Broadband Nature of Power Spectra for Intermittent Maps with Summable and Nonsummable Decay of Correlations

Georg A. Gottwald* Ian Melbourne†

31 October 2015

Abstract

We present results on the broadband nature of the power spectrum $S(\omega)$, $\omega \in (0, 2\pi)$, for a large class of nonuniformly expanding maps with summable and nonsummable decay of correlations. In particular, we consider a class of intermittent maps $f : [0,1] \to [0,1]$ with $f(x) \approx x^{1+\gamma}$ for $x \approx 0$, where $\gamma \in (0,1)$. Such maps have summable decay of correlations when $\gamma \in (0, \frac{1}{2})$, and $S(\omega)$ extends to a continuous function on $[0, 2\pi]$ by the classical Wiener-Khintchine Theorem. We show that $S(\omega)$ is typically bounded away from zero for Hölder observables.

Moreover, in the nonsummable case $\gamma \in [\frac{1}{2}, 1)$, we show that $S(\omega)$ is defined almost everywhere with a continuous extension $\tilde{S}(\omega)$ defined on $(0, 2\pi)$, and $\tilde{S}(\omega)$ is typically nonvanishing.

1 Introduction

Let $f : X \to X$ be a measure preserving transformation of a probability space (X, μ) and let $v : X \to \mathbb{R}$ be an L^2 observable. The power spectrum $S : [0, 2\pi] \to \mathbb{R}$ is given by

$$S(\omega) = \lim_{n\to\infty} \frac{1}{n} \int_X \Big| \sum_{j=0}^{n-1} e^{ij\omega} v \circ f^j \Big|^2 \, d\mu.$$

By the Wiener-Khintchine Theorem [24], $S(\omega) = \sum_{k=-\infty}^{\infty} e^{ik\omega} \rho(k)$ where $\rho(k) = \int_X v \circ f^k \, v \, d\mu - \Big(\int_X v \, d\mu\Big)^2$ is the autocorrelation function of v. In particular, the power spectrum is analytic if and only if the autocorrelations decay exponentially.

*School of Mathematics and Statistics, University of Sydney, Sydney 2006 NSW, Australia
†Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

More generally, the power spectrum is well-defined and continuous provided the autocorrelations are summable.

The power spectrum is often used by experimentalists to distinguish periodic and quasiperiodic dynamics (discrete power spectrum with peaks at the harmonics and subharmonics) and chaotic dynamics (broadband power spectra). See for example [10]. In the atmospheric and oceanic sciences and in climate science, power spectra have been widely used to detect variability in particular frequency bands (see, for example, [9, 7]). Spectral analysis was successful in detecting dominant time scales in teleconnection patterns, revealing intraseasonal variability in time series of the global atmospheric angular momentum [6], interannual variability in the El Niño/Southern Oscillation system [4, 14], and the Atlantic Multidecadal Variability [5]. On millenial temporal scales the power spectrum was instrumental in unraveling dominant cycles in paleoclimatic records [25, 26].

Despite this widespread applicability across these disparate temporal scales, there are surprisingly few rigorous results on the nature of power spectra of complex systems. The (quasi)periodic case with its peaks at discrete frequencies is well understood. The nature of power spectra for chaotic systems was first treated in [23] in the case of uniformly hyperbolic (Axiom A) systems. In our previous paper [16], we considered in more detail the broadband nature of power spectra for chaotic dynamical systems and showed that for certain classes of dynamical systems f and observables v, the power spectrum is bounded away from zero.

The main results in [16] are for nonuniformly expanding/hyperbolic dynamical systems with exponential decay of correlations. These results are summarised below in Subsection 1.1. The current paper is concerned with systems possessing subexponential — even nonsummable — decay of correlations. A prototypical example is the class of Pomeau-Manneville intermittent maps [22], specifically the class considered in [15]. These are maps $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} x(1 + 2^\gamma x^\gamma), & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1] \end{cases}, \tag{1.1}$$

where $\gamma > 0$ is a parameter. For $\gamma \in (0, 1)$, there is a unique absolutely continuous invariant probability measure μ, and autocorrelations decay at the rate $O(1/n^\beta)$ for Hölder observables where $\beta = \gamma^{-1} - 1$. In particular, if $\gamma \in (0, \frac{1}{2})$, then the autocorrelation function is summable and the Wiener-Khintchine Theorem assures that the power spectrum is well-defined and continuous. We show that the power spectrum is typically bounded away from zero. Moreover, we show that the same result holds for all $\gamma \in (0, 1)$ provided the Hölder exponent[1] of v is sufficiently large. More precisely, we prove:

[1] Recall that if (X, d) is a metric space and $\eta \in (0, 1]$, then $v : X \to \mathbb{R}$ is C^η (Hölder with exponent η) if $|v|_\eta = \sup_{x \neq y} |v(x) - v(y)|/d(x, y)^\eta < \infty$.

Theorem 1.1 *Suppose that $f : [0,1] \to [0,1]$ is of the type (1.1) where $\gamma \in (0,1)$. Suppose that $v : [0,1] \to \mathbb{R}$ is C^η where $\eta > 0$.*

If $\gamma \in (0, \frac{1}{2})$, then the power spectrum is continuous on $(0, 2\pi)$ with a continuous extension to $[0, 2\pi]$, and is typically[2] bounded away from zero.

If $\gamma \in [\frac{1}{2}, 1)$, and $\eta > (3\gamma - 1)/2$, then the power spectrum is defined almost everywhere with a continuous extension to $(0, 2\pi)$. Typically this extension is nonvanishing.

Remark 1.2 When $\gamma \in (0, \frac{1}{2})$, the result is a special case of a more general result, Theorem 1.3, stated below. The only part of the result that is new for $\gamma \in (0, \frac{1}{2})$ is that the power spectrum is typically bounded below.

The case $\gamma \in [\frac{1}{2}, 1)$ depends more strongly on the details of the system, and seems to be a new result in its entirety.

1.1 The results in [16] for systems with exponential decay

The simplest case considered in [16] is when f is either a (noninvertible) uniformly expanding map or a uniformly hyperbolic (Axiom A) diffeomorphism and Λ is a locally maximal transitive subset of X. Suppose first that Λ is mixing. Then $f : \Lambda \to \Lambda$ has exponential decay of correlations for Hölder observables; in particular, the autocorrelation function of v decays exponentially provided v is Hölder. In this situation, we showed [16, Theorem 1.3] that the power spectrum is bounded away from zero for typical Hölder observables.

Still in the uniformly expanding/hyperbolic setting, it was shown in [16] that the mixing condition is unnecessary. There is an integer $q \geq 1$ such that (i) Λ is a disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_q$, (ii) $f(\Lambda_i) = \Lambda_{i+1}$ (computing subscripts $\bmod q$), and (iii) $f^q : \Lambda_i \to \Lambda_i$ is mixing for each i. Moreover, $f^q : \Lambda_i \to \Lambda_i$ has exponential decay of correlations for Hölder observables. By [16], the power spectrum is analytic with removable singularities at $2\pi j/q$, $j = 0, \ldots, q$, and typically bounded away from zero.

Large classes of nonuniformly expanding/hyperbolic systems can be treated in the same manner, namely those modelled by the tower construction of Young [27]. These systems include the logistic family, Hénon-like attractors, and planar periodic dispersing billiards. In such cases, the power spectrum is again analytic (up to removable singularities) and typically bounded away from zero.

1.2 Systems with subexponential but summable decay

The current paper is concerned with the case when exponential decay of correlations fails. Young [28] considers nonuniformly expanding maps with subexponential decay

[2]Throughout this paper "typically" means lying outside a closed subspace of infinite codimension within the Banach space of Hölder observables of a given exponent. Thus Theorem 1.1 fails only for infinitely degenerate observables.

of correlations. The precise definition of nonuniformly expanding map is given in Section 2. Although the power spectrum is no longer analytic, we may still discuss its boundedness properties.

If the decay is summable, then the power spectrum is continuous and extends to a continuous (and hence bounded) function on $[0, 2\pi]$. We prove:

Theorem 1.3 *Let f be a nonuniformly expanding map with polynomial decay of correlations at a rate $O(1/n^\beta)$ where $\beta > 1$. Then the power spectrum is bounded away from zero for typical Hölder observables.*

Remark 1.4 This result was claimed in [16, Section 4] but the proof sketched there is incomplete (the iterates of $L^k\hat{v}$ are summable as claimed, but only in L^p spaces, so the step that involves evaluation at periodic data is problematic). A full proof is given in this paper. Moreover, we consider also the case $\beta \in (0, 1]$.

The remainder of this paper is organised as follows. In Section 2, we give the definition for nonuniformly expanding map, and state Theorem 2.3 which implies Theorem 1.3. Also, we show how Theorem 1.1 follows from Theorem 2.3. In Section 3, we prove Theorem 2.3.

2 Nonuniformly expanding maps

Let (X, d) be a locally compact separable bounded metric space with Borel probability measure m_0 and let $f : X \to X$ be a nonsingular transformation for which m_0 is ergodic. Let $Y \subset X$ be a measurable subset with $m_0(Y) > 0$, and let α be an at most countable measurable partition of Y. We suppose that there is an L^1 *return time* function $r : Y \to \mathbb{Z}^+$, constant on each $a \in \alpha$ with value $r(a) \geq 1$, and constants $\lambda > 1$, $\eta \in (0, 1)$, $C \geq 1$, such that for each $a \in \alpha$,

(1) $F = f^{r(a)} : a \to Y$ is a measure-theoretic bijection.

(2) $d(Fx, Fy) \geq \lambda d(x, y)$ for all $x, y \in a$.

(3) $d(f^\ell x, f^\ell y) \leq Cd(Fx, Fy)$ for all $x, y \in a$, $0 \leq \ell < r(a)$.

(4) $g_a = \frac{d(m_0|a \circ F^{-1})}{dm_0|_Y}$ satisfies $|\log g_a(x) - \log g_a(y)| \leq Cd(x, y)^\eta$ for all $x, y \in Y$.

Such a dynamical system $f : X \to X$ is called *nonuniformly expanding.* The induced map $F = f^r : Y \to Y$ is uniformly expanding and there is a unique F-invariant probability measure μ_Y on Y equivalent to $m_0|_Y$ with density bounded above and below. Moreover μ_Y is mixing. This leads to a unique f-invariant probability measure μ on X equivalent to m_0 (see for example [28, Theorem 1]).

We assume throughout that $\gcd\{r(a) - r(b) : a, b \in \alpha\} = 1$. In particular, μ is mixing. The assumption is trivially satisfied for the maps (1.1) since $\{r(a) : a \in \alpha\} =$

$\mathbb{Z}^+$. Such a restriction is not completely avoidable, since the power spectrum has a finite number of (removable) singularities [16] when F is not mixing.

The results in [16] apply directly when $\mu_Y(y \in Y : r(y) > n)$ decays exponentially. In this paper, we show that the same results hold when $r \in L^{2+}(Y)$, thus proving Theorem 1.3.[3] In addition, we obtain results in the case $r \in L^{1+}(Y)$.

Given $v \in L^\infty(X)$ and $\omega \in [0, 2\pi]$, we define the *induced observable* $V_\omega : Y \to \mathbb{C}$,

$$V_\omega(y) = \sum_{\ell=0}^{r(y)-1} e^{i\ell\omega} v(f^\ell y).$$

Proposition 2.1 *For all $\omega_0, \omega \in [0, 2\pi]$, $a \in \alpha$,*

$$|1_a V_\omega|_\infty \le |v|_\infty r(a), \quad |1_a V_\omega - 1_a V_{\omega_0}|_\infty \le 2|v|_\infty r(a)^2 |\omega - \omega_0|.$$

Proof The first estimate is immediate. Also, for $y \in a$,

$$|V_\omega(y) - V_{\omega_0}(y)| \le \sum_{\ell=0}^{r(a)-1} |e^{i\ell\omega} - e^{i\ell\omega_0}||v|_\infty \le 2 \sum_{\ell=0}^{r(a)-1} |\ell(\omega - \omega_0)||v|_\infty \le 2|v|_\infty r(a)^2 |\omega - \omega_0|,$$

as required. ∎

Corollary 2.2 *Let $p \ge 1$. If $r \in L^p(Y)$ then $V_\omega \in L^p(Y)$ for all $\omega \in [0, 2\pi]$. Moreover, $\omega \mapsto V_\omega$ is a continuous map from $[0, 2\pi]$ to $L^p(Y)$.*

Proof We have $|V_\omega|_p^p \le \sum_{a\in\alpha} \mu_Y(a)|1_a V_\omega|_\infty^p \le \sum_{a\in\alpha} \mu_Y(a)|v|_\infty^p r(a)^p = |v|_\infty^p |r|_p^p < \infty$, so $V_\omega \in L^p(Y)$.

Next we prove continuity. Let $\omega_0 \in [0, 2\pi]$. Then

$$\begin{aligned}
|V_\omega - V_{\omega_0}|_p^p &\le \sum_{a\in\alpha} \mu_Y(a)|1_a V_\omega - 1_a V_{\omega_0}|_\infty^p \\
&\le \sum_{a\,:\,r(a)\le R} \mu_Y(a) 2^p |v|_\infty^p R^{2p} |\omega - \omega_0|^p + \sum_{a\,:\,r(a)>R} \mu_Y(a) 2^p |v|_\infty^p r(a)^p \\
&\le 2^p |v|_\infty^p \Big(R^{2p} |\omega - \omega_0|^p + \int_{\{r(a)>R\}} r^p \, d\mu_Y \Big).
\end{aligned}$$

Fix R large so that the second term is as small as desired. For this fixed R, the first term converges to zero as $\omega \to \omega_0$, proving continuity at ω_0. ∎

Define $V_\omega^* : Y \to \mathbb{C}$,

$$V_\omega^*(y) = \max_{0\le j\le r(y)-1} \Big| \sum_{\ell=0}^{j} e^{i\ell\omega} v(f^\ell y) \Big|.$$

Again $|1_a V_\omega^*|_\infty \le |v|_\infty r(a)$, so if $r \in L^p(Y)$, then $V_\omega^* \in L^p(Y)$ for all $\omega \in [0, 2\pi]$.

We now state our main result; this is proved in Section 3.

[3] Throughout, we write $\phi \in L^{p+}(Y)$ as shorthand for $\phi \in L^{p+\epsilon}(Y)$ for some $\epsilon > 0$.

Theorem 2.3 *Let $f : X \to X$ be a nonuniformly expanding map and let $v : X \to \mathbb{R}$ be a Hölder observable.*

(a) Suppose that $r \in L^{2+}(Y)$. Then the limit $S(\omega) = \lim_{n\to\infty} n^{-1} \int_X |e^{ij\omega} v\circ f^j|^2\, d\mu$ exists for all $\omega \in (0, 2\pi)$ and extends to a continuous function on $[0, 2\pi]$. Typically, $\inf_{\omega\in(0,2\pi)} S(\omega) > 0$.

(b) Suppose that $r \in L^a(Y)$ and that $V^*_\omega \in L^{bp}(Y)$ for all $\omega \in (0, 2\pi)$, where $a \in (1, \infty]$, $1/a + 1/b = 1$ and $p > 2$. Suppose further that $\omega \mapsto V_\omega$ is a continuous map from $(0, 2\pi)$ to $L^2(Y)$. Then the limit $S(\omega) = \lim_{n\to\infty} n^{-1} \int_X |e^{ij\omega} v \circ f^j|^2\, d\mu$ exists for almost every $\omega \in (0, 2\pi)$ and extends to a continuous function $\tilde{S}(\omega)$ on $(0, 2\pi)$. Typically, $\tilde{S}(\omega)$ is nonvanishing on $(0, 2\pi)$.

Remark 2.4 (i) Young [28] considers the case where $\mu_Y(r > n) = O(1/n^{\beta+1})$ for some $\beta > 0$ and deduces decay of correlations at rate $O(1/n^\beta)$. The case $\beta > 1$ is the setting of Theorem 1.3. It is easily seen that $r \in L^{p+}(Y)$ if and only if $\mu(r > n) = O(1/n^{p+})$, so Theorem 1.3 is a restatement of Theorem 2.3(a).

(ii) If $r \in L^2(Y)$, then summable decay of correlations follows from [20, Corollary 1.3] and hence the Wiener-Khintchine Theorem guarantees that the power spectrum extends to a continuous function on $[0, 2\pi]$. However, our proof that the spectrum is typically bounded below requires that $r \in L^{2+}(Y)$.

(iii) If $r \notin L^2(Y)$, then we expect (but have been unable to prove) that $S(\omega) \to \infty$ as $\omega \to 0$ and $\omega \to 2\pi$. It would then follow that typically the power spectrum is bounded below also in Theorem 2.3(b).

(iv) The proof of Theorem 2.3(b) shows that the limit $S(\omega)$ exists and is continuous on the set of irrational angles ω. A different argument, which we have not included, shows that the limit exists also for rational angles $\omega \in (0, 2\pi)$ and we conjecture that the resulting function $\omega \to S(\omega)$ is continuous on $(0, 2\pi)$.

2.1 Application to intermittent maps

For the intermittent maps (1.1), it is convenient to take $Y = [\frac{1}{2}, 1]$ and to let $r : Y \to \mathbb{Z}^+$ be the first return time $r(y) = \inf\{n \geq 1 : f^n y \in Y\}$. It is standard that f is a nonuniformly expanding map and that $r \in L^p(Y)$ for any $p < \frac{1}{\gamma}$. Hence if $\gamma \in (0, \frac{1}{2})$, then Theorem 2.3 applies directly. This completes the proof of Theorem 1.1 for $\gamma \in (0, \frac{1}{2})$.

For $\gamma \in [\frac{1}{2}, 1)$, we still have that $r \in L^{1+}(Y)$. To apply Theorem 2.3, we require the next result.

Proposition 2.5 *Suppose that $\gamma \in (0, 1)$ and $v \in C^\eta$ where $0 < \eta < \gamma$. Let $p < 1/(\gamma - \eta)$. Then $V^*_\omega \in L^p(Y)$ for all $\omega \in (0, 2\pi)$ and $\omega \mapsto V_\omega$ is a continuous map from $(0, 2\pi)$ to $L^p(Y)$.*

Proof This is analogous to the situation in [12, Theorem 4.1] (see also [13]).

Writing $v = (v - v(0)) + v(0)$, we may consider the cases $v(0) = 0$ and $v \equiv v(0)$ separately. For $v(0) = 0$, we show that $V_\omega^* \in L^p(Y)$ for all $\omega \in [0, 2\pi]$ and that $\omega \mapsto V_\omega$ is a continuous map from $[0, 2\pi]$ to $L^p(Y)$. For $v \equiv v(0)$, we show that $V_\omega^* \in L^\infty(Y)$ for all $\omega \in (0, 2\pi)$ and that $\omega \mapsto V_\omega$ is a continuous map from $(0, 2\pi)$ to $L^q(Y)$ for all $q < \infty$.

First, suppose that $v(0) = 0$, so $|v(y)| \le |v|_\eta |y|^\eta$. It is well known that $|f^\ell y| \ll (r(y) - \ell)^{-1/\gamma}$ for $\ell = 1, \dots, r(y)$. (See for example [15].) Hence for $y \in Y$, $\omega \in [0, 2\pi]$,

$$|V_\omega^*(y)| \le \sum_{\ell=0}^{r(y)-1} |v(f^\ell y)| \le \sum_{\ell=0}^{r(y)-1} |v|_\eta |f^\ell y|^\eta \ll \sum_{\ell=1}^{r(y)-1} (r(y) - \ell)^{-\eta/\gamma} \ll r(y)^{1-\eta/\gamma}.$$

If $p < 1/(\gamma - \eta)$, then $p(1 - \eta/\gamma) < 1/\gamma$ and

$$\int_Y |V_\omega^*(y)|^p \, d\mu_Y \ll \int_Y r^{p(1-\eta/\gamma)} \, d\mu_Y < \infty.$$

Next, we recall the estimate $|e^{ix} - 1| \le 2 \min\{1, |x|\} \le 2|x|^\epsilon$, which holds for all $x \in \mathbb{R}$, $\epsilon \in [0, 1]$. For all $y \in Y$, $\omega_0, \omega \in [0, 2\pi]$,

$$\begin{aligned} |V_\omega(y) - V_{\omega_0}(y)| &\le \max_{0 \le \ell < r(y)} |e^{i\ell\omega} - e^{i\ell\omega_0}| \sum_{\ell=0}^{r(y)-1} |v|_\eta |v(f^\ell y)|^\eta \\ &\ll r(y)^\epsilon |\omega - \omega_0|^\epsilon \sum_{\ell=0}^{r(y)-1} (r(y) - \ell)^{-\eta/\gamma} \ll r(y)^{1-\eta/\gamma+\epsilon} |\omega - \omega_0|^\epsilon, \end{aligned}$$

so for ϵ sufficiently small, $|V_\omega - V_{\omega_0}|_p \ll |\omega - \omega_0|^\epsilon$.

It remains to consider the case $v \equiv v(0)$. We have

$$V_\omega^*(y) = |v(0)| \max_{0 \le \ell < r(y)} |(1 - e^{i\omega\ell})/(1 - e^{i\omega})| \le 2|v(0)||1 - e^{i\omega}|^{-1},$$

for all $y \in Y$, $\omega \in (0, 2\pi)$. Moreover, for $\omega_0, \omega \in (0, 2\pi)$, regarding ω_0 as fixed, $|V_\omega(y) - V_{\omega_0}(y)| \le g_1(\omega, y) + g_2(\omega)$ where

$$\begin{aligned} g_1(\omega, y) &= |v(0)||1 - e^{i\omega_0}|^{-1} |e^{i\omega r(y)} - e^{i\omega_0 r(y)}|, \\ g_2(\omega) &= 2|v(0)||(1 - e^{i\omega})^{-1} - (1 - e^{i\omega_0})^{-1}|. \end{aligned}$$

Clearly, $g_2(\omega) \to 0$ as $\omega \to \omega_0$. Also, taking $\epsilon = 1/q$, we have that $g_1(\omega, y) \ll r(y)^{1/q} |\omega - \omega_0|^{1/q}$, so $\int_Y |g_1(\omega, y)|^q \, d\mu_Y \ll |r|_1 |\omega - \omega_0|$. It follows that $g_1(\omega, \cdot) \to 0$ in $L^q(Y)$, and hence similarly for $|V_\omega - V_{\omega_0}|$, as $\omega \to \omega_0$. This completes the proof. ∎

Proof of Theorem 1.1 As already mentioned, the case $\gamma \in (0, \frac{1}{2})$ follows directly from Theorem 2.3(a).

When $\gamma \in [\frac{1}{2}, 1)$, we require that $\eta > (3\gamma - 1)/2$. Without loss, $\eta \in ((3\gamma-1)/2, \gamma)$. Note that $1/(\gamma - \eta) > 2/(1-\gamma) > 2$. By Proposition 2.5, $\omega \mapsto V_\omega$ is a continuous map from $(0, 2\pi)$ to $L^2(Y)$.

Let $a = 1/(\gamma + \epsilon)$, $b = 1/(1-\gamma-\epsilon)$ where $\epsilon \in (0, 1-\gamma)$. Then $r \in L^a(Y)$. Since $1/(\gamma - \eta) > 2/(1-\gamma)$, it follows from Proposition 2.5 that $V_\omega^* \in L^{2/(1-\gamma-\delta)}(Y)$ for $\delta > 0$ sufficiently small. Hence we can choose $p > 2$, $\epsilon > 0$ such that $V_\omega^* \in L^{bp}(Y)$ for all $\omega \in (0, 2\pi)$. Now apply Theorem 2.3(b). ∎

3 Proof of Theorem 2.3

In Subsection 3.1, we prove a version of our main results for the induced system $F = f^r : Y \to Y$. This result is lifted to the original system $f : Y \to Y$ in Subsection 3.2. In Subsection 3.3, we complete the proof of Theorem 2.3.

3.1 The induced system

Let $f : X \to X$ be a nonuniformly expanding map as in Section 2 with induced map $F = f^r : Y \to Y$ and partition α. Define $r_n = \sum_{j=0}^{n-1} r \circ F^j$. The *induced power spectrum* $S^Y : (0, 2\pi) \to \mathbb{R}$ is given by

$$S^Y(\omega) = \lim_{n\to\infty} n^{-1} \int_Y \Big| \sum_{j=0}^{n-1} e^{i\omega r_j} V_\omega \circ F^j \Big|^2 \, d\mu_Y.$$

We can now state the main result in this subsection.

Lemma 3.1 *Suppose that $r \in L^1(Y)$ and that $\omega \mapsto V_\omega$ is continuous as a function from $(0, 2\pi)$ to $L^2(Y)$.*

(a) The pointwise limit $S^Y : (0, 2\pi) \to [0, \infty)$ exists and is continuous.

(b) Typically, $\{\omega \in (0, 2\pi) : S^Y(\omega) = 0\} = \emptyset$.

In the remainder of this subsection, we prove Lemma 3.1.

If $a_0, \ldots, a_{n-1} \in \alpha$, we define the n-cylinder $[a_0, \ldots, a_{n-1}] = \bigcap_{j=0}^{n-1} F^{-j} a_j$. Fix $\theta \in (0, 1)$ and define the symbolic metric $d_\theta(x, y) = \theta^{s(x,y)}$ where the *separation time* $s(x, y)$ is the least integer $n \geq 0$ such that x and y lie in distinct n-cylinders. For convenience we rescale the metric d on X so that $\operatorname{diam}(Y) \leq 1$.

Proposition 3.2 *Let $\eta \in (0, 1]$ and fix $\theta \in [\lambda^{-\eta}, 1)$. Then $d(x, y)^\eta \leq d_\theta(x, y)$ for all $x, y \in Y$.*

Proof Let $n = s(x, y)$. By condition (2),

$$1 \geq \operatorname{diam} Y \geq d(F^n x, F^n y) \geq \lambda^n d(x, y) \geq (\theta^{1/\eta})^{-n} d(x, y).$$

Hence $d(x, y)^\eta \leq \theta^n = d_\theta(x, y)$. ∎

An observable $\phi : Y \to \mathbb{R}$ is *Lipschitz* if $\|\phi\|_\theta = |\phi|_\infty + |\phi|_\theta < \infty$ where $|\phi|_\theta = \sup_{x \neq y} |\phi(x) - \phi(y)|/d_\theta(x, y)$. The set $F_\theta(Y)$ of Lipschitz observables is a Banach space. More generally, we say that $\phi : Y \to \mathbb{R}$ is *locally Lipschitz*, and write $\phi \in F_\theta^{\mathrm{loc}}(Y)$, if $\phi|_a \in F_\theta(a)$ for each $a \in \alpha$. Accordingly, we define $D_\theta \phi(a) = \sup_{x,y \in a:\, x \neq y} |\phi(x) - \phi(y)|/d_\theta(x, y)$.

Proposition 3.3 *Let $v : X \to \mathbb{R}$ be a C^η function, $\eta \in (0, 1]$. Set $\theta = \lambda^{-\eta}$. Then $V_\omega \in F_\theta^{\mathrm{loc}}(Y)$ for all $\omega \in [0, 2\pi]$, and there is a constant $C \geq 1$ such that*

$$D_\theta V_\omega(a) \leq C|v|_\eta r(a),$$

for all $\omega \in [0, 2\pi]$, $a \in \alpha$.

Proof Let $y, y' \in a$. Then $r(y) = r(y') = r(a)$. By condition (3) and Proposition 3.2,

$$\begin{aligned} |V_\omega(y) - V_\omega(y')| &\leq \sum_{\ell=0}^{r(a)-1} |v(f^\ell y) - v(f^\ell y')| \leq |v|_\eta \sum_{\ell=0}^{r(a)-1} d(f^\ell y, f^\ell y')^\eta \\ &\leq C^\eta |v|_\eta r(a) d(Fy, Fy')^\eta \leq C^\eta |v|_\eta r(a) d_\theta(Fy, Fy') = C^\eta \theta^{-1} |v|_\eta r(a) d_\theta(y, y'), \end{aligned}$$

yielding the required estimate for $D_\theta V_\omega(a)$. ∎

The transfer operator $P : L^1(Y) \to L^1(Y)$ corresponding to F is given by $\int_Y P\phi\, \psi\, d\mu_Y = \int_Y \phi\, \psi \circ F\, d\mu_Y$ for all $\psi \in L^\infty$. It can be shown that $(P\phi)(y) = \sum_{a \in \alpha} g(y_a)\phi(y_a)$ where y_a denotes the unique preimage of y in a under F and $\log g$ is the potential. Moreover, there exists a constant C_1 such that

$$g(y) \leq C_1 \mu_Y(a), \quad \text{and} \quad |g(y) - g(y')| \leq C_1 \mu_Y(a) d_\theta(y, y'), \tag{3.1}$$

for all $y, y' \in a$, $a \in \alpha$.

For $\omega \in [0, 2\pi]$, we define the twisted transfer operator $P_\omega : L^1(Y) \to L^1(Y)$ given by $P_\omega v = P(e^{-i\omega r} v)$.

Proposition 3.4 *Let $J \subset (0, 2\pi)$ be a closed subset. Viewing P_ω as an operator on $F_\theta(Y)$, there exists $C \geq 1$ and $\tau \in (0, 1)$ such that $\|P_\omega^n\| \leq C\tau^n$ for all $\omega \in J$, $n \geq 1$.*

Proof This result is a combination of standard and elementary observations. By [1, Theorem 2.4], $\omega \mapsto P_\omega$ is a continuous map from $[0, 2\pi]$ to $F_\theta(Y)$. Hence it suffices to show that the spectral radius of P_ω is less than 1 for $\omega \in (0, 2\pi)$.

It is easily checked that the spectral radius of P_ω is at most 1, and that the essential spectral radius is at most θ (see for example [1, Proposition 2.1]). Hence it remains to rule out eigenvalues on the unit circle.

Suppose for contradiction that $P_\omega v = e^{i\psi}v$ for some eigenfunction $v \in F_\theta(Y)$ and some $\psi \in [0, 2\pi]$. A calculation using the fact that $v \mapsto e^{i\omega r}v \circ F$ is the L^2 adjoint of P_ω (see for example [17, p. 429]) shows that $e^{i\omega r}v \circ F = e^{-i\psi}v$. By ergodicity of F, $|v|$ is constant and hence v is nonvanishing.

Since $F|_a : a \to Y$ is onto for each a, there exists $y_a \in a$ with $Fy_a = y_a$. Evaluating at y_a and using the fact that $v(y_a) \neq 0$, we obtain that $e^{i\omega r(a)} = e^{-i\psi}$ for each $a \in \alpha$. Hence $\omega(r(a) - r(b)) = 0 \bmod 2\pi$ for all $a, b \in \alpha$. Since $\omega \in (0, 2\pi)$, it is immediate that $\omega = 2\pi p/q$ where p, q are integers with $\gcd(p, q) = 1$ and $1 \leq p < q$. But then $\frac{p}{q}(r(a) - r(b) = 0 \bmod \mathbb{Z}$ and so q divides $r(a) - r(b)$ for all $a, b \in \alpha$. This contradicts the assumption that $\gcd\{r(a) - r(b) : a, b \in \alpha\} = 1$. ∎

Proposition 3.5 *Suppose that $r \in L^1(Y)$. Then*

(a) $P_\omega V_\omega \in F_\theta(Y)$ for all $\omega \in [0, 2\pi]$ and $\sup_{\omega\in[0,2\pi]} \|P_\omega V_\omega\|_\theta < \infty$.

(b) $\sum_{n=1}^\infty \sup_{\omega\in J}\big|\int_Y P_\omega^n V_\omega \bar{V}_\omega \, d\mu_Y\big| < \infty$ for any closed subset $J \subset (0, 2\pi)$.

Proof (a) Write $(P_\omega\phi)(y) = \sum_{a\in\alpha} g(y_a)e^{-i\omega r(a)}\phi(y_a)$. We use Propositions 2.1 and 3.3 and the estimates (3.1). First,

$$|P_\omega V_\omega|_\infty \leq \sum_{a\in\alpha} |1_a g|_\infty |1_a V_\omega|_\infty \leq C_1 \sum_{a\in\alpha} \mu_Y(a)|v|_\infty r(a) = C_1|v|_\infty|r|_1.$$

Also, $P_\omega V_\omega(y) - P_\omega V_\omega(y') = I_1 + I_2 + I_2$ where

$$I_1 = \sum_{a\in\alpha}(g(y_a) - g(y'_a))e^{-i\omega r(a)}V_\omega(y_a), \quad I_2 = \sum_{a\in\alpha} g(y'_a)e^{-i\omega r(a)}(V_\omega(y_a) - V_\omega(y'_a)).$$

We have

$$|I_1| \leq C_1 \sum_{a\in\alpha} \mu_Y(a)d_\theta(y_a, y'_a)|v|_\infty r(a) = \theta C_1|v|_\infty|r|_1 d_\theta(y, y'),$$

and

$$\begin{aligned}|I_2| &\leq C_1 \sum_{a\in\alpha} \mu_Y(a)D_\theta V_\omega(a)d_\theta(y_a, y'_a) \\ &\leq \theta C_1 C \sum_{a\in\alpha} \mu_Y(a)|v|_\eta r(a)d_\theta(y, y') = \theta C_1 C|v|_\eta|r|_1 d_\theta(y, y').\end{aligned}$$

We deduce that $|P_\omega V_\omega|_\theta \ll (|v|_\infty + |v|_\eta)|r|_1$, completing the proof of part (a).

(b) For $n \ge 1$,

$$\Big|\int_Y P_\omega^n V_\omega \, \bar V_\omega \, d\mu_Y\Big| \le |P_\omega^n V_\omega|_\infty \, |V_\omega|_1 \le \|P_\omega^{n-1}\| \|P_\omega V_\omega\|_\theta \, |V_\omega|_1.$$

The result follows from Corollary 2.2, Proposition 3.4 and part (a), ∎

Proof of Lemma 3.1(a) Let $\omega \in (0, 2\pi)$. For $0 \le j < k$,

$$\begin{aligned}\int_Y e^{-i\omega(r_k - r_j)} V_\omega \circ F^j \, \bar V_\omega \circ F^k \, d\mu_Y &= \int_Y e^{-i\omega r_{k-j} \circ F^j} V_\omega \circ F^j \, \bar V_\omega \circ F^k \, d\mu_Y \\ &= \int_Y e^{-i\omega r_{k-j}} V_\omega \, \bar V_\omega \circ F^{k-j} \, d\mu_Y = \int_Y P_\omega^{k-j} V_\omega \, \bar V_\omega \, d\mu_Y.\end{aligned}$$

Hence,

$$\begin{aligned}\int_Y \Big|\sum_{j=0}^{n-1} e^{i\omega r_j} V_\omega \circ F^j\Big|^2 \, d\mu_Y &= \sum_{j,k=0}^{n-1} \int_Y e^{i\omega(r_j - r_k)} V_\omega \circ F^j \, \bar V_\omega \circ F^k \, d\mu_Y \\ &= \sum_{j=0}^{n-1} \int_Y |V_\omega \circ F^j|^2 \, d\mu_Y + 2 \sum_{0 \le j < k < n} \operatorname{Re} \int_Y e^{-i\omega(r_k - r_j)} V_\omega \circ F^j \, \bar V_\omega \circ F^k \, d\mu_Y \\ &= n \int_Y |V_\omega|^2 \, d\mu_Y + 2 \sum_{0 \le j < k < n} \operatorname{Re} \int_Y P_\omega^{k-j} V_\omega \, \bar V_\omega \, d\mu_Y \\ &= n \int_Y |V_\omega|^2 \, d\mu_Y + 2 \sum_{m=1}^{n-1} (n - m) \operatorname{Re} \int_Y P_\omega^m V_\omega \, \bar V_\omega \, d\mu_Y.\end{aligned}$$

Hence

$$S^Y(\omega) = \int_Y |V_\omega|^2 \, d\mu_Y + 2 \lim_{n\to\infty} \sum_{m=1}^{n-1} \big(1 - \frac{m}{n}\big) \operatorname{Re} \int_Y P_\omega^m V_\omega \, \bar V_\omega \, d\mu_Y.$$

By Proposition 3.5(b), this converges uniformly on compact subsets of $(0, 2\pi)$ to the sum

$$\begin{aligned}S^Y(\omega) &= \int_Y |V_\omega|^2 \, d\mu_Y + 2 \sum_{n=1}^{\infty} \operatorname{Re} \int_Y P_\omega^n V_\omega \, \bar V_\omega \, d\mu_Y \\ &= \int_Y |V_\omega|^2 \, d\mu_Y + 2 \sum_{n=1}^{\infty} \operatorname{Re} \int_Y e^{-i\omega r_n} V_\omega \, \bar V_\omega \circ F^n \, d\mu_Y.\end{aligned} \tag{3.2}$$

Fix $n \ge 1$ and let $I_\omega = e^{-i\omega r_n} V_\omega \, \bar V_\omega \circ F^n$. Note that $|I_\omega| = |V_\omega| \, |V_\omega| \circ F^n$ and $\int_Y |I_\omega| \, d\mu_Y \le |V_\omega|_2 |V_\omega \circ F^n|_2 = |V_\omega|_2^2 < \infty$ for each ω. We claim that $\omega \mapsto \int_Y I_\omega \, d\mu_Y$ is continuous on $(0, 2\pi)$. It then follows from uniform convergence of the series (3.2) that $S^Y : (0, 2\pi) \to [0, \infty)$ is continuous.

To prove the claim, fix $n \geq 1$ and $\omega_* \in (0, 2\pi)$. Let ω_k be a sequence in $(0, 2\pi)$ converging to ω_*. We show that $\int_Y I_{\omega_k}\, d\mu_Y \to \int_Y I_{\omega_*}\, d\mu_Y$ as $k \to \infty$. Certainly $I_{\omega_k} \to I_{\omega_*}$ pointwise. Moreover,

$$\begin{aligned}\int_Y |I_{\omega_k}|\, d\mu_Y - \int_Y |I_{\omega_*}|\, d\mu_Y = \int_Y (|V_{\omega_k}| - |V_{\omega_*}|)\, |V_{\omega_k}| \circ F^n\, d\mu_Y \\ + \int_Y |V_{\omega_*}|\, (|V_{\omega_k}| - |V_{\omega_*}|) \circ F^n\, d\mu_Y,\end{aligned}$$

and so

$$\begin{aligned}\Big|\int_Y |I_{\omega_k}|\, d\mu_Y - \int_Y |I_{\omega_*}|\, d\mu_Y\Big| &\leq \big||V_{\omega_k}| - |V_{\omega_*}|\big|_2 |V_{\omega_k}|_2 + |V_{\omega_*}|_2 \big||V_{\omega_k}| - |V_{\omega_*}|\big|_2 \\ &\leq |V_{\omega_k} - V_{\omega_*}|_2 (|V_{\omega_k}|_2 + |V_{\omega_*}|_2) \\ &\leq |V_{\omega_k} - V_{\omega_*}|_2 (|V_{\omega_k} - V_{\omega_*}|_2 + 2|V_{\omega_*}|_2).\end{aligned}$$

It follows that $\int_Y |I_{\omega_k}|\, d\mu_Y \to \int_Y |I_{\omega_*}|\, d\mu_Y$ as $k \to \infty$. By the dominated convergence theorem, $\int_Y I_{\omega_k}\, d\mu_Y \to \int_Y I_{\omega_*}\, d\mu_Y$ completing the proof of the claim. ∎

Proof of Lemma 3.1(b) Let $\omega \in (0, 2\pi)$ and define $\chi_\omega = \sum_{j=1}^\infty P_\omega^j V_\omega$ and $\tilde{V}_\omega = V_\omega + \chi_\omega - e^{i\omega r} \chi_\omega \circ F$. Now $\|P_\omega^j V_\omega\|_\theta \leq \|P_\omega^{j-1}\| \|P_\omega V_\omega\|_\theta$, so it follows from Propositions 3.4 and 3.5(a) that χ_ω is absolutely summable in $F_\theta(Y)$. In particular $\tilde{V}_\omega \in L^2(Y)$.

A calculation shows that $\tilde{V}_\omega \in \ker P_\omega$ and so $|\sum_{j=1}^{n-1} e^{i\omega r_j} \tilde{V}_\omega \circ F^j|_2^2 = n|\tilde{V}_\omega|_2^2$. Moreover, $\sum_{j=1}^{n-1} e^{i\omega r_j} \tilde{V}_\omega \circ F^j = \sum_{j=1}^{n-1} e^{i\omega r_j} V_\omega \circ F^j + \chi_\omega - e^{i\omega r_n} \chi_\omega \circ F_\omega^n$ and so

$$\Big| \big|\sum_{j=1}^{n-1} e^{i\omega r_j} V_\omega \circ F^j\big|_2 - \big|\sum_{j=1}^{n-1} e^{i\omega r_j} \tilde{V}_\omega \circ F^j\big|_2 \Big| \leq 2|\chi_\omega|_2.$$

Hence $S^Y(\omega) = |\tilde{V}_\omega|_2^2$ for all $\omega \in (0, 2\pi)$. In particular, for a fixed $\omega \in (0, 2\pi)$ we have that if $S^Y(\omega) = 0$, then equivalently $\tilde{V}_\omega = 0$ and so

$$V_\omega = e^{i\omega r} \chi_\omega \circ F - \chi_\omega. \tag{3.3}$$

It remains to exclude the possibility that (3.3) holds for some ω. Following [16, Section 3], let $y_0 \in Y$ be a periodic point of period p and let y_n be a sequence with $F y_n = y_{n-1}$ and such that $d_\theta(y_{np}, y_0) \leq \theta^{np}$. This ensures in particular that $r_p(y_{jp}) = r_0(y_0)$ for all j.

Set $A_\omega = \sum_{j=0}^{p-1} e^{i\omega r_j} V_\omega \circ F^j$ and define $g(\omega) = \sum_{j=1}^\infty e^{-ij\omega r_p(y_0)} (A_\omega(y_{jp}) - A_\omega(y_0))$.

Note that for each fixed y_n, the function $\omega \mapsto A_\omega(y_n)$ is a finite trigonometric polynomial and hence is analytic on $[0, 2\pi]$. We claim that $g : [0, 2\pi] \to \mathbb{C}$ is analytic. Suppose that $d_\theta(y, y') \geq p$. Then

$$\begin{aligned}|A_\omega(y) - A_\omega(y')| &\leq \sum_{j=0}^{p-1} \sum_{\ell=0}^{r(F^j y)-1} |v|_\theta d_\theta(F^j y, F^j y') = |v|_\theta \sum_{j=0}^{p-1} r(F^j y) \theta^{-j} d_\theta(y, y') \\ &\leq |v|_\theta\, \theta^{-p} (1-\theta)^{-1} r_p(y) d_\theta(y, y'),\end{aligned}$$

Hence

$$|A_\omega(y_{jp}) - A_\omega(y_0)| \le |v|_\theta\, \theta^{-p}(1-\theta)^{-1} r_p(y_0)\theta^j,$$

proving the claim.

If (3.3) holds, then $A_\omega = e^{i\omega r_p}\chi_\omega \circ F^p - \chi_\omega$ and so

$$\sum_{j=1}^{n} e^{-ij\omega r_p(y_0)} A_\omega(y_{jp}) = \chi_\omega(y_0) - e^{-in\omega r_p(y_0)}\chi_\omega(y_{np}),$$
$$\sum_{j=1}^{n} e^{-ij\omega r_p(y_0)} A_\omega(y_0) = \chi_\omega(y_0) - e^{-in\omega r_p(y_0)}\chi_\omega(y_0).$$

Hence

$$g(\omega) = \lim_{n\to\infty} e^{-in\omega r_p(y_0)}(\chi_\omega(y_0) - \chi_\omega(y_{np})) = 0.$$

Still keeping ω fixed, we can perturb the value of v at y_1 (say), independently of any other values of v involved in the computation of g, so that $g(\omega) \ne 0$. Hence typically (3.3) does not hold (and so $S^Y(\omega)$ is nonzero) for any fixed value of ω.

Considering two such analytic functions g_1 and g_2 (for two distinct periodic points) we can perturb so that g_1 and g_2 have no common zeros on $[0,2\pi]$ and hence S^Y is nonvanishing on $(0,2\pi)$. By considering infinitely many periodic points, and hence infinitely many functions of the form g, we obtain infinitely many independent obstructions to the existence of an $\omega \in (0,2\pi)$ such that $S^Y(\omega) = 0$. ∎

3.2 Relation between S^Y and S

In this subsection, we relate the power spectrum S^Y of the induced map $F : Y \to Y$ with the power spectrum S of the underlying nonuniformly expanding map $f : X \to X$. Let $\bar r = \int_Y r\, d\mu_Y$. We say that ω is an irrational angle if $\omega \in [0,2\pi] \setminus \pi\mathbb{Q}$.

Lemma 3.6 *Let ω be an irrational angle. Suppose either that $r \in L^{2+}(Y)$, or that $r \in L^a(Y)$ and that $V_\omega^* \in L^{bp}(Y)$ for all $\omega \in (0,2\pi)$, where $a \in (1,\infty]$, $1/a + 1/b = 1$ and $p > 2$. Then $S(\omega) = S^Y(\omega)/\bar r$.*

From now on, we work with a fixed irrational angle $\omega \in (0,2\pi) \setminus \pi\mathbb{Q}$. We suppose throughout that $v : X \to \mathbb{R}$ is Hölder.

Let $d\varphi$ denote Haar measure on S^1, Consider the circle extensions

$$f_\omega : X \times S^1 \to X \times S^1, \qquad f_\omega(x,\varphi) = (fx, \varphi + \omega),$$
$$F_\omega : Y \times S^1 \to Y \times S^1, \qquad F_\omega(y,\varphi) = (Fy, \varphi + \omega r(y)),$$

with invariant probability measures $\nu = \mu \times d\varphi$ and $\nu_Y = \mu_Y \times d\varphi$ respectively.

Recall that $F = f^r : Y \to Y$ is the induced map obtained from $f : X \to X$ with return time $r : Y \to \mathbb{Z}^+$. Extend r to a return time on $Y \times S^1$ by setting $r(y, \varphi) = r(y)$. Then $F_\omega = f_\omega^r$ is the induced map obtained from f_ω.

Let $v : X \to \mathbb{R}$ be an observable. We associate to v the observable $u : X \times S^1 \to \mathbb{C}$ given by $u(x, \varphi) = e^{i\varphi} v(x)$. This leads to the induced observable $U_\omega : Y \times S^1 \to \mathbb{C}$ given by $U_\omega(q) = \sum_{\ell=0}^{r(q)-1} u(f_\omega^\ell q)$. Note that

$$U_\omega(y, \varphi) = \sum_{\ell=0}^{r(y)-1} u(f^\ell y, \varphi + \ell\omega) = e^{i\varphi} \sum_{\ell=0}^{r(y)-1} e^{i\ell\omega} v(f^\ell y) = e^{i\varphi} V_\omega(y),$$

and similarly that

$$\sum_{j=0}^{n-1} U_\omega \circ F_\omega^j(y, \varphi) = e^{i\varphi} \sum_{j=0}^{n-1} e^{i\omega r_j(y)} V_\omega \circ F^j(y).$$

Proposition 3.7 *Let ω be an irrational angle. Suppose that $r \in L^{1+}(Y)$ and that $V_\omega^* \in L^2(Y)$. Then*

$$n^{-1} \Big| \sum_{j=0}^{n-1} u \circ f_\omega^j \Big|^2 \to_d Z_\omega \quad \textit{on } (X \times S^1, \nu),$$

where Z_ω is a random variable with $\mathbb{E} Z_\omega = S^Y(\omega)/\bar{r}$.

Proof Since $\omega \in [0, 2\pi] \setminus \pi\mathbb{Q}$, the circle extension $f_\omega : X \times S^1 \to X \times S^1$ is ergodic, and we are in a position to apply [12, Theorem 1.10]. Note that G, ϕ, V, V^*, f_h in [12] correspond to $S^1, u, V_\omega, V_\omega^*, f_\omega$ here.

By [12, Theorem 1.10], we obtain a functional central limit theorem (weak invariance principle) as follows. Define $W_{n,\omega}(t) = n^{-1/2} \sum_{j=0}^{[nt]-1} u \circ f_\omega^j$ for $t = 0, 1/n, \ldots, 1$ and linearly interpolate to obtain $W_{n,\omega} \in C([0,1], \mathbb{R}^2)$. Then $W_{n,\omega} \to_w W_\omega$ in $C([0,1], \mathbb{R}^2)$ on $(X \times S^1, \nu)$, where W_ω is a two-dimensional Brownian motion with some covariance matrix Σ_ω. Moreover, it follows from the proof of [12, Theorem 1.10] (see the statements of [12, Theorems 2.1 and 3.3]) that $\Sigma_\omega = \hat{\Sigma}_\omega / \bar{r}$ where

$$\hat{\Sigma}_\omega = \lim_{n \to \infty} n^{-1} \int_{Y \times S^1} \Big(\sum_{j=0}^{n-1} U_\omega \circ F_\omega^j \Big) \otimes \Big(\sum_{j=0}^{n-1} U_\omega \circ F_\omega^j \Big) \, d\nu_Y.$$ [4]

Consider the functional $\chi : C([0,1], \mathbb{R}^2) \to \mathbb{R}$, $\chi(g) = |g(1)|^2$. By the continuous mapping theorem, $\chi(W_{n,\omega}) \to_d \chi(W_\omega)$, so

$$n^{-1} \Big| \sum_{j=0}^{n-1} u \circ f_\omega^j \Big|^2 \to_d Z_\omega, \quad \text{where } Z_\omega = |W_\omega(1)|^2.$$

[4] For $a, b \in \mathbb{C} \cong \mathbb{R}^2$, we define $a \otimes b = ab^T = \begin{pmatrix} \operatorname{Re} a \operatorname{Re} b & \operatorname{Re} a \operatorname{Im} b \\ \operatorname{Im} a \operatorname{Re} b & \operatorname{Im} a \operatorname{Im} b \end{pmatrix}$.

In particular,

$$\mathbb{E}Z_\omega = \mathbb{E}|W_\omega(1)|^2 = \Sigma_\omega^{11} + \Sigma_\omega^{22} = (\hat{\Sigma}_\omega^{11} + \hat{\Sigma}_\omega^{22})/\bar{r}.$$

Finally,

$$\begin{aligned} S^Y(\omega) &= \lim_{n\to\infty} n^{-1} \int_Y |\sum_{j=0}^{n-1} e^{i\omega r_j} V_\omega \circ F^j|^2 \, d\mu_Y \\ &= \lim_{n\to\infty} n^{-1} \int_{Y\times S^1} |\sum_{j=0}^{n-1} U_\omega \circ F_\omega^j|^2 \, d\nu_Y = \hat{\Sigma}_\omega^{11} + \hat{\Sigma}_\omega^{22}, \end{aligned}$$

completing the proof. ∎

Proposition 3.8 *Suppose that either*

(a) $r \in L^{2+}(Y)$ *and choose* $p > 2$ *such that* $r \in L^{\frac{p}{2}+1+}(Y)$, *or*

(b) $r \in L^a(Y)$ *and* $V_\omega^* \in L^{bp}(Y)$ *for all* $\omega \in (0, 2\pi)$, *where* $a \in (1, \infty]$, $1/a + 1/b = 1$, *and* $p > 2$.

Then there is a constant $C \geq 1$ *such that* $|\sum_{j=0}^{n-1} u \circ f_\omega^j|_p \leq C n^{1/2}$.

Proof The method in both cases is to obtain a martingale coboundary decomposition [11] and then to apply Burkholder's inequality [3] to the martingale part. In case (a), the decomposition is done on $X \times S^1$ following [19]. In case (b), we pass to a tower extension and reduce to the induced system on $Y \times S^1$.

Case (a): By Markov's inequality, the assumption $r \in L^{\frac{p}{2}+1+}(Y)$ guarantees that $\mu(r > n) = O(n^{-(\beta+1)})$ for some $\beta > p/2$. Let $L : L^1(X) \to L^1(X)$ and $L_\omega : L^1(X \times S^1) \to L^1(X \times S^1)$ be the transfer operators corresponding to f and f_ω respectively. Regard u as fixed, and let $u' \in L^\infty(X \times S^1)$ be of the form $u'(y, \varphi) = e^{i\varphi} v'(x)$ where $v' \in L^\infty(X)$. By [2], $\int_{X\times S^1} L_\omega^n u \, \bar{u}' \, d\nu = \int_{X\times S^1} u \, \bar{u}' \circ f_\omega^n \, d\nu = O(n^{-\beta}|u'|_\infty)$. Equivalently $\int_X L^n v \, \bar{v}' \, d\mu = O(n^{-\beta}|v'|_\infty)$. By duality $\int_{X\times S^1} |L_\omega^n u| \, d\nu = \int_X |L^n v| \, d\mu = O(n^{-\beta})$.

Now we proceed as in the proof of [19, Theorem 3.1]. Since $|L_\omega^n u|_1 = O(n^{-\beta})$ and $|L_\omega^n u|_\infty = O(1)$, it follows by interpolation that $|L_\omega^n u|_q$ is summable for $q < \beta$, in particular for $q = p/2$. Hence $\chi_\omega = \sum_{n=1}^\infty L_\omega^n u \in L^{p/2}(X \times S^1)$ and we obtain $u = \tilde{u}_\omega + \chi_\omega \circ f_\omega - \chi_\omega$ where $\tilde{u}_\omega \in \ker L_\omega$. Continuing as in [19] (in particular, see [19, equation (3.1)]) we obtain the desired result.

Case (b): Define the Young tower [28]

$$\Delta = \{(y, \varphi, \ell) \in Y \times S^1 \times \mathbb{Z} : 0 \leq \ell < r(y)\},$$

and the tower map $\hat f_\omega:\Delta\to\Delta$,

$$\hat f_\omega(y,\varphi,\ell)=\begin{cases}(y,\varphi+\omega,\ell+1), & \ell\le r(y)-2\\ (Fy,\varphi+\omega,0), & \ell=r(y)-1\end{cases}$$

with invariant probability measure $\nu_\Delta=(\nu\times\text{counting})/\bar r$.

The projection $\pi:\Delta\to X\times S^1$ given by $\pi(y,\varphi,\ell)=(f^\ell y,\varphi)$ is a measure-preserving semiconjugacy between $\hat f_\omega$ and f_ω, with $\nu=\pi_*\nu_\Delta$.

Let $\hat u=u\circ\pi$. Then $\int_{X\times S^1}|\sum_{j=0}^{n-1}u\circ f_\omega^j|^p\,d\nu=\int_\Delta|\sum_{j=0}^{n-1}\hat u\circ\hat f_\omega^j|^p\,d\nu_\Delta$.

Next, let $N_n:\Delta\to\{0,1,\dots,n\}$ be the number of laps by time n,

$$N_n(y,\varphi,\ell)=\#\{j\in\{1,\dots,n\}:\hat f_\omega^j(y,\varphi,\ell)\in Y\times S^1\times\{0\}\}.$$

Then

$$e^{i\ell\omega}\sum_{j=0}^{n-1}\hat u\circ\hat f_\omega^j(y,\varphi,\ell)=\sum_{k=0}^{N_n(y,\varphi,\ell)-1}U_\omega\circ F_\omega^k(y,\varphi)+H_\omega\circ\hat f_\omega^n(y,\varphi,\ell)-H_\omega(y,\varphi,\ell)$$

where $H_\omega(y,\varphi,\ell)=e^{i\varphi}\sum_{\ell'=0}^{\ell-1}e^{i\ell'\omega}v(f^{\ell'}y)$. Note that $|H_\omega(y,\varphi,\ell)|\le V_\omega^*(y)$.

Now

$$\begin{aligned}\int_\Delta|H_\omega\circ\hat f_\omega^n|^p\,d\mu_\Delta&=\int_\Delta|H_\omega|^p\,d\mu_\Delta=(1/\bar r)\int_Y\sum_{\ell=0}^{r(y)-1}|H_\omega(y,\varphi,\ell)|^p\,d\nu_Y(y,\varphi)\\ &\le\int_Y r\;|V_\omega^*|^p\,d\mu_Y\le|r|_a|V_\omega^{*p}|_b=|r|_a|V_\omega^*|_{bp}^p.\end{aligned}\tag{3.4}$$

Next, by Hölder's inequality,

$$\begin{aligned}\int_\Delta\Big|\sum_{k=0}^{N_n-1}U_\omega\circ F_\omega^k\Big|^p\,d\nu_\Delta&\le\int_\Delta\max_{j\le n}\Big|\sum_{k=0}^{j-1}U_\omega\circ F_\omega^k\Big|^p\,d\nu_\Delta\\ &=(1/\bar r)\int_{Y\times S^1}r\;\max_{j\le n}\Big|\sum_{k=0}^{j-1}U_\omega\circ F_\omega^k\Big|^p\,d\nu_Y\le|r|_{L^a(Y)}\;\Big|\max_{j\le n}\Big|\sum_{k=0}^{j-1}U_\omega\circ F_\omega^k\Big|\Big|_{L^{bp}(Y\times S^1)}^p.\end{aligned}$$

Let $Q_\omega:L^1(Y\times S^1)\to L^1(Y\times S^1)$ denote the transfer operator corresponding to $F_\omega:Y\times S^1\to Y\times S^1$. For an observable $U:Y\times S^1\to\mathbb C$ of the form $U(y,\varphi)=e^{i\varphi}V(y)$, we have $(Q_\omega U)(y,\varphi)=e^{i\varphi}(P_\omega V)(y)$.

By Propositions 3.4 and 3.5(a),

$$\chi_\omega=\sum_{n=1}^\infty Q_\omega^nU_\omega=e^{i\varphi}\sum_{n=1}^\infty P_\omega^nV_\omega\in L^\infty(Y\times S^1).$$

Hence we can write

$$U_\omega=m_\omega+\chi_\omega\circ F_\omega-\chi_\omega,$$

where $m_\omega \in \ker Q_\omega$. Note also by the hypothesis on V_ω^* that $U_\omega \in L^{bp}(Y \times S^1)$ and hence $m_\omega \in L^{bp}(Y \times S^1)$ where $bp > 2$. By Burkholder's inequality [3],

$$\Big|\max_{j\le n}\Big|\sum_{k=0}^{j-1} m_\omega \circ F_\omega^k\Big|\Big|_{L^{bp}(Y\times S^1)} \ll |m_\omega|_{L^{bp}(Y\times S^1)}\; n^{1/2},$$

and so $\big|\max_{j\le n}|\sum_{k=0}^{j-1} U_\omega \circ F_\omega^k|\big|_{L^{bp}(Y\times S^1)} \ll n^{1/2}$. Hence we have shown that

$$\Big|\sum_{k=0}^{N_n-1} U_\omega \circ F_\omega^k\Big|_{L^p(\Delta)} \le |r|_a^{1/p}\,\Big|\max_{j\le n}\Big|\sum_{k=0}^{j-1} U_\omega \circ F_\omega^k\Big|\Big|_{L^{bp}(Y\times S^1)} \ll n^{1/2}. \tag{3.5}$$

By the triangle inequality, it follows from (3.4) and (3.5) that

$$\Big|\sum_{j=0}^{n-1} u \circ f_\omega^j\Big|_{L^p(X\times S^1)} = \Big|\sum_{j=0}^{n-1} \hat{u} \circ \hat{f}_\omega^j\Big|_{L^p(\Delta)} \ll n^{1/2},$$

as required. ∎

Proof of Lemma 3.6 The bound on moments in Proposition 3.8 together with the distributional limit law in Proposition 3.7 implies convergence of lower moments, (see for example [21]), and so

$$\lim_{n\to\infty} n^{-1}\Big|\sum_{j=0}^{n-1} u \circ f_\omega^j\Big|^2_{L^2(X\times S^1)} = S^Y(\omega)/\bar{r}.$$

Now $\sum_{j=0}^{n-1} u(f_\omega^j(x,\varphi)) = e^{i\varphi}\sum_{j=0}^{n-1} e^{ij\omega} v(f^j(x))$ and so $|\sum_{j=0}^{n-1} u \circ f_\omega^j|_{L^2(X\times S^1)} = |\sum_{j=0}^{n-1} e^{ij\omega} v \circ f^j|_{L^2(X)}$. Hence

$$S(\omega) = \lim_{n\to\infty} n^{-1}\Big|\sum_{j=0}^{n-1} e^{ij\omega} v \circ f^j\Big|^2_{L^2(X)} = S^Y(\omega)/\bar{r},$$

as required. ∎

3.3 Completion of the proof

First assume the hypotheses of Theorem 2.3(b). By Lemma 3.1, S^Y exists and is continuous on $(0, 2\pi)$, and typically S^Y is nonvanishing on $(0, 2\pi)$. By Lemma 3.6, S coincides with $S^Y/\bar{r}$ almost everywhere on $(0, 2\pi)$. Theorem 2.3(b) follows immediately.

Under the hypotheses of Theorem 2.3(a) we have the same properties, but in addition $S(\omega)$ is continuous on $(0, 2\pi)$ by the Wiener-Khintchine Theorem, so it follows that $S(\omega) = S^Y(\omega)/\bar{r}$ for all $\omega \in (0, 2\pi)$. Hence S is typically nonvanishing on $(0, 2\pi)$. Moreover, by Wiener-Khintchine, S extends to a continuous function $S_0(\omega) = \sum_{k=-\infty}^{\infty} e^{ik\omega}\rho(k)$ on $[0, 2\pi]$. By the Green-Kubo formula, $S_0(0) = S_0(2\pi)$ coincides with the variance $\sigma^2 = \lim_{n\to\infty} n^{-1} \int_X |\sum_{j=0}^{n-1} v_0 \circ f^j|^2 \, d\mu$ where $v_0 = v - \int_X v \, d\mu$. Typically $\sigma^2 > 0$, see for example [18, Remark 2.11]. Hence typically S_0 is bounded away from zero on $[0, 2\pi]$ and so S is bounded away from zero on $(0, 2\pi)$. This completes the proof of Theorem 2.3(a).

Acknowledgements This research was supported in part by an International Research Collaboration Award at the University of Sydney. GAG acknowledges funding from the Australian Research Council. The research of IM was supported in part by a European Advanced Grant *StochExtHomog* (ERC AdG 320977).

References

[1] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* **1** (2001) 193–237.

[2] H. Bruin, M. Holland and I. Melbourne. Subexponential decay of correlations for compact group extensions of nonuniformly expanding systems. *Ergodic Theory Dynam. Systems* **25** (2005) 1719–1738.

[3] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability* **1** (1973) 19–42.

[4] W. Y. Chen. Assessment of Southern Oscillation sea-level pressure indices. *Monthly Weather Review* **110** (1982) 800–807.

[5] T. L. Delworth and M. E. Mann. Observed and simulated multidecadal variability in the Northern Hemisphere. *Climate Dynamics* **16** (2000) 661–676.

[6] J. O. Dickey, M. Ghil and S. L. Marcus. Extratropical aspects of the 40-50 day oscillation in length-of-day and atmospheric angular momentum. *J. Geophys. Res.* **96**(D12) (1991) 22643–22658.

[7] H. A. Dijkstra. Nonlinear Climate Dynamics. Cambridge University Press (2013).

[8] S. B. Feldstein. The timescale, power spectra, and climate noise properties of teleconnection patterns. *J. Cimate* **13** (2000) 4430–4440.

[9] M. Ghil, M. R. Allen, M. D Dettinger, K. Ide, D. Kondrashov, M. E. Mann, A. W. Robertson, A. Saunders, Y. Tian, F. Varadi and P. Yiou. Advanced spectral methods for climatic time series. *Reviews of Geophysics* **40** (2002) 3–41.

[10] J. P. Gollub and H. L. Swinney. Onset of turbulence in a rotating fluid. *Phys. Rev. Lett.* **35** (1975) 927–930.

[11] M. I. Gordin. The central limit theorem for stationary processes. *Soviet Math. Dokl.* **10** (1969) 1174–1176.

[12] G. A. Gottwald and I. Melbourne. Central limit theorems and suppression of anomalous diffusion for systems with symmetry. Preprint. arXiv:1404.0770.

[13] S. Gouëzel. Central limit theorem and stable laws for intermittent maps. *Probab. Theory Relat. Fields* **128** (2004) 82–122.

[14] N. Jiang, J. D. Neelin and M. Ghil. Quasi-quadrennial and quasi-biennial variability in the equatorial Pacific. *Climate Dynamics* **12** (1995) 101–112.

[15] C. Liverani, B. Saussol and S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999) 671–685.

[16] I. Melbourne and G. A. Gottwald. Power spectra for deterministic chaotic dynamical systems. *Nonlinearity* **21** (2008) 179–189.

[17] I. Melbourne and M. Nicol. Statistical properties of endomorphisms and compact group extensions. *J. London Math. Soc.* **70** (2004) 427–446.

[18] I. Melbourne and M. Nicol. Almost sure invariance principle for nonuniformly hyperbolic systems. *Commun. Math. Phys.* **260** (2005) 131–146.

[19] I. Melbourne and M. Nicol. Large deviations for nonuniformly hyperbolic systems. *Trans. Amer. Math. Soc.* **360** (2008) 6661–6676.

[20] I. Melbourne and D. Terhesiu. Decay of correlations for nonuniformly expanding systems with general return times. *Ergodic Theory Dynam. Systems* **34** (2014) 893–918.

[21] I. Melbourne and A. Török. Convergence of moments for Axiom A and nonuniformly hyperbolic flows. *Ergodic Theory Dynam. Systems* **32** (2012) 1091–1100.

[22] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.* **74** (1980) 189–197.

[23] D. Ruelle. Resonances of chaotic dynamical systems. *Phys. Rev. Lett.* **56** (1986) 405–407.

[24] N. G. van Kampen. *Stochastic processes in physics and chemistry.* Lecture Notes in Mathematics **888**, North-Holland Publishing Co., Amsterdam-New York, 1981.

[25] C. Wunsch. On sharp spectral lines in the climate record and the millennial peak. *Paleoceanography* **15**(4) (2000) 417–424.

[26] C. Wunsch. The spectral description of climate change including the 100 ky energy. *Climate Dynamics* **20** (2003) 353–363.

[27] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* **147** (1998) 585–650.

[28] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* **110** (1999) 153–188.