STABILITY OF MIXING AND RAPID MIXING FOR HYPERBOLIC FLOWS

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Abstract. We obtain general results on the stability of mixing and rapid mixing (superpolynomial decay of correlations) for hyperbolic flows. Amongst C^r Axiom A flows, $r \geq 2$, we show that there is a C^2-open, C^r-dense set of flows for which each nontrivial hyperbolic basic set is rapid mixing. This is the first general result on the stability of rapid mixing (or even mixing) for Axiom A flows that holds in a C^r, as opposed to Hölder, topology.

1. Introduction

Let M be a compact connected differential manifold and let Φ_t be a C^1 flow on M. A Φ_t-invariant set Λ is *(topologically) mixing* if for any nonempty open sets $U, V \subset \Lambda$ there exists $T > 0$ such that $\Phi_t(U) \cap V \neq \emptyset$ for all $t > T$. The flow is *stably mixing* if all nearby flows (in an appropriate topology) are mixing.

In this work we are interested in the C^r-stability of mixing, and of the rate of mixing, for Axiom A and Anosov flows.

There is a quite extensive literature on mixing and rates of mixing for certain classes of Anosov flows. In particular, Anosov [1] showed that geodesic flows for negatively curved compact Riemannian manifolds are always mixing. Anosov also proved the *Anosov alternative*: a transitive volume preserving Anosov flow is either mixing or the suspension of an Anosov diffeomorphism by a constant roof function. Plante [25] generalized the Anosov alternative to general equilibrium states and proved that codimension one Anosov flows are mixing if and only if they are stably mixing (for this class, mixing is equivalent to the joint nonintegrability of the stable and unstable foliations which is a C^1-open condition). Anosov's results on geodesic flows were generalized to contact flows by Katok & Burns [19]. More recently, Chernov [11], Dolgopyat [14] and Liverani [21] have obtained results on exponential rates of mixing for restricted classes of Anosov flows. Bowen [6] showed that if a mixing Anosov flow is the suspension of an Anosov diffeomorphism of an infranilmanifold then it is stably mixing. However, the question of the existence of mixing but not stably mixing Anosov flows is still open. As far as the authors are aware, there are no known examples of Anosov flows that are stably exponentially mixing.

We turn now to Axiom A flows. Let $\mathcal{A}_r(M)$ denote the set of C^r flows $(1 \leq r \leq \infty)$ on M satisfying Axiom A and the no cycle property [31, 28]. The nonwandering set Ω of such a flow admits the *spectral decomposition* $\Omega = \Lambda_1 \cup \cdots \cup \Lambda_k$, where the Λ_i are disjoint closed topologically transitive locally maximal hyperbolic sets. The sets Λ_i are called *(hyperbolic) basic sets*. A basic set is *nontrivial* if it is neither an equilibrium nor

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a periodic solution. Bowen [4, 6] proved that nontrivial basic sets are generically mixing and gave an important characterization of mixing.

Theorem 1.1 (Bowen, 1972, 1976). (1) *For $1 \le r \le \infty$, there is a residual subset of flows in $\mathcal{A}_r(M)$ in the C^r topology for which each nontrivial basic set is mixing.*

(2) *A flow $\Phi_t \in \mathcal{A}_r(M)$ is not mixing on a basic set Λ if and only if there exists $c > 0$ such that every periodic orbit in Λ has period which is an integer multiple of c.*

Remark 1.2. If Λ is a basic set for an Axiom A flow, then a consequence of the work of Sinai, Ruelle and Bowen in the 1970's is that the following topological and measure-theoretic notions of mixing are equivalent: (a) topological mixing, (b) measure-theoretic weak mixing, and (c) measure-theoretic mixing (for (b,c) it is assumed that the measure is an equilibrium state corresponding to a Hölder continuous potential). Moreover, such flows are Bernoulli. (See [7] and references therein.) In this paper, mixing will refer to any and all of these properties.

For general Axiom A flows it is well-known that a mixing flow need not be stably mixing. Hence, the best one can hope for is to show that $\mathcal{A}_r(M)$ contains an open and dense set of mixing flows. Our first main result shows that this is true for $r \ge 2$.

Theorem 1.3. *(a) Suppose $2 \le r \le \infty$. There is a C^2-open, C^r-dense subset of flows in $\mathcal{A}_r(M)$ for which each nontrivial basic set is mixing.*

(b) Suppose $1 \le r \le \infty$. There is a C^1-open, C^r-dense subset of flows in $\mathcal{A}_r(M)$ for which each nontrivial attracting basic set is mixing.

Remark 1.4. Rather little hyperbolicity is required for our methods to apply. It is enough that (a) Λ is a locally maximal transitive set, (b) Λ contains a transverse homoclinic point, and (c) there is sufficient (Livšic) regularity of solutions of cohomology equations for Theorem 1.1(2) to be valid.

In order to quantify rates of mixing, we need to introduce correlation functions. Suppose then that Λ is a basic set for an Axiom A flow Φ_t and let μ be an equilibrium state for a Hölder potential [7]. Given $A, B \in L^2(\Lambda, \mu)$, we define the correlation function

$$\rho_{A,B}(t) = \int_\Lambda A \circ \Phi_t \, B \, d\mu - \int_\Lambda A \, d\mu \int_\Lambda B \, d\mu.$$

The flow Φ_t is mixing if and only if $\rho_{A,B}(t) \to 0$ as $t \to \infty$ for all $A, B \in L^2(\Lambda, \mu)$. Bowen and Ruelle [7] asked whether $\rho_{A,B}(t)$ decays at an exponential rate when A, B are restrictions of smooth functions. (For Axiom A diffeomorphisms, mixing hyperbolic basic sets automatically have exponential decay of correlations for Hölder observations.) Subsequently, Ruelle [30] found examples of mixing Axiom A flows which did not mix exponentially. Moreover, Pollicott [26] showed that the decay rates for mixing basic sets could be arbitrarily slow. On the other hand, exponential mixing is proved for the aforementioned restricted classes of Anosov flows and also for certain uniformly hyperbolic attractors with one-dimensional unstable manifolds (Pollicott [27]). The authors are unaware of any other examples of smooth exponentially mixing Axiom A flows.

A weaker notion of decay is superpolynomial decay (called rapid mixing for the remainder of this paper) where for any $n > 0$, there is a constant $C \ge 1$ such that

$$|\rho_{A,B}(t)| \le C\|A\| \, \|B\| t^{-n}, \;\; t > 0,$$

for all observations A, B that are sufficiently smooth in the flow direction. Here $\| \ \|$ denotes the appropriate C^s-norm. The constants C and s depend on the flow Φ_t, the equilibrium state μ and the polynomial degree n. It turns out that rapid mixing is independent of the choice of equilibrium state μ [15, Theorems 2, 4].

Remark 1.5. Suppose that Φ_t is a rapid mixing Axiom A flow and that A, B are observations. If Φ_t, A, B are C^∞ then $\rho_{A,B}$ decays faster than any polynomial rate for any equilibrium state. (Indeed, $\rho_{A,B} \in \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions.) If Φ_t is C^r, $r < \infty$, then the definition of rapid mixing admits the possibility that $s > r$ for certain equilibrium states. In this situation, the condition that A, B are sufficiently smooth in the flow direction is not automatic even if A, B are C^∞.

Dolgopyat [15] proved that typical (in the measure-theoretic sense of prevalence) Axiom A flows are rapid mixing. However, the set of rapid mixing flows obtained in [15] is nowhere dense, and there is no uniformity in the constant C.

Our second main result (which extends Theorem 1.3) shows that typical Axiom A flows are *stably* rapid mixing in the sense that rapid mixing is robust to C^2-small perturbations of the underlying flow. In addition, it follows from our arguments that the constant C can be chosen uniformly for flows close to the given one, which is important for applications to statistical physics (see [11, Introduction]).

Theorem 1.6. *(a) Suppose $2 \le r \le \infty$. There is a C^2-open, C^r-dense subset of flows in $\mathcal{A}_r(M)$ for which each nontrivial basic set is rapid mixing.*

(b) Suppose $1 \le r \le \infty$. There is a C^1-open, C^r-dense subset of flows in $\mathcal{A}_r(M)$ for which each nontrivial attracting basic set is rapid mixing.

Remark 1.7. It follows from our proof of Theorem 1.6(a) that we obtain a $C^{1,1}$-open set of rapid mixing flows (here $C^{1,1}$ means C^1 with Lipschitz derivative). Details are provided in Remark 4.10.

The proof of Theorem 1.6 relies on the following result which should be contrasted with Theorem 1.1(2).

Theorem 1.8 (Dolgopyat [15]). *Let Λ be a basic set for a flow $\Phi_t \in \mathcal{A}_r(M)$ and suppose that Λ is not rapid mixing. Then there exists $c > 0$ and $C > 0$ such that for every $\alpha > 0$, there exists $\beta > 0$ and a sequence $|b_k| \to \infty$ such that for each $k \ge 1$ and each period τ corresponding to a periodic orbit in Λ,*

$$\operatorname{dist}(b_k n_k \tau\,,\, c\mathbb{Z}) \le C\tau |b_k|^{-\alpha}, \tag{1.1}$$

where $n_k = [\beta \ln |b_k|]$ and dist *denotes Euclidean distance.*

This result is implicit in [15] and seems of independent interest, so we indicate the proof at the end of Section 2.

Remark 1.9. It follows as in [22] that the almost sure invariance principle holds for the time-one map of rapid mixing Axiom A flows (for sufficiently smooth observables). Hence we obtain a strengthened version of [22, Theorem 1]. The standard consequences of the almost sure invariance principle include the central limit theorem and law of the iterated logarithm [24]. (The corresponding results for the flow itself hold for all Axiom A flows [13, 23, 29] but time-one maps are more delicate.)

Remark 1.10. In the survey article [10], it is mistakenly claimed that the open and denseness of rapid mixing for Axiom A flows was proved in Dolgopyat [15]. In fact, the only result on openness claimed in [14, 15] is [14, Theorem 3] where it is proved that Anosov flows with jointly nonintegrable foliations (which is an open condition) are rapid mixing. The density of joint nonintegrability for Anosov flows (and Axiom A attractors) is a consequence of methods of Brin [8, 9]. Hence Theorem 1.6(b) is implicit in previous work, though we have not seen this result stated elsewhere. For completeness, we give an alternative proof of Theorem 1.6(b) in this paper.

In [12, Theorem 4.14], it is incorrectly claimed that mixing Anosov flows are automatically rapid mixing. This remains an open question. Plante [25] conjectured that mixing is equivalent to joint nonintegrability of the stable and unstable foliations. If the conjecture were true then mixing would be equivalent to rapid mixing (and stable rapid mixing) for Anosov flows.

We briefly outline the remainder of the paper. In Section 2, we introduce the key new idea in this paper, namely the notion of *good asymptotics.* Then we show that good asymptotics implies part (a) of Theorem 1.6. In Section 3, we prove Theorem 1.6(b). In Section 4, we prove that good asymptotics holds for an open and dense set of flows.

2. Good asymptotics and rapid mixing

We start by specifying the topologies we shall be assuming on spaces of Axiom A and Anosov flows.

C^s topology on the space of C^r-flows. Let $\mathcal{F}^r(M)$ denote the space of C^r-flows on M, $r \geq 2$. Let $t_0 > 0$. Every flow $\Phi_t \in \mathcal{F}^r(M)$ restricts to a C^r map $\Phi^{[t_0]} : M \times [0, t_0] \to M$. Let $1 \leq s \leq r$. Since $M \times [0, t_0]$ is compact, we may take the usual C^s topology on C^r maps $M \times [0, t_0] \mapsto M$, and thereby define a C^s topology on $\mathcal{F}^r(M)$. Using the one-parameter group property of flows, it is easy to see that the C^s topology we have defined on $\mathcal{F}^r(M)$ is independent of $t_0 > 0$. We topologize $\mathcal{A}_r(M)$ as a subspace of $\mathcal{F}^r(M)$.

2.1. **Good asymptotics.** Let Λ be a basic set for a flow $\Phi_t \in \mathcal{A}_r(M)$. Choose a periodic point $p \in \Lambda$ with period τ_0 and let x_H be a transverse homoclinic point for p. Associated to p and x_H are certain constants $\gamma \in (0, 1)$ and $\kappa \in \mathbb{R}$, see Section 4. Using a shadowing argument, it is shown in Section 4 that under certain C^1-open and C^r-dense nondegeneracy conditions it is possible to choose a sequence of periodic points $p_N \in \Lambda$ with $p_N \to x_H$ such that the periods $\tau(N)$ of p_N satisfy

$$\tau(N) = N\tau_0 + \kappa + E_N\gamma^N \cos(N\theta + \varphi_N) + o(\gamma^N), \tag{2.1}$$

where (E_N) is a bounded sequence of real numbers, and either (i) $\theta = 0$ and $\varphi_N \equiv 0$, or (ii) $\theta \in (0, \pi)$ and $\varphi_N \in (\theta_0 - \pi/12, \theta_0 + \pi/12)$ for some θ_0.

Definition 2.1. (Assumptions and notation as above.)

(1) The sequence (p_N) of periodic points has *good asymptotics* if $\liminf_{N\to\infty} |E_N| > 0$.
(2) The basic set Λ has *good asymptotics* if Λ contains a transverse homoclinic point x_H such that the corresponding sequence of periodic points (p_N) has good asymptotics.
(3) The flow $\Phi_t \in \mathcal{A}_r(M)$ has good asymptotics if every nontrivial basic set of Φ_t has good asymptotics.

The main technical result of this paper is the following lemma which is proved in Section 4.

Lemma 2.2. *For $r \geq 2$, $\mathcal{A}_r(M)$ contains a C^2-open, C^r-dense subset $\mathcal{U}$ consisting of flows with good asymptotics.*

2.2. **Genericity of stable rapid mixing.** In the remainder of this section we show how the genericity of stable rapid mixing for Axiom A flows (Theorem 1.6(a)) follows from good asymptotics, Lemma 2.2 and the periodic data criterion of Theorem 1.8. Theorem 1.3(a) is obtained by a similar, but simpler, calculation using good asymptotics and Theorem 1.1(2).

We note that our argument relies only on the set of periods of the flow, and not the location of the periodic orbits.

Proof of Theorem 1.6(a). It suffices by Lemma 2.2 to show that good asymptotics implies rapid mixing. Choose periodic points p, p_N in Λ with periods $\tau_0, \tau(N)$ satisfying (2.1). We show that if Λ is not rapid mixing, then $\liminf |E_N| = 0$ so that there is no good asymptotics.

Fix $\alpha > 0$ (our proof works for any positive value of α). Let $c > 0$, $\beta > 0$ and $|b_k| \to \infty$ be as in Theorem 1.8. Recall that $n_k = [\beta \ln |b_k|]$. The set of periods includes $\tau(N)$ and $N\tau_0$, and $\tau(N) = O(N)$, so

$$\mathrm{dist}(b_k n_k \tau(N)\,,\, c\mathbb{Z}) = O(N|b_k|^{-\alpha}), \quad \mathrm{dist}(b_k n_k N\tau_0\,,\, c\mathbb{Z}) = O(N|b_k|^{-\alpha}).$$

Using formula (2.1) for $\tau(N)$, eliminating τ_0, dividing by c and relabeling, we obtain

$$\mathrm{dist}(b_k n_k(\kappa + E_N\gamma^N \cos(N\theta + \varphi_N) + o(\gamma^N))\,,\, \mathbb{Z}) = O(N|b_k|^{-\alpha}).$$

Set $N = N(k) = [\rho \ln |b_k|]$. For large enough $\rho > 0$, we have $b_k n_k E_{N(k)}\gamma^{N(k)} = O(|b_k|^{-\alpha} \ln |b_k|)$. It follows that $\mathrm{dist}(b_k n_k \kappa\,,\, \mathbb{Z}) = O(|b_k|^{-\alpha} \ln |b_k|)$ and so

$$\mathrm{dist}(b_k n_k(E_N\gamma^N \cos(N\theta + \varphi_N) + o(\gamma^N))\,,\, \mathbb{Z}) = O(N|b_k|^{-\alpha}) + O(|b_k|^{-\alpha} \ln |b_k|). \tag{2.2}$$

Let $S = \sup_N |E_N|$ and set $M(k) = [(\ln(|b_k|n_k) + \ln S + \ln 2)/(-\ln \gamma)] + 1$. Then $S b_k n_k \gamma^{M(k)} = \pm\frac{1}{2}\gamma^{\rho_k}$, with $\rho_k \in (0, 1]$. In particular, $|S b_k n_k \gamma^{M(k)}| \le \frac{1}{2}$ and so taking $N = M(k) + j$ with $j \ge 0$ fixed, condition (2.2) implies that

$$\lim_{k\to\infty} b_k n_k E_{M(k)+j}\gamma^{M(k)} \cos((M(k) + j)\theta + \varphi_{M(k)+j}) = 0.$$

Moreover, $|b_k n_k \gamma^{M(k)}| \ge \gamma/2S$ and it follows that

$$\lim_{k\to\infty} E_{M(k)+j} \cos((M(k) + j)\theta + \varphi_{M(k)+j}) = 0.$$

The proof is complete once we show that there is a choice of $j \ge 0$ for which $\cos((M(k) + j)\theta + \varphi_{M(k)+j})$ does not converge to 0 as $k \to \infty$. Assume by contradiction that for each integer $j \ge 0$

$$\lim_{k\to\infty} (M(k) + j)\theta + \varphi_{M(k)+j} = \pi/2 \mod \pi. \tag{2.3}$$

Recall that if $\theta = 0$ then $\varphi_N \equiv 0$, hence (2.3) fails (with $j = 0$). Otherwise, $\theta \in (0, \pi)$ and $|\varphi_N - \theta_0| < \pi/12$. Taking differences of (2.3) for various values of j we obtain that $\ell\theta \in [-\pi/6, \pi/6] \mod \pi$ for all ℓ, which is impossible. □

Proof of Theorem 1.8. Let $\mathcal{T}(\Lambda)$ denote the set of all periods τ corresponding to periodic orbits in Λ. Note that we do not restrict to prime periods and so $m\mathcal{T}(\Lambda) \subset \mathcal{T}(\Lambda)$ for all positive integers m.

First, we prove the theorem for symbolic semiflows. Let $\sigma : X_+ \to X_+$ be a one-sided subshift of finite type and let $f : X_+ \to \mathbb{R}$ be a roof function that is Lipschitz with respect to the usual metric on X_+. Let X_+^f be the corresponding suspension semiflow and define the set of periods $\mathcal{T}(X_+^f)$.

Define $\mathcal{V}_b : C^0(X_+) \to C^0(X_+)$, $b \in \mathbb{R}$, by $(\mathcal{V}_b w)(x) = e^{ibf(x)} w(\sigma x)$. For $n \geq 1$, define $f_n(x) = \sum_{j=0}^{n-1} f(\sigma^j x)$. Then $(\mathcal{V}_b^n w)(x) = e^{ibf_n(x)} w(\sigma^n x)$.

Suppose that X_+^f is not rapid mixing, and let $\alpha > 0$. By [15, Theorems 1 and 2] (specifically, [15, Theorem 2(v)]), there exist $\beta > 0$ and a sequence $|b_k| \to \infty$, such that for each k there exists $w_k : X_+ \to \mathbb{C}$ continuous and of modulus 1 such that

$$|\mathcal{V}_{b_k}^{n_k} w_k - w_k|_\infty \leq |b_k|^{-\alpha},$$

where $n_k = [\beta \ln |b_k|]$. Since $|\mathcal{V}_b|_\infty \leq 1$, it is immediate that $|\mathcal{V}_{b_k}^{qn_k} w_k - w_k|_\infty \leq q|b_k|^{-\alpha}$, for all $k, q \geq 1$. In other words,

$$|e^{ib_k f_{qn_k}(x)} w_k(\sigma^{qn_k} x) - w_k(x)| \leq q|b_k|^{-\alpha}, \tag{2.4}$$

for all $x \in X_+$, $k, q \geq 1$.

Let $\tau \in \mathcal{T}(X_+^f)$. There exists a periodic point $p \in X_+^f$ with prime period τ/ℓ for some $\ell \geq 1$ and a corresponding point $x \in X_+$ of prime period N such that $f_N(x) = \tau/\ell$. Take $q = \ell N$. Then

$$f_{qn_k}(x) = \ell n_k f_N(x) = n_k \tau,$$

and so (2.4) reduces to

$$\mathrm{dist}(b_k n_k \tau\,,\, 2\pi\mathbb{Z}) \leq 2\ell N |b_k|^{-\alpha}.$$

On the other hand, $\tau = \ell f_N(x) \geq \ell N \min f$, so we obtain

$$\mathrm{dist}(b_k n_k \tau\,,\, 2\pi\mathbb{Z}) \leq C\tau |b_k|^{-\alpha}, \tag{2.5}$$

for all $k \geq 1$ and $\tau \in \mathcal{T}(X_+^f)$.

Now suppose that Λ is a hyperbolic basic set. Bowen [5] showed that there is a symbolic flow $X^{f'}$, where X is a two-sided subshift of finite type, and a bounded-to-one semiconjugacy $\pi : X^{f'} \to \Lambda$. Moreover, there are standard techniques for passing from $X^{f'}$ to X_+^f where X_+ is a one-sided subshift of finite type (for example [26, page 419]). It is easily verified that there is an integer $\ell \geq 1$ such that $\ell\mathcal{T}(\Lambda) \subset \mathcal{T}(X_+^f)$. (The integer ℓ takes into account the fact that the projection $\pi : X^{f'} \to \Lambda$ is bounded-to-one.) Some tedious but standard arguments show that if X_+^f is rapid mixing, then $X^{f'}$ is rapid mixing and it is immediate that Λ is rapid mixing.

It follows from this discussion that if Λ is not rapid mixing, then the estimate (2.5) holds for all $k \geq 1$ and $\tau \in \mathcal{T}(X_+^f)$. Moreover, if $\tau \in \mathcal{T}(\Lambda)$, then $\ell\tau \in \mathcal{T}(X_+^f)$ and so dividing throughout by ℓ in (2.5) yields the required result. □

3. Rapid mixing for hyperbolic attractors

In this section, we prove Theorem 1.6(b). We start by recalling the definitions of local product structure and the temporal distance function [11, 21].

Let Λ be a basic set for the flow $\Phi_t \in \mathcal{A}_1(M)$. Then Λ has a local product structure. That is, there exist an open neighborhood U of the diagonal of Λ in M^2 and $\epsilon > 0$ such that if $(x,y) \in U_\Lambda = U \cap \Lambda^2$, then $W_\epsilon^{uc}(x) \cap W_\epsilon^s(y)$ and $W_\epsilon^{sc}(x) \cap W_\epsilon^u(y)$ each consist of a single point lying in Λ. We define the continuous maps $[\ ,\ ]_s, [\ ,\ ]_u : U_\Lambda \to \Lambda$ by $W_\epsilon^{uc}(x) \cap W_\epsilon^s(y) = \{[x,y]_s\}$, and $W_\epsilon^u(x) \cap W_\epsilon^{sc}(y) = \{[x,y]_u\}$. Given Φ_t, we may choose U, ϵ to be constant on a C^1-neighborhood of Φ_t.

Definition 3.1. Let Λ be a basic set for $\Phi_t \in \mathcal{A}_1(M)$. Choose U, ϵ as above and set $U_\Lambda = U \cap \Lambda^2$. We define the *temporal distance function* $\Delta : U_\Lambda \to \mathbb{R}$ by $[x, y]_u = \Phi_{\Delta(x,y)}([x, y]_s)$.

Proposition 3.2. *The temporal distance function $\Delta(x, y)$ is continuous with respect to x, y, and the flow Φ_t (C^1-topology on $\mathcal{A}_1(M)$).*

Proof. The result follows from the continuity of the foliations $W^a_\epsilon(x)$, $a \in \{s, sc, u, uc\}$, with respect to both the flow and the point. Note that by changing the flow we are also modifying the domain of Δ, but in a continuous manner. □

The following result is well known.

Proposition 3.3. *If the temporal distance function is locally constant (that is, for x and y close enough, $\Delta(x, y) = 0$), then the flow is (bounded-to-one) semiconjugate to a locally constant suspension over a subshift of finite type.*

Sketch of proof. By [5], the flow is realized as (the quotient of) a suspension over a Markov partition. One can assume that the roof function is constant along the stable leaves spanning the rectangles of the partition (to achieve this, replace the smooth transversals used in [5] by Hölder transversals of the form $T_x = \{z \mid z \in W^s_{\mathrm{loc}}(y),\ y \in W^u_{\mathrm{loc}}(x)\}$). Refine the partition so that the temporal distance function is identically zero on each rectangle. The vanishing of the temporal distance function means that the stable and unstable foliations of the flow commute over each rectangle, that is, the rectangles are also spanned by the unstable foliation. This implies that the roof function is locally constant along the unstable foliation as well, proving the claim. □

Corollary 3.4. *If the basic set Λ has good asymptotics (in the sense of Definition 2.1) then the temporal distance function is not locally constant.*

Proof. If the temporal distance function is locally constant then, by Proposition 3.3, Λ is a suspension with locally constant roof function. Therefore the sequence $(\tau(N))$ of periods in (2.1) satisfies $\tau(N + 1) - \tau(N) = \tau_0$ for all sufficiently large N and so Λ does not have good asymptotics. □

The following result is a slight modification of Dolgopyat [14, Theorem 3].

Lemma 3.5. *Let Λ be a hyperbolic attractor such that there exist $x, y \in U_\Lambda$ such that $\Delta(x, y) \neq 0$. Then Λ is rapid mixing.*

Proof. Set $z = [x, y]_s$. Clearly $\Delta(z, y) = 0$. Since Λ is an attractor, $W^{uc}(x) \subset \Lambda$. Consider a path $\alpha \in [0, 1] \mapsto x^\alpha \in W^{uc}_\epsilon(x) \subset \Lambda$ joining x to z. By the intermediate value theorem, Proposition 3.2 implies that $\alpha \mapsto \Delta(x^\alpha, y)$ contains a nontrivial interval. The claim then follows from [15, Theorem 6], which states that for flows that are not rapid mixing, the range of the temporal distance function has zero lower box counting dimension. (See also [14] and [17, Theorem 9.3].) □

Proof of Theorem 1.6(b). We only have to show that the hypotheses of Lemma 3.5 hold for a C^1-open, C^r-dense set of attractors in $\mathcal{A}_r(M)$. The openness follows from Proposition 3.2. The density follows from Lemma 2.2 and Corollary 3.4 (if $r < 2$, first approximate the flows by smoother ones). □

Remarks 3.6. (1) It follows from the proof of Theorem 1.6(b), see also [8, 9], that joint nonintegrability of the stable and unstable foliations is a C^1-open and C^r-dense property for transitive C^r Anosov flows. It is well-known that joint nonintegrability implies mixing, but the converse remains an open question (as discussed in Remark 1.10).
(2) Parts (b) of Theorems 1.3 and 1.6 require only the density part of Lemma 2.2.

4. Openness and density of good asymptotics

In this section, we prove Lemma 2.2, thus showing that there is an open and dense set of Axiom A flows with good asymptotics.

The sequence of periodic points $\{p_N\}$ implicit in Lemma 2.2 is constructed in subsection 4.1. The calculations depend on whether the eigenvalues of a certain linear map are real or complex. Focusing first on the case of real eigenvalues, we formulate Lemma 4.2 which gives the required estimates on the periods of the periodic points p_N. Equation (2.1) and Lemma 2.2 are immediate consequences. Lemma 4.2 is proved in subsection 4.2. In subsection 4.3, we indicate the modifications that are required when there are complex eigenvalues.

4.1. **Construction of the periodic point sequence.**

In this section we give the construction of the sequence (p_N) used in Definition 2.1.

Local sections for a flow containing a transverse homoclinic orbit. Let $\Gamma \subset \Lambda$ be a periodic orbit for the C^r flow Φ_t, $r \geq 2$, and fix $p \in \Gamma$. Assume that $x_H \in W^s_{\mathrm{loc}}(p)$ is a transverse homoclinic point for Γ. Let Σ be a smooth local transverse cross section to the flow such that $\Gamma \cap \Sigma = \{p\}$. Choose an open neighborhood Σ_1 of p in Σ such that the Poincaré return map $\Psi : \Sigma_1 \to \Sigma$ is well-defined and C^r. Modifying and extending Σ_1, Σ away from p, we may suppose that the Ψ-orbit of x_H is contained in Σ and so x_H is a transverse homoclinic point for the fixed point p of Ψ. The closure of the Ψ-orbit of x_H is a compact hyperbolic invariant subset of Σ_1. The first return time to Σ determines a C^r map $f : \Sigma_1 \to \mathbb{R}$ such that $\Psi(x) = \Phi_{f(x)}(x)$, $x \in \Sigma_1$.

We may choose a C^1-open neighborhood $\mathcal{U}$ of $\Phi_t \in \mathcal{F}^r(M)$, such that Σ_1, Σ define a local section for flows $\Phi'_t \in \mathcal{U}$ and the properties described above continue to hold for Φ'_t. More precisely, for each $\Phi'_t \in \mathcal{U}$, there exists a periodic orbit Γ' such that $\Gamma' \cap \Sigma = \{p'\}$, the Poincaré return map $\Psi' : \Sigma_1 \to \Sigma$ is well-defined with a homoclinic point $x'_H \in \Sigma_1$, and the closure of the Ψ'-orbit of x_H is a compact invariant hyperbolic subset of Σ_1. Furthermore, p' and x'_H depend continuously on Φ'_t, C^1-topology, and Ψ' and $f' : \Sigma_1 \to \mathbb{R}$ depend continuously on Φ'_t, C^s-topology, $1 \leq s \leq r$.

Nondegeneracy conditions on Ψ. We shall need to assume a number of nondegeneracy conditions on the closure of the Ψ-orbit of x_H. These are labeled (N1)–(N4) below.

Let $D\Psi(p)$ denote the differential of Ψ at p, with eigenvalues μ_i, λ_j where

$$|\mu_S| \leq \cdots \leq |\mu_1| < 1 < |\lambda_1| \leq \cdots \leq |\lambda_T|.$$

Define

$$\gamma = \max\{|\mu_1|, |\lambda_1|^{-1}\} \in (0, 1).$$

We assume

(N1) If ν_i and ν_j are distinct eigenvalues of $D\Psi(p)$ which are not complex conjugates, then $|\nu_i| \neq |\nu_j|$.

(N2) $|\nu_i\nu_j| \neq |\nu_k|$ for all eigenvalues ν_i, ν_j, ν_k of $D\Psi(p)$.

It follows from (N1) that the eigenvalues of $D\Psi(p)$ are distinct and $D\Psi(p)$ is semisimple. Since we are assuming Φ_t, and therefore Ψ, is at least C^2, it follows from (N2) and Belickii's linearization theorem [2, 3] that Ψ is C^1-linearizable at p.

Since Ψ is C^r, there are C^r local stable and unstable manifolds through p. We use these invariant manifolds as the basis for a local C^r-coordinate system at p. Thus we regard p as the origin of the vector space $\mathbb{R}^n = \mathbb{E}^s \oplus \mathbb{E}^u$ with the local stable (respectively, unstable) manifold through p contained in $\mathbb{E}^s$ (respectively, $\mathbb{E}^u$). We choose coordinates on $\mathbb{E}^s, \mathbb{E}^u$ so that $D\Psi(p) = \boldsymbol{\mu} \oplus \boldsymbol{\lambda}$ is in real Jordan normal form (1×1 blocks for real eigenvalues, 2×2 blocks for complex eigenvalues). Let $x_H = (A, 0) \in \mathbb{E}^s$ be the transverse homoclinic point for p. Let $\tilde{x}_H = (0, B) \in \mathbb{E}^u$ be the point corresponding to x_H, now regarded as lying on the unstable manifold of p — see Figure 1. Note that the forward orbit of x_H is contained in $\mathbb{E}^s$, while the backward orbit of $\tilde{x}_H$ is contained in $\mathbb{E}^u$, and that we regard x_H and $\tilde{x}_H$ as identified. We assume there exists $C > 0$ such that

(N3) $|\Psi^n(x_H)\mu_1^{-n}|, |\Psi^{-n}(\tilde{x}_H)\lambda_1^n| \geq C$, all $n \geq 0$.

Another way of viewing (N3) is to note that by (N2) we may C^1 linearize Ψ. If, in the linearized coordinates, $A = (A_1, \ldots, A_S)$, $B = (B_1, \ldots, B_T)$, then $(N3)$ is equivalent to requiring $A_1, B_1 \neq 0$.

Let W_A and W_B be neighborhoods of x_H and $\tilde{x}_H$ chosen so that the orbit of x_H intersects W_A and W_B only in the points x_H and $\tilde{x}_H$. We regard W_A and W_B as identified (in the ambient manifold). Choose an open set $\hat{K}$ disjoint from W_A and W_B, such that $K = \hat{K} \cup W_A \cup W_B$ contains p and the homoclinic orbit through x_H. We may choose K so that $\Psi(W_A) \subset \hat{K}$ and $\Psi^{-1}(W_B) \subset \hat{K}$.

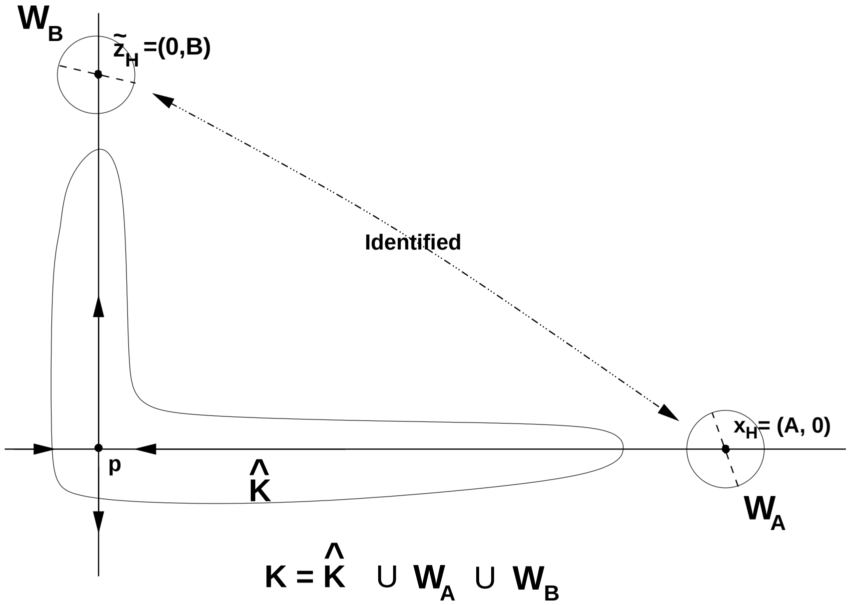

FIGURE 1. Basic local setup near the Ψ-orbit of x_H

From now on, we regard Ψ as defined on K with the understanding that if $z \in K$ then $\Psi^n(z)$ is defined provided that the iterates of z up to and including $\Psi^n(z)$ all lie in

K. Henceforth all our computations, perturbations and estimates will be done inside K. Of course, everything translates back to the ambient manifold M and we may regard K (with W_A, W_B identified) as an open subset of Σ_1. In particular, C^r functions $f : \Sigma_1 \to \mathbb{R}$ determine C^r functions on K, $r \geq 0$. The converse also holds providing we take account of the identification of W_A and W_B.

We shall also assume $|\mu_1| \neq |\lambda_1|^{-1}$. Since the case $|\mu_1| < |\lambda_1|^{-1}$ follows from $|\mu_1| > |\lambda_1|^{-1}$ by time-reversal, it is no loss of generality to write our final assumption as

(N4) $|\mu_1| > |\lambda_1|^{-1}$.

It follows that $|\mu_1 \lambda_j| > 1$ for all j.

Denote the eigenspace associated to μ_1 by $\mathbb{E}_1$. If μ_1 is real, $\mathbb{E}_1$ is one-dimensional, and if μ_1 is complex, then $\mathbb{E}_1$ is a two-dimensional $D\Psi(p)$-invariant real subspace of $\mathbb{E}^s$. In the latter case, there is a natural choice of complex structure on $\mathbb{E}_1$ so that $\boldsymbol{\mu}|\mathbb{E}_1$ is $\mathbb{C}$-linear and $\boldsymbol{\mu}(u) = \mu_1 u$ for $u \in \mathbb{E}_1$. We denote the $\mathbb{E}_1$ component of $X \in \mathbb{E}^s$ by X_1 and regard X_1 as a complex number. Note that if instead of μ_1, we had used $\bar{\mu}_1$, then we would obtain the conjugate complex structure on $\mathbb{E}_1$. For this reason, we make a fixed choice of eigenvalue μ_1 from the complex conjugate pair $\{\mu_1, \bar{\mu}_1\}$. Similar comments and conventions apply to all of the real eigenspaces associated to complex eigenvalues of $D\Psi(p)$.

We remark that conditions (N1)–(N4) are open in the C^1-topology (on $\mathcal{F}^r(M)$) and, allowing both inequalities in (N4), dense in the C^r-topology. The open neighborhood $\mathcal{U}$ of Φ_t described above may be chosen so that all of the constructions and conventions we have given above continue to hold for flows Φ_t' lying in $\mathcal{U}$. Let $\mathcal{V}$ be the corresponding set of C^r diffeomorphisms $\Psi : \Sigma_1 \to \Sigma$.

Lemma 4.1. *Let $\Psi \in \mathcal{V}$. There exists $N_0 \geq 1$ and a sequence of periodic points $p_N \to x_H$, $N \geq N_0$, such that p_N is of period N and, in the coordinates defined above, $p_N = \Psi^N p_N$ has the representations*

$$p_N = \Big(A + C(\mu_1^N) + o(\gamma^N)\,,\, O(\gamma^N)\Big) \;\; \text{on } W_A,$$

$$\Psi^N p_N = \Big(O(\gamma^N)\,,\, B + D(\mu_1^N) + o(\gamma^N)\Big) \;\; \text{on } W_B,$$

where A, B, C, D are constants that depend continuously on Ψ (C^2 topology). Here, $C : \mathbb{E}_1 \to \mathbb{E}^s$, $D : \mathbb{E}_1 \to \mathbb{E}^u$ are $\mathbb{R}$-linear maps, and C is injective.

Proof. It follows from condition (N2) and Belickii's linearization theorem [3] that Ψ can be C^1-linearized in a neighborhood of p and that the linearization depends continuously on Ψ in the C^2 topology (in fact in the $C^{1,1}$ topology). After C^1-linearizing, we may suppose that Ψ coincides with the linear map $D\Psi(p)$ in a neighborhood of p. Using Ψ, we may extend the domain of the linearized coordinates along $\mathbb{E}^s$ and $\mathbb{E}^u$. Hence we may shrink K so that in the linearized coordinates $\Psi|(\hat{K} \cup W_A) = D\Psi(p)$ and $\Psi^{-1}|(\hat{K} \cup W_B) = D\Psi(p)^{-1}$. The nonlinearity of Ψ is pushed into the C^1 diffeomorphism identifying W_A and W_B. In the operator norm derived from the induced Euclidean norms on $\mathbb{E}^s$ and $\mathbb{E}^u$, the linear maps $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}^{-1}$ are contractions with $\|\boldsymbol{\mu}\|, \|\boldsymbol{\lambda}^{-1}\| \leq \gamma$.

Set $a_N = (A, \boldsymbol{\lambda}^{-N} B)$. For N sufficiently large, $(\boldsymbol{\mu}^j A, \boldsymbol{\lambda}^{j-N} B) \in K$, $0 \leq j \leq N$, and so $\Psi^N(a_N) = (\boldsymbol{\mu}^N A, B)$.

Note that $a_N \to x_H$ and $\Psi^N a_N \to \tilde{x}_H \sim x_H$ in K as $N \to \infty$. Moreover, setting $M_1 = |A| + |B|$ we have that $\{\Psi^j(a_N) \mid j = 0, \ldots, N\}$ is a periodic $M_1 \gamma^N$ pseudo-orbit in K (see [20, §18.1]). It follows from the Anosov Closing Lemma [20, §6.4] that there is a constant $M_2 > 0$ such that for N sufficiently large, there is a periodic point $p_N \in K$ of period N such that $|\Psi^j(p_N) - \Psi^j(a_N)| < M_2\gamma^N$ for $0 \le j \le N$. Taking $j = 0$, we may write $p_N = (A + C_N\gamma^N, \boldsymbol{\lambda}^{-N}B + E_N\gamma^N)$, where $|C_N|, |E_N| \le M_2$. Since $\Psi^N(p_N) = (\boldsymbol{\mu}^N(A+C_N\gamma^N), B+\boldsymbol{\lambda}^N E_N\gamma^N)$, it follows taking $j = N$ that we can write $\boldsymbol{\lambda}^N E_N = D_N$, where $|D_N| \le M_2$.

Hence, in the linearized coordinates, we have periodic points $\Psi^N p_N = p_N$ for N sufficiently large, with

$$p_N = \Big(A + C_N\gamma^N\,,\, \boldsymbol{\lambda}^{-N}(B + D_N\gamma^N)\Big),\;\; \Psi^N p_N = \Big(\boldsymbol{\mu}^N(A + C_N\gamma^N)\,,\, B + D_N\gamma^N\Big).$$

The identification between W_A and W_B is given by a C^1-diffeomorphism χ. Since $\Psi^N(p_N) = p_N$, we have

$$\chi(A + \gamma^N C_N, \boldsymbol{\lambda}^{-N}(B + \gamma^N D_N)) = (\boldsymbol{\mu}^N(A + \gamma^N C_N), B + \gamma^N D_N).$$

Writing $\chi(A + x, y) = (0, B) + (E_{11}x + E_{12}y, E_{21}x + E_{22}y) + o(|(x, y)|)$, we obtain

$$E_{11}(\gamma^N C_N) = \mu_1^N A_1 + o(\gamma^N), \qquad E_{21}(\gamma^N C_N) = \gamma^N D_N + o(\gamma^N),$$

where $\mu_1^N A_1$ is defined by complex multiplication in case μ_1 is complex (see the remarks above). It follows by the transversality of the stable and unstable manifolds at x_H that $E_{11} : \mathbb{E}^s \to \mathbb{E}^s$ is non-singular. Set $L = E_{11}^{-1}$. Define $C(u) = L(uA_1)$, $u \in \mathbb{E}_1$, and $D = E_{21}C$. It follows that in the linearized coordinates

$$p_N = \Big(A + C(\mu_1^N) + o(\gamma^N)\,,\, \boldsymbol{\lambda}^{-N}(B + D(\mu_1^N) + o(\gamma^N))\Big)$$

Since the change of coordinates is C^1, we have the required expression for p_N in the original coordinate system (with different values of A, C). Similarly for $\Psi^N p_N$. □

Associated to each flow $\Phi_t \in \mathcal{U}$ are the C^r pair (Ψ, f) where $\Psi \in \mathcal{V}$ and $f : \Sigma_1 \to \mathbb{R}$. Set $\tilde{f} = f - f(p) \in C^r(\Sigma_1)$ and define

$$A_N(\Psi, f) = \sum_{i=0}^{N-1} \tilde{f}(\Psi^i p_N) - \sum_{i=-\infty}^{\infty} \tilde{f}(\Psi^i x_H). \tag{4.1}$$

The bi-infinite sum converges since $\tilde{f}(p) = 0$ and f is C^1 (it is enough that f be Hölder). We suppress the dependence of A_N on the choices of p, x_H, p_N and local section $\Sigma_1 \subset \Sigma$.

If we let τ_0 denote the period of the Φ_t-orbit through p and $\tau(N)$ denote the period of the Φ_t-orbit through p_N, then

$$\tau(N) = N\tau_0 + \kappa + A_N(\Psi, f),$$

where $\kappa = \sum_{i=-\infty}^{\infty} \tilde{f}(\Psi^i x_H)$. In order to show that the basic set Λ for Φ_t has good asymptotics, we need to obtain precise asymptotic estimates of $A_N(\Psi, f)$.

By Sternberg's linearization theorem [2, 32], there is a C^r-dense subset $\mathcal{V}_\infty \subset \mathcal{V}$ consisting of C^∞ maps that are C^2-linearizable at p. We carry out our estimates on a C^2-open neighborhood of $\mathcal{V}_\infty$ inside $\mathcal{V}$. We define the distance $\|\Psi_1 - \Psi_2\|_r$ between $\Psi_1, \Psi_2 \in \mathcal{V}$ to be $\max_{|\alpha| \le r} |\partial^\alpha \Psi_1 - \partial^\alpha \Psi_2|_\infty$.

We begin by making the simplifying assumption that the eigenvalues of $D\Psi(p)$ are real. In the next lemma we write $\mu_{1,\Psi}$ and γ_Ψ to emphasize the dependence of the eigenvalues on Ψ.

Lemma 4.2. *Let $r \geq 2$. Let $\Psi_0 \in \mathcal{V}_\infty$ and assume that the the eigenvalues of $D\Psi_0(p)$ are real. Then we may find a C^2-open neighborhood $\mathcal{V}_0$ of Ψ_0 in $\mathcal{V}$ and a continuous (linear) map $E(\Psi_0, \cdot) : C^2(\Sigma_1) \to \mathbb{R}$ such that*

(1) *$A_N(\Psi_0, f) = E(\Psi_0, f)\mu_{1,\Psi_0}^N + o(\gamma_{\Psi_0}^N)$, for all $f \in C^r(\Sigma_1)$.*
(2) *$E(\Psi_0, f) \neq 0$ for a C^r-dense set of $f \in C^r(\Sigma_1)$.*
(3) *$A_N(\Psi, f) = E_N(\Psi, f)\mu_{1,\Psi}^N + o(\gamma_\Psi^N)$, for all $(\Psi, f) \in \mathcal{V}_0 \times C^r(\Sigma_1)$, where $|E_N(\Psi, f) - E(\Psi_0, f)| = O(\|f\|_2 \|\Psi - \Psi_0\|_2)$ uniformly in N.*

We indicate how Lemma 2.2 follows from Lemma 4.2 in the real eigenvalue case.

Proof of Lemma 2.2, real eigenvalue case. Let $\Phi_t \in \mathcal{A}_r(M)$ have nontrivial hyperbolic basic set Λ containing the transverse homoclinic point x_H. Associated to Φ_t is the C^r Poincaré map $\Psi_0 : \Sigma_1 \to \Sigma$ and C^r map $f_0 : \Sigma_1 \to \mathbb{R}$. After a C^r small perturbation of Φ_t, we may suppose that Ψ_0 lies in $\mathcal{V}_\infty$. It follows from Lemma 4.2(3) that for all C^r pairs (Ψ, f) sufficiently C^2-close to (Ψ_0, f_0), there is a bounded sequence $E_N(\Psi, f)$ such that $A_N(\Psi, f) = E_N(\Psi, f)\mu_{1,\Psi}^N + o(\gamma_\Psi^N)$. Hence equation (2.1) is valid for all C^r flows sufficiently C^2-close to Φ_t with $\theta = \varphi_N \equiv 0$. (Note that E_N here and in (2.1) differ by a factor of $(-1)^N$ when $\mu_{1,\Psi} < 0$.)

By Lemma 4.2(1), we can write $A_N(\Psi_0, f_0) = E(\Psi_0, f_0)\mu_{1,\Psi_0}^N + o(\gamma_{\Psi_0}^N)$. It follows from Lemma 4.2(2) that, after a C^r small perturbation of f_0, we may suppose that $E(\Psi_0, f_0) \neq 0$.

By continuity of $E(\Psi_0, \cdot)$, it follows that $E(\Psi_0, f)$ is bounded away from zero for all $f \in C^r(\Sigma_1)$ sufficiently C^2-close to f_0. By Lemma 4.2(3), $E_N(\Psi, f)$ is bounded away from zero, uniformly in N, for all C^r pairs (Ψ, f) sufficiently C^2-close to (Ψ_0, f_0). Therefore the good asymptotics property holds for all C^2-small perturbations of the flow corresponding to (Ψ_0, f_0). □

In the next subsection, we prove Lemma 4.2 by carrying out explicit and quite lengthy calculations. However, we should emphasize that the proof of density in Lemma 2.2 is somewhat simpler and moreover sufficient for the results on attractors in Section 3 (though *not* for the results on general Axiom A flows). Thus, in order to prove density, it suffices to verify that

(1′) $A_N(\Psi_0, f) = E(\Psi_0, f)\mu_{1,\Psi_0}^N + o(\gamma_{\Psi_0}^N)$, where $E(\Psi_0, f) \in \mathbb{R}$, for all $(\Psi_0, f) \in \mathcal{V}_\infty \times C^r(\Sigma_1)$.
(2′) For any $\Psi_0 \in \mathcal{V}_\infty$ and any $\epsilon > 0$, there exists $f \in C^r(\Sigma_1)$ with $\|f\|_r < \epsilon$ such that $E(\Psi_0, f) \neq 0$.

The proof of (2′) is particularly simple as f can be chosen to be supported in a small neighborhood of x_H so that $A_N(\Psi_0, f) = f(p_N) - f(x_H)$. For the proof of (1′), one can work in a C^2-linearized coordinate system (see also the following subsection).

4.2. **Proof of Lemma 4.2.** Let $\Psi_0 \in \mathcal{V}_\infty$, so that Ψ_0 can be C^2-linearized at its fixed point p. Fix coordinates on $K \subset \mathbb{R}^n = \mathbb{E}^s \oplus \mathbb{E}^u$ in which Ψ_0 is given by the diagonal matrix

$$\Psi_0(x, y) = \big(\boldsymbol{\mu}_0 x, \boldsymbol{\lambda}_0 y\big)$$

(recall that the nonlinearity of Ψ_0 is concentrated in the diffeomorphism identifying W_B to W_A). Since local stable and unstable manifolds through a hyperbolic point depend continuously on the flow [18, Theorem 6.23], for any C^r-diffeomorphism $\Psi : \Sigma_1 \to \Sigma$

that is C^2-close to Ψ_0, there is a C^2 coordinate map $h_\Psi : \Sigma_1 \to \mathbb{E}^s \oplus \mathbb{E}^u$, which depends continuously on Ψ (C^2 topology), such that through this identification

$$\Psi(x,y) = \big(\boldsymbol{\mu}(I + \mathbf{a}(x,y))x\,,\, \boldsymbol{\lambda}(I+\mathbf{b}(x,y))y\big). \tag{4.2}$$

Here, $\boldsymbol{\mu} = \operatorname{diag}(\mu_1,\dots,\mu_p)$, $\boldsymbol{\lambda} = \operatorname{diag}(\lambda_1,\dots,\lambda_q)$ are diagonal matrices, and $\mathbf{a} : \mathbb{E}^s \times \mathbb{E}^u \to L(\mathbb{E}^s,\mathbb{E}^s)$, $\mathbf{b} : \mathbb{E}^s \times \mathbb{E}^u \to L(\mathbb{E}^u,\mathbb{E}^u)$ are C^1 matrix-valued maps, with $\mathbf{a}(0,0) = \mathbf{b}(0,0) = 0$. The maps $\mathbf{a}$, $\mathbf{b}$ (C^1-topology) and matrices $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$ depend continuously on Ψ (C^2 topology). Similarly, we may write $\Psi^{-1}(x,y) = \big(\boldsymbol{\mu}^{-1}(I+\tilde{\mathbf{b}}(x,y))x\,,\, \boldsymbol{\lambda}^{-1}(I+\tilde{\mathbf{a}}(x,y))y\big)$, with $\tilde{\mathbf{a}}(0,0) = \tilde{\mathbf{b}}(0,0) = 0$.

We may choose $\tilde{C}, \epsilon_0 > 0$ such that if $\epsilon \in (0,\epsilon_0]$ and $\|\Psi - \Psi_0\|_2 < \tilde{C}\epsilon$, then $\|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|$, $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\|$, $\|\mathbf{a}\|_1$, $\|\mathbf{b}\|_1$, $\|\tilde{\mathbf{a}}\|_1$, $\|\tilde{\mathbf{b}}\|_1$, $\|\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}\|$, $\|\boldsymbol{\lambda}^{-1} - \boldsymbol{\lambda}_0^{-1}\| \le \epsilon$.

We begin by obtaining more accurate estimates of the periodic points p_N. This is done in Lemmas 4.3, 4.4, 4.5 and 4.6. Using these estimates, we compute $E_N = E_N(\Psi, f)$ in Propositions 4.7, 4.8 and 4.9, and then complete the proof of Lemma 4.2.

Set $\hat{\boldsymbol{\mu}} = \mu_1^{-1}\boldsymbol{\mu}$. For $n \ge 0$, define $Q_n \in L(\mathbb{E}^s,\mathbb{E}^s)$ by $Q_0 = I$ and

$$Q_n = \prod_{m=0}^{n-1} \hat{\boldsymbol{\mu}}(I + \mathbf{a}(\Psi^m x_H)),\ n \ge 1.$$

Here, as elsewhere in this section, we adopt the convention that $\prod_{m=a}^{b} y_m = y_b \cdots y_a$. It follows from the definition of Q_n that $\Psi^n x_H = (\mu_1^n Q_n A, 0)$, $n \ge 0$.

Similarly, we set $\hat{\boldsymbol{\lambda}} = \lambda_1^{-1}\boldsymbol{\lambda}$ and for $n \ge 0$ define $R_n \in L(\mathbb{E}^u,\mathbb{E}^u)$ by $R_0 = I$ and

$$R_n = \prod_{m=0}^{n-1} \hat{\boldsymbol{\lambda}}^{-1}(I + \tilde{\mathbf{a}}(\Psi^{-m}\tilde{x}_H)),\ n \ge 1.$$

Note that $\Psi^{-n}\tilde{x}_H = (0, \lambda_1^{-n} R_n B)$, $n \ge 0$.

Choose $\epsilon \in (0,\epsilon_0]$ sufficiently small so that $\beta = \gamma(1+\epsilon) < 1$. Define

$$K = [\Pi_{m=0}^{\infty}(1 + 2\epsilon(|A| + |B|)\beta^m)]^2,$$

so $1 \le K < \infty$.

Lemma 4.3. *For all $n \ge 1$,*

$$\|Q_n\|,\ \|R_n\| \le K, \qquad |\Psi^n x_H| \le K|A||\mu_1|^n,\ |\Psi^{-n}\tilde{x}_H| \le K|B||\lambda_1|^{-n}.$$

Proof. It is immediate that $\|Q_n\| \le (1+\epsilon)^n$. In particular, $|\Psi^n x_H| \le \beta^n|A|$. But then $\|Q_n\| \le \prod_{m=0}^{n-1}(1 + \|\mathbf{a}\|_1|A|\beta^m) \le K$. Hence $|\Psi^n x_H| \le K|A||\mu_1|^n$. Similar arguments give the required estimates on $\|R_n\|$ and $|\Psi^{-n}\tilde{x}_H|$. □

Lemma 4.4. *We may choose N_0 such that if $N \ge N_0$ and $0 \le n \le N$, then*

$$\Psi^n p_N = (\mu_1^n Q_{N,n}[A + C\mu_1^N + o(\gamma^N)]\,,\, \lambda_1^{n-N} R_{N,N-n}[B + D\mu_1^N + o(\gamma^N)]),$$

where $\|Q_{N,n}\|$, $\|R_{N,N-n}\| \le K$.

Proof. We use the representations of p_N and $\Psi^N p_N$ given in Lemma 4.1. Working forwards from p_N and backwards from $\Psi^N p_N$, we obtain

$$Q_{N,n} = \prod_{m=0}^{n-1} \hat{\boldsymbol{\mu}}(I + \mathbf{a}(\Psi^m p_N)), \qquad R_{N,n} = \prod_{m=0}^{n-1} \hat{\boldsymbol{\lambda}}^{-1}(I + \tilde{\mathbf{a}}(\Psi^{-m} p_N)).$$

(Our convention is that $Q_{N,0}$, $R_{N,0}$ are the identity maps on $\mathbb{E}^s$, $\mathbb{E}^u$ respectively.) Just as in the previous proposition, we have $\|Q_{N,n}\| \leq (1+\epsilon)^n$ and $\|R_{N,n}\| \leq (1+\epsilon)^n$. Hence, for sufficiently large N, $|\Psi^n p_N| \leq 2|A|\beta^n + 2|B|\beta^{N-n}$. It follows that $\|Q_{N,n}\|$ can be bounded by a product $\prod_{m=0}^{n-1}(1+\ell\beta^m+\ell\beta^{N-m})$, where $\ell = 2(|A|+|B|)\epsilon$. Since $(1+\ell\beta^m+\ell\beta^{N-m}) \leq (1+\ell\beta^m)(1+\ell\beta^{N-m})$, it follows easily that

$$\prod_{m=0}^{n-1}(1+\ell\beta^m+\ell\beta^{N-m}) \leq [\prod_{m=0}^{N}(1+\ell\beta^m)]^2 \leq K$$

A similar estimate applies for $R_{N,n}$. □

Lemma 4.5. *There exists $J>0$ such that*

$$\|Q_n - \hat{\boldsymbol{\mu}}^n\| \leq \epsilon J, \quad \|Q_{N,n} - \hat{\boldsymbol{\mu}}^n\| \leq \epsilon J, \quad \|R_n - \hat{\boldsymbol{\lambda}}^{-n}\| \leq \epsilon J, \quad \|R_{N,n} - \hat{\boldsymbol{\lambda}}^{-n}\| \leq \epsilon J,$$

for all $0 \leq n \leq N$.

Proof. We prove the estimate for $Q_{N,n}$. The result for Q_n is simpler, and $R_{N,n}$, R_n are treated similarly.

Using the definition of $Q_{N,n}$ and the estimates of Lemma 4.4 we obtain that

$$\|Q_{N,n} - \hat{\boldsymbol{\mu}}^n\| = \|\sum_{m=0}^{n-1}\hat{\boldsymbol{\mu}}^{n-m}\mathbf{a}(\Psi^m p_N)\prod_{\ell=0}^{m-1}\hat{\boldsymbol{\mu}}(I+\mathbf{a}(\Psi^\ell p_N))\|$$

$$\leq K\sum_{m=0}^{n-1}\|\hat{\boldsymbol{\mu}}\mathbf{a}(\Psi^m p_N))\| \leq \epsilon K\sum_{m=0}^{n-1}|\Psi^m p_N| \leq \epsilon K_1\sum_{m=0}^{n-1}(|\mu_1^m| + |\lambda_1^{m-N}|) \leq \epsilon J,$$

where $J = K_1((1-|\mu_1|)^{-1} + (1-|\lambda_1|^{-1})^{-1})$. □

Lemma 4.6. *There exists $N_1 \geq N_0$ and $L>0$ such that for $\epsilon>0$ sufficiently small, $N \geq N_1$, and $0 \leq n \leq N$*

$$\|Q_{N,n} - Q_n\| \leq \epsilon L(\gamma^N + |\lambda_1|^{n-N}), \qquad \|R_{N,n} - R_n\| \leq \epsilon L(\gamma^N + \gamma^{N-n}).$$

Proof. We prove only the first formula, the proof of the second being similar.

It follows easily from the definitions of $Q_{N,n}$ and Q_n, together with the estimates of Lemmas 4.3, 4.4, that

$$\|Q_{N,n} - Q_n\| \leq K\sum_{m=0}^{n-1}|\mathbf{a}(\Psi^m p_N) - \mathbf{a}(\Psi^m x_H)| \leq \epsilon K\sum_{m=0}^{n-1}|\Psi^m p_N - \Psi^m x_H|.$$

The claim is true for $n=0$. Assume inductively that the lemma holds for $m<n$. Then for $N \geq N_1$ large enough (independent of n), there is a constant $K_1>0$ such that

$$\begin{aligned}|\Psi^m p_N - \Psi^m x_H| &= \big|\big(\mu_1^m[(Q_{N,m}-Q_m)A + Q_{N,m}(C\mu_1^N + o(\gamma^N))],\\ &\qquad \lambda_1^{m-N}R_{N,N-m}[B + D\mu_1^N + o(\gamma^N)]\big)\big|\\ &\leq K_1|\mu_1|^m(L\epsilon+1)(\gamma^N + |\lambda_1|^{m-N}) + K_1|\lambda_1|^{m-N}\\ &\leq K_1(L\epsilon+1)\gamma^N(|\mu_1|^m + \rho^{N-m}) + K_1|\lambda_1|^{m-N},\end{aligned}$$

where $\rho = |\mu_1\lambda_1|^{-1} \in (0,1)$. Hence we can choose a constant $K_2>0$, independent of n, N, such that

$$\|Q_{N,n} - Q_n\| \leq \epsilon K_2(L\epsilon+1)(\gamma^N + |\lambda_1|^{n-N}).$$

Therefore the induction step works with $L = 2K_2$, $\epsilon < 1/L$. $\square$

We are now in a position to estimate $A_N(\Psi, f)$ for $f \in C^r(\Sigma_1)$. It is convenient to split up $f : \Sigma_1 \to \mathbb{R}$ into pure terms $x_i\alpha(x)$, $y_j\alpha(y)$, and mixed terms $x_iy_jH(x,y)$. We compute $A_N(\Psi, f)$ for mixed terms f in Proposition 4.7 and pure x-terms in Proposition 4.8. The similar calculations for pure y-terms are stated without proof in Proposition 4.9.

Proposition 4.7 (Mixed terms). *Let $f(x,y) = x_iy_jH(x,y)$ where $H : \mathbb{E}^s \times \mathbb{E}^u \to \mathbb{R}$ is C^0. Suppose that $\|\Psi - \Psi_0\|_2 \le \tilde{C}\epsilon$ where Ψ_0 is linear. Then $A_N(\Psi, f) = E_N\mu_1^N + o(\gamma^N)$ where $E_N = E + O(\epsilon\|f\|_2)$ and*

$$E = \begin{cases} A_1B_j\sum_{k=1}^{\infty}(\mu_1\lambda_j)^{-k}H(\Psi^{-k}x_H) & \text{if } i = 1, \\ 0 & \text{if } i \ge 2. \end{cases}$$

Proof. We have $A_N(\Psi, f) = \sum_{n=0}^{N-1} f(\Psi^n p_N) = \sum_{n=0}^{N-1} t_n$, where

$$t_n = \mu_1^n(Q_{N,n}[A + C\mu_1^N + o(\gamma^N)])_i\lambda_1^{n-N}(R_{N,N-n}[B + D\mu_1^N + o(\gamma^N)])_jH(\Psi^n p_N).$$

By Lemma 4.5, $Q_{N,n} = \hat{\boldsymbol{\mu}}^n + O(\epsilon)$ and $R_{N,N-n} = \hat{\boldsymbol{\lambda}}^{n-N} + O(\epsilon)$ uniformly in n, N, and so we have

$$t_n = \mu_i^n\lambda_j^{n-N}[A_iB_jH(\Psi^n p_N) + O(\gamma^N)] + O(\epsilon\|H\|_0)|\mu_1|^n|\lambda_1|^{n-N}.$$

Now $\sum_{n=0}^{N-1}\mu_i^n\lambda_j^{n-N} = O(|\mu_i|^N + |\lambda_j|^{-N})$. Hence $A_N(\Psi, f) = O(\epsilon\|H\|_0)\gamma^N + o(\gamma^N)$, $i \ge 2$.

If $i = 1$, write $\rho = (\mu_1\lambda_j)^{-1}$, $|\rho| < 1$, and set $k = N - n$. It follows that

$$t_n = \mu_1^N\rho^k[A_1B_jH(\Psi^{N-k}p_N) + O(\gamma^N) + O(\epsilon\|H\|_0)].$$

Also, $\Psi^{N-k}p_N = \Psi^{-k}p_N \to \Psi^{-k}\tilde{x}_H$ and H is continuous, so

$$\begin{aligned} A_N(\Psi, f) &= \mu_1^N\sum_{k=1}^{N}\rho^k[A_1B_jH(\Psi^{-k}p_N) + O(\gamma^N) + O(\epsilon\|H\|_0)] \\ &= [E + O(\epsilon\|H\|_0)]\mu_1^N + o(\gamma^N) = [E + O(\epsilon\|f\|_2)]\mu_1^N + o(\gamma^N), \end{aligned}$$

where $E = A_1B_j\sum_{k=1}^{\infty}\rho^kH(\Psi^{-k}x_H)$. $\square$

Proposition 4.8 (Pure x-terms). *Let $f(x,y) = x_i\alpha(x)$ where $\alpha : \mathbb{E}^s \to \mathbb{R}$ is C^1. Suppose that $\|\Psi - \Psi_0\|_2 \le \tilde{C}\epsilon$ where Ψ_0 is linear. Then $A_N(\Psi, f) = E_N\mu_1^N + o(\gamma^N)$ where $E_N = E + O(\epsilon\|f\|_1)$ and*

$$E = \begin{cases} \sum_{n=0}^{\infty}(df)_{\Psi^n x_H}(\boldsymbol{\mu}^nC) - A_1(1-\mu_1)^{-1}\alpha(0) & \text{if } i = 1, \\ \sum_{n=0}^{\infty}(df)_{\Psi^n x_H}(\boldsymbol{\mu}^nC) & \text{if } i \ge 2. \end{cases}$$

Proof. Ideas and notation already used in Proposition 4.7 will be used without comment. Write $A_N(\Psi, f) = \sum_{n=0}^{N-1}[f(\Psi^n p_N) - f(\Psi^n x_H)] - \sum_{n=N}^{\infty} f(\Psi^n x_H)$. The second summation for $A_N(\Psi, f)$ has n^{th} term

$$-\mu_1^n(Q_nA)_i\alpha(\Psi^n x_H) = -\mu_i^nA_i\alpha(\Psi^n x_H) + O(\epsilon\|\alpha\|_0)\gamma^n.$$

Hence the contribution to $A_N(\Psi, f)$ is

$$-\sum_{n=N}^{\infty}[\mu_i^nA_i\alpha(\Psi^n x_H) + O(\epsilon\|\alpha\|_0)\gamma^n] = -\mu_i^NA_i\sum_{n=0}^{\infty}\mu_i^n\alpha(\Psi^{n+N}x_H) + O(\epsilon\|f\|_1)\mu_i^N.$$

The contribution to E from this sum is zero if $i \geq 2$ and $-A_1(1-\mu_1)^{-1}\alpha(0)$ when $i=1$.

By Lemma 4.4, for $0 \leq n \leq N-1$, we have the following expression for the difference of the $\mathbb{E}^s$-components $(\Psi^n p_N)_x$ and $(\Psi^n x_H)_x$:

$$(\Psi^n p_N)_x - (\Psi^n x_H)_x = \mu_1^n(Q_{N,n}-Q_n)A + \mu_1^n(Q_{N,n}-\hat{\boldsymbol{\mu}}^n)(C\mu_1^N + o(\gamma^N)) + \boldsymbol{\mu}^n(C\mu_1^N + o(\gamma^N)).$$

It follows from Lemmas 4.5 and 4.6 that for sufficiently large N we have, for $0 \leq n \leq N-1$,

$$\|\mu_1^n(Q_{N,n} - \hat{\boldsymbol{\mu}}^n)\| \leq \gamma^n \epsilon J,$$
$$\|\mu_1^n(Q_{N,n} - Q_n)\| \leq \gamma^n \epsilon L(\gamma^N + |\lambda_1|^{n-N}) = \epsilon\gamma^N L(\gamma^n + \rho^{N-n}),$$

where $\rho = |\mu_1\lambda_1|^{-1} < 1$. Therefore

$$|(\Psi^n p_N)_x - (\Psi^n x_H)_x - \boldsymbol{\mu}^n C\mu_1^N| \leq \epsilon\gamma^N[L(\gamma^n + \rho^{N-n})|A| + \gamma^n J(|C|+1)] \tag{4.3}$$

for sufficiently large N. Hence

$$|(\Psi^n p_N)_x - (\Psi^n x_H)_x| = O(\gamma^N). \tag{4.4}$$

Since for C^2 functions u we have the estimate

$$|u(y) - u(x) - (du)_x(y-x)| \leq \frac{1}{2}\|du\|_{\mathrm{Lip}}|y-x|^2 \leq \frac{1}{2}\|u\|_2|y-x|^2,$$

and f depends only on the $\mathbb{E}^s$-component, we obtain from (4.4) that

$$|f(\Psi^n p_N) - f(\Psi^n x_H) - (df)_{\Psi^n x_H}\left((\Psi^n p_N)_x - (\Psi^n x_H)_x\right)| = O(\gamma^{2N}\|f\|_2).$$

It follows using (4.3) that

$$\begin{aligned}
\sum_{n=0}^{N-1}[f(\Psi^n p_N) - f(\Psi^n x_H)] &= \sum_{n=0}^{N-1}\left[(df)_{\Psi^n x_H}\left[(\Psi^n p_N)_x - (\Psi^n x_H)_x\right] + O(\gamma^{2N})\right] \\
&= \left(\sum_{n=0}^{N-1}(df)_{\Psi^n x_H}\left(\boldsymbol{\mu}^n C\mu_1^N\right)\right) + O\left(\|f\|_1\sum_{n=0}^{N-1}\epsilon\gamma^N(\gamma^n + \rho^{N-n})\right) + o(\gamma^N) \\
&= \left(\sum_{n=0}^{N-1}(df)_{\Psi^n x_H}\left(\boldsymbol{\mu}^n C\right) + O(\epsilon\|f\|_1)\right)\mu_1^N + o(\gamma^N),
\end{aligned}$$

and so contributes $\sum_{n=0}^{\infty}(df)_{\Psi^n x_H}(\boldsymbol{\mu}^n C) + O(\epsilon\|f\|_1)$ to E_N. □

Proposition 4.9 (Pure y-terms). *Let $f(x,y) = y_j\beta(y)$ where $\beta : \mathbb{E}^u \to \mathbb{R}$ is C^1. Suppose that $\|\Psi - \Psi_0\|_2 \leq \tilde{C}\epsilon$ where Ψ_0 is linear. Then $A_N(\Psi, f) = E_N\mu_1^N + o(\gamma^N)$ where $E_N = E + O(\epsilon\|f\|_1)$ and*

$$E = \sum_{n=1}^{\infty}(df)_{\Psi^{-n}x_H}\left(\boldsymbol{\lambda}^{-n}D\right).$$

Proof. The proof is similar to that of Proposition 4.8, except that there is no special case (all eigenvalues of $\boldsymbol{\lambda}^{-1}$ have absolute value less than γ) and the infinite sum starts at $n=1$ because of the convention regarding the identification of W_A and W_B. □

Remark 4.10. We explain here why these computations hold for a $C^{1,1}$ neighborhood of flows whose return map around the periodic orbit Γ (see beginning of Section 4.1) is C^2-linearizable and satisfies the non-degeneracy conditions (N1)-(N4). In the proof of Lemma 4.1, Belickii's C^1 linearization theorem holds in the $C^{1,1}$-topology. The subsequent estimates of the orbits of p_N and x_H depend only on $C^{0,1}$-bounds of $\mathbf{a}$, $\mathbf{b}$. The proof of Proposition 4.7 (mixed terms) can also be carried out in the $C^{1,1}$ setting; see [16, Lemma 4.13(1)]. Finally, the proofs of Propositions 4.8 and 4.9 are valid for $f \in C^{1,1}$.

Proof of Lemma 4.2. Let $\Psi \in \mathcal{V}$ be sufficiently C^2-close to $\Psi_0 \in \mathcal{V}_\infty$ and let $f \in C^r(\Sigma_1)$. Statements (1) and (2) of Lemma 4.2, and the continuity of $E(\Psi_0, \cdot)$, are immediate from the explicit formulas for E in Propositions 4.7, 4.8 and 4.9 (with $\epsilon = 0$).

The three previous propositions give $E_N(\Psi, f) = E(\Psi, f) + \epsilon_N$ where $\epsilon_N \to 0$ uniformly in N as $\|\Psi - \Psi_0\|_2 \to 0$. Moreover, it is clear from the explicit formulas that $E(\Psi, f) \to E(\Psi_0, f)$ as $\|\Psi - \Psi_0\|_2 \to 0$. This proves Lemma 4.2(3). $\square$

4.3. **Complex eigenvalues.** Finally, we indicate the changes that are required when $D\Psi(p)$ has complex eigenvalues. Suppose for simplicity that all the eigenvalues are complex. We then have $S+T$ real two-dimensional eigenspaces each of which admits a natural complex structure (see the remarks preceding Lemma 4.1). If we have associated real coordinates (u_j, v_j) on an eigenspace $\mathbb{E}_i$ and α, β are real functions, we may write $u_j\alpha + v_j\beta$ uniquely in the form $z_j a + \bar{z}_j \bar{a}$, where $z_j = u_j + \imath v_j$, and $a = (\alpha - \imath\beta)/2$. Similar formulas hold for mixed terms $x_j y_k H$. It follows just as before that we can write $f \in C^r(\Sigma_1)$, $f(0) = 0$, as a sum of real valued functions defined using complex coordinates.

We define the differential operators $\partial_{z_j} = \frac{1}{2}(\frac{\partial}{\partial u_j} - \imath\frac{\partial}{\partial v_j})$, $\partial_{\bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial u_j} + \imath\frac{\partial}{\partial v_j})$. With respect to these operators we have the usual derivative formula

$$a(z_0 + z) = a(z_0) + \sum\nolimits_j \Big(\partial_{z_j} a(z_0) z + \partial_{\bar{z}_j} a(z_0) \bar{z}\Big) + o(|z|).$$

Let $1 \le j \le S$. We define $\bar{C}_j : \mathbb{E}_1 \to \mathbb{E}_j$ by $\bar{C}_j(u) = \overline{C_j(u)}$, $u \in \mathbb{E}_1$, where C_j are the components of $C : \mathbb{E}_1 \to \mathbb{E}^s$. (In terms of coordinates on $\mathbb{E}_1, \mathbb{E}_j$, this amounts to multiplying the second row of the matrix of C_j by -1.) We similarly define $\bar{D}_j$, $1 \le j \le T$.

With these preliminaries out of the way, the computations used to prove Lemma 4.2 go through much as before. For $\Psi_0 \in \mathcal{V}_\infty$ we find that $A_N(\Psi_0, f) = \mathrm{Re}(\mathbf{E}(\mu_1^N)) + o(\gamma^N)$, where the $\mathbb{R}$-linear map $\mathbf{E} : \mathbb{E}_1 \to \mathbb{C}$ can be computed explicitly as in Subsection 4.2. In particular, $\mathbf{E}$ depends continuously on f and is typically nonvanishing. Since $\mu_1 = \gamma e^{\imath\theta}$ is complex, we may write $\mathrm{Re}(\mathbf{E}(\mu_1^N)) = E\gamma^N \cos(N\theta + \theta_0)$, where $\theta, \theta_0 \in [0, 2\pi)$ and $E \in \mathbb{R}$ with $\theta \ne 0, \pi$. For Ψ sufficiently C^2-close to Ψ_0, we obtain $A_N(\Psi, f) = \mathrm{Re}(\mathbf{E}_N(\mu_1^N)) + o(\gamma^N)$, where the $\mathbb{R}$-linear maps $\mathbf{E}_N : \mathbb{E}_1 \to \mathbb{C}$ converge uniformly to $\mathbf{E}$ as $\|\Psi - \Psi_0\|_2 \to 0$. Hence we may write $\mathrm{Re}(\mathbf{E}_N(\mu_1^N)) = E_N\gamma^N \cos(N\theta + \varphi_N)$, where $|\varphi_N - \theta_0| \le \pi/12$ and $|E_N|$ is bounded away from zero.

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