we label $L_1, \ldots, L_m$.

**Proposition 5.1** *Suppose that $f$ is $\mathbf{D}_m$-equivariant with a $\Sigma$-symmetric topologically mixing attractor. Then $\Sigma = \mathbf{D}_m$, $\mathbf{D}_1$ or $\mathbf{1}$.*

**Proof:** Suppose that $\Sigma \neq \mathbf{D}_m$. Then there are reflections in $\mathbf{D}_m$ that do not lie in $\Sigma$ so that $L_\Sigma \neq \emptyset$. Observe that any nontrivial rotation in $\mathbf{D}_m$ cannot preserve a connected component of $\mathbb{R}^2 - L_\Sigma$. It follows from Proposition 4.9 that $\Sigma = \mathbf{D}_1$ or $\mathbf{1}$. ∎

**Theorem 5.2** *Suppose that $f$ is $\mathbf{D}_m$-equivariant, $m \geq 2$. Suppose further that $A$ is an attractor for $f$ and $\Sigma_A = \mathbf{D}_k$.*

*(a) If $m$ is odd, then $k = 1$ or $m$.*

*(b) If $m$ is even, then $k = 1, 2$ or $m$.*

We note that the first nontrivial consequence of Theorem 5.2 occurs when $m = 6$.

**Proof:** By Theorem 4.10, $\mathbf{D}_k$ must have a subgroup $\Sigma$ satisfying conditions (a)–(c) of that theorem. Condition (c) together with Proposition 4.9 and Proposition 5.1 implies that $\Sigma = \mathbf{D}_m$, $\mathbf{D}_1$ or 1. Now $\mathbf{D}_m$ is not a subgroup unless $k = m$, and $\mathbf{D}_1$ is normal in $\mathbf{D}_k$ (condition (a)) only if $k = 1$ or 2. Finally $\mathbf{D}_k/1$ is cyclic (condition (b)) only if $k = 1$. ∎

**Remark 5.3** King and Stewart [9] prove that there exist attractors with cyclic symmetry for any cyclic subgroup. It is shown in Ashwin and Melbourne [1] that there exist attractors with $\mathbf{D}_m$ and $\mathbf{D}_2$ symmetry.

**Lemma 5.4** *Suppose that $A$ is a $\mathbf{D}_m$-symmetric attractor for a $\mathbf{D}_m$-equivariant mapping $f$. If $m \geq 3$ then $A$ intersects each half-line emanating from the origin. If $m = 2$ then $A$ intersects at least one line of symmetry.*

**Proof:** Let $S_0$ be any half-line emanating from the origin and define $S = \bigcup_{\gamma \in \mathbf{D}_m} \gamma S_0$. The set $S$ generally consists of $2m$ half-lines and is illustrated for the case $m = 4$ in Figure 1. Note that when $S_0$ lies on an axis of symmetry, then $S$ consists of $m$ half-lines. When $m \geq 3$ it is easy to check that for any choice of $S_0$, the collection $\mathcal{D}$ of connected components of $\mathbb{C} - S$ is indeed a

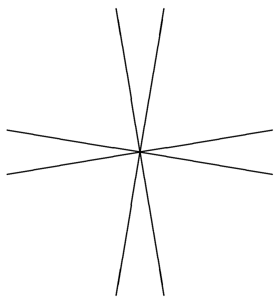

Figure 1: half-lines for $m = 4$.

fundamental decomposition. It follows immediately from Proposition 4.14, that if $A$ is an attractor with $\Sigma_A = \mathbf{D}_m$ and $m \geq 3$, then for any choice of $S_0$, $A \subset \overline{\mathcal{P}_S}$ and $A \cap S_0 \neq \emptyset$.

In the case $m = 2$ let $S$ be the union of the two axes of symmetry. Then the connected components of $\mathbb{C} - S$ form a fundamental decomposition. It follows that $A \subset \overline{\mathcal{P}_S}$, and $A \cap S \neq \emptyset$. ∎

**Corollary 5.5** *Let $A$ be a $\mathbf{D}_m$-symmetric $\omega$-limit set for a $\mathbf{D}_m$-equivariant mapping $f$. Then $A$ has weak dependence. Moreover, if $A$ has positive measure, then $A$ has sensitive dependence.*

**Proof:** Let $S$ be the union of the axes of symmetry for $\mathbf{D}_m$. Choosing $S_0$ to be a half-axis of symmetry in the proof of Lemma 5.4 shows that $A \subset \overline{\mathcal{P}_S}$. It follows that $A$ and $S$ satisfy the hypotheses of Proposition 2.13. ∎

**Proposition 5.6** *Let $f$ be $\mathbf{D}_m$-equivariant where $m$ is even and $m \geq 4$. Let $A$ be an attractor with $\Sigma_A = \mathbf{D}_2$. Then $A$ intersects precisely one axis of symmetry.*

**Proof:** Let $L$ and $M$ denote the two axes of symmetry for the subgroup $\mathbf{D}_2$. By Lemma 5.4, $A$ intersects at least one of the axes, say $L$. Moreover, by Proposition 4.8, $A$ does not intersect any other axis of symmetry with the possible exception of $M$.

Let $S$ denote the union of the two axes of symmetry for $\mathbf{D}_m$ that are adjacent to $L$. Then $A \cap S = \emptyset$. Since $A$ intersects $L$, $A$ intersects a connected component of $\mathbb{C} - S$ that intersects $L$. But such a connected component cannot intersect $M$. Therefore $A \cap M = \emptyset$ by Lemma 2.5. ∎

As promised, we can improve Theorem 4.6 for the faithful representations of $\mathbf{D}_m$ on $\mathbb{C}$.

**Theorem 5.7** *If $A$ is a $\mathbf{D}_m$ symmetric attractor, $m \geq 3$, and $A$ is the disjoint union of two compact subsets $A_1$ and $A_2$, then these subsets are $\mathbf{D}_m$-symmetric.*

**Proof:** Let $S_0$ be a closed set with the property that $A_1$ and $A_2$ lie in distinct connected components of $\mathbb{C} - S_0$. Define $S = \bigcup_{\gamma \in \mathbf{D}_m} \gamma S_0$. By Corollary 2.4, $A$ is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $\mathbb{C} - \mathcal{P}_S$.

We claim that these components are $\mathbf{D}_m$-invariant. It is sufficient to show that reflections leave the components invariant since $\mathbf{D}_m$ is generated by reflections. Let $L$ be an axis of symmetry corresponding to a reflection $\kappa$ and observe that $\kappa$ permutes connected components by the $\mathbf{D}_m$-invariance of $S$. But $A$ intersects $L$ by Lemma 5.4 and hence $L$ intersects one of the connected components, say $C_0$. Since $\kappa$ fixes $L$ pointwise we have $\kappa C_0 = C_0$. In addition, the equivariant map $f$ permutes the connected components so that $\kappa C_j = C_j$ for each $j$ thus verifying the claim.

Now let $\gamma \in \mathbf{D}_m$. Then $\gamma A = A$ and by the claim $\gamma C_j = C_j$ for each $j$. Hence $\gamma(A \cap C_j) = A \cap C_j$. But $S$ is constructed so that only one of $A_1$ or $A_2$ may intersect a given $C_j$. If $A_1$ say intersects $C_j$ then we have $\gamma(A_1 \cap C_j) = A_1 \cap C_j$. Thus $A_1$ and $A_2$ are unions of $\mathbf{D}_m$-symmetric subsets and are themselves $\mathbf{D}_m$-symmetric. ∎

The following result is useful for computing symmetry-increasing bifurcations. See [4].

**Theorem 5.8** *Let $f$ be a $\mathbf{D}_m$-equivariant mapping, $m \geq 3$, with an attractor $A$.*

*(a) If $S_A = \mathbf{D}_m$ then $A \subset \overline{\mathcal{P}_S}$ where $S$ is the union of any two lines through the origin.*

*(b) If $\Sigma_A = \mathbf{D}_2$, then $A \subset \overline{\mathcal{P}_L}$ for some line of symmetry $L$.*

**Proof:** (a) Suppose that $A \not\subset \overline{\mathcal{P}_S}$. Then by Lemma 2.1,

$$A \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}},$$

where $C_0, \ldots, C_{r-1}$ are connected components of $\mathbb{C} - \mathcal{P}_S$, and these connected components are permuted cyclically by $f$.

Since $m \geq 3$ there is an axis of symmetry $M$, $M \not\subset S$. By Lemma 5.4 $A \cap M \neq \emptyset$. Hence one of the connected components, $C_0$ say, intersects $M$. It follows that $\overline{C_j} \cap M \neq \emptyset$ for $j = 0, \ldots, r-1$. In particular, $A$ intersects only the two connected components of $\mathbb{C} - S$ that intersect $M$. But $A$ is $\mathbf{D}_m$-symmetric and hence intersects all four connected components of $\mathbb{C} - S$ giving a contradiction.
(b) Let $S = L_1 \cup L_2$ where $L_1$ and $L_2$ are the axes of symmetry for $\mathbf{D}_2$. By the proof of Lemma 5.4, $A \subset \overline{\mathcal{P}_S}$. It follows from Proposition 2.3 that $A \subset \overline{\mathcal{P}_{A\cap S}}$. By Proposition 5.6, $A$ intersects precisely one of these axes $L_1$ say. In particular $A \cap S \subset L_1$. Therefore $A \subset \overline{\mathcal{P}_{L_1}}$. ∎

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