NATURAL MEASURES AND STATISTICAL PROPERTIES OF NON-STATISTICAL MAPS WITH MULTIPLE NEUTRAL FIXED POINTS

DOUGLAS COATES, IAN MELBOURNE, AND AMIN TALEBI

Abstract. In this article we show that a large class of infinite measure preserving dynamical systems that do not admit physical measures nevertheless exhibit strong statistical properties. In particular, we give sufficient conditions for existence of a distinguished natural measure ν such that the pushforwards of any absolutely continuous probability measure converge to ν. Moreover, we obtain a distributional limit law for empirical measures. We also extend existing results on the characterisation of the set of almost sure limit points for empirical measures. Our results apply to various intermittent maps with multiple neutral fixed points preserving an infinite σ-finite absolutely continuous measure.

1. Introduction

In this article, we show that a large class of infinite measure preserving dynamical systems that do not admit physical measures nevertheless exhibit strong statistical properties. Our results apply to a class of intermittent maps [PM80] such as those considered by Thaler [Tha80] and Coates *et al.* [CLM23]. Specifically, we study maps that preserve an infinite σ-finite absolutely continuous measure and have several "equally sticky" neutral fixed points.

For the sake of clarity, we first introduce our results for Thaler maps with two neutral fixed points [Tha02] before discussing more general situations. Let $f : [0,1] \to [0,1]$ be an interval map with the following properties for some $\alpha \in (0,1]$, b_1, $b_2 > 0$, $c \in (0,1)$:

(1) $0, 1$ are neutral fixed points: $f'(0) = f'(1) = 1$;

(2) $f|_{(0,c)}$ and $f|_{(c,1)}$ are C^2 diffeomorphisms onto $(0,1)$ admitting C^2 extensions onto $[0,1]$;

(3) $f'(x) > 1$ for all $x \in (0,1)$ and f is convex (resp. concave) on a neighbourhood of 0 (resp. 1);

(4) $fx - x \sim b_1 x^{1+1/\alpha}$ as $x \to 0$ and $fx - x \sim b_2(1-x)^{1+1/\alpha}$ as $x \to 1$.

By [Tha83], f is conservative and ergodic with a unique (up to scaling) invariant σ-finite absolutely continuous measure μ, and moreover $\mu([0,1]) = \infty$. However, it was shown in [ATZ05] that for almost every $x \in [0,1]$, the sequence of empirical measures

$$e_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j x} = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \delta_x, \tag{1.1}$$

does not converge in the weak-$*$ topology. Indeed, the set of limit points of $e_n(x)$ is almost surely equal to the set

$$\mathcal{S} := \{\nu_p := p\delta_0 + (1-p)\delta_1 : p \in [0,1]\}$$

of all convex combinations of the Dirac masses at the neutral fixed points.

Date: July 10, 2024.

Douglas Coates was partially supported by the São Paulo Research Foundation (FAPESP) grant 2022/16259-2.
Amin Talebi is partially supported by INSF grant no. 4001845.

Remark 1.1. Here and throughout, convergence of sequences of measures is with respect to the weak-$*$ topology unless otherwise specified.

Recall that an invariant probability measure ν is a ***physical measure*** if $e_n \to \nu$ with positive probability. Since e_n diverges almost everywhere for the Thaler maps described above, there exist no physical measures and such maps are said to be ***non-statistical***, see [CYZ20; Tal20] and references therein.

One may ask whether convergence holds when the point mass δ_x in (1.1) is replaced by other types of probability measure λ and whether there is a distinguished limit measure $\nu_{\bar{p}} \in \mathcal{S}$. A probability measure ν is called ***natural*** [BB03; CE12; JT05; Mis05] if there exists an absolutely continuous probability measure λ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\lambda = \nu$. (By the dominated convergence theorem, physical measures are natural.)

Our first result shows that there exists a unique natural measure $\nu_{\bar{p}} \in \mathcal{S}$. Moreover, $\nu_{\bar{p}}$ attracts all absolutely continuous probability measures λ and it is not necessary to take Cesáro averages. (See [Kel04] for similar results in the simpler symmetric setting with $a_1 = a_2$, where $\bar{p} = \frac{1}{2}$. See also [Zwe02] for results on nonexistence of natural measures when f is badly behaved near the neutral fixed points.)

Theorem 1.2 (Existence and uniqueness of natural measures). *There exists a* $\bar{p} \in [0,1]$, *with corresponding measure* $\nu_{\bar{p}} \in \mathcal{S}$, *such that*

$$\lim_{n\to\infty} f_*^n\lambda = \nu_{\bar{p}} \quad \textit{for every absolutely continuous probability measure } \lambda.$$

Remark 1.3. A formula for $\bar{p}$ is given in Section 4.1, namely $\bar{p} = c_1/(c_1+c_2)$ where c_1, c_2 are as in (4.4).

An almost immediate consequence of Theorem 1.2 is the following decay of correlations type result.

Corollary 1.4. *Suppose that* $\varphi : [0,1] \to \mathbb{R}$ *is bounded, measurable, and continuous at* 0 *and* 1. *Then*

$$\lim_{n\to\infty} \int \psi \cdot \varphi \circ f^n \, d\,\mathrm{Leb} = \int \psi \, d\,\mathrm{Leb} \int \varphi \, d\nu_{\bar{p}} \quad \textit{for all } \psi \in L^1(\mathrm{Leb}).$$

Equivalently, $\lim_{n\to\infty} \int \psi \cdot \varphi \circ f^n \, d\mu = \int \psi \, d\mu \int \varphi \, d\nu_{\bar{p}}$ *for all* $\psi \in L^1(\mu)$.

We now return to consideration of the sequence e_n of empirical measures. Since e_n fails to converge pointwise, it is natural to consider alternative modes of convergence such as distributional convergence.

To describe the distributional convergence of e_n, we recall the classical arcsine law for occupation times of [Lév39] as adapted by [Lam58] and then by [Tha02] to the current context. Define the sequence of occupation times

$$S_n := \sum_{j=0}^{n-1} 1_B \circ f^j,$$

where $B \subset [0,1)$ is a closed interval containing 0. Also, we define the $[0,1]$-valued random variable $Z_{\alpha,p}$ for $\alpha \in (0,1]$, $p \in [0,1]$, as follows: When $\alpha = 1$, we set $Z_{1,p} \equiv p$. When $\alpha \in (0,1)$, let $Z_{\alpha,p}$ be the random variable with continuous density

$$\frac{\hat{p}\sin\alpha\pi}{\pi} \frac{t^\alpha(1-t)^{\alpha-1} + t^{\alpha-1}(1-t)^\alpha}{\hat{p}^2 t^{2\alpha} + 2\hat{p}t^\alpha(1-t)^\alpha\cos\alpha\pi + (1-t)^{2\alpha}}, \tag{1.2}$$

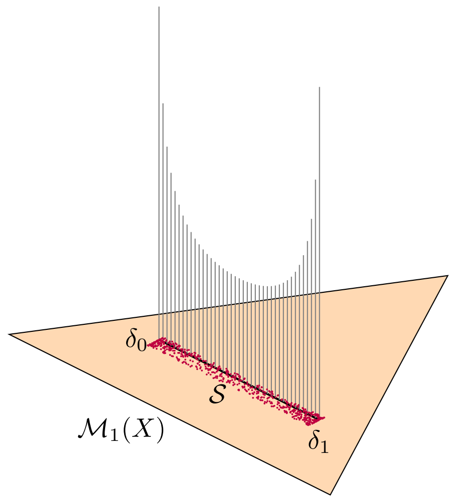

Figure 1. A schematic picture for the distributional limit law for empirical measures. The small dots around the set $\mathcal{S}$ are empirical measures $e_n(x)$ of an ensemble of points in the phase space chosen randomly (w.r.t. an absolutely continuous probability measure) for some fixed large n. As n increases, each individual dot oscillates between δ_0 and δ_1 ([ATZ05]), but in such a way as to make together an asymptotic distribution on $\mathcal{S}$ (Theorem 1.5).

where $\hat{p} = p^{-1} - 1$. Thaler [Tha02] proved a distributional limit law for occupation times

$$\tfrac{1}{n}S_n \text{ converges strongly in distribution to } Z_{\alpha,\bar{p}}.$$

When $\alpha \in (0,1)$, this means that $\lambda(\frac{1}{n}S_n \in I) \to \mathbb{P}(Z_{\alpha,\bar{p}} \in I)$ for all absolutely continuous probability measures λ on $[0,1]$ and all open intervals $I \subset [0,1]$. (When $\alpha = 1$, we exclude intervals I with an endpoint at $\bar{p}$.) Equivalently, $\frac{1}{n}S_{n*}\lambda \to \omega$ where ω is the distribution $\omega(I) = \mathbb{P}(Z_{\alpha,\bar{p}} \in I)$ of $Z_{\alpha,\bar{p}}$.

To each $[0,1]$-valued random variable Z, we can associate the $\mathcal{S}$-valued random variable $\nu_Z = Z\delta_0 + (1-Z)\delta_1$.

Theorem 1.5 (Distributional limit law for empirical measures).

$$e_n \textit{ converges strongly in distribution to } \nu_{Z_{\alpha,\bar{p}}}.$$

As mentioned before, the set of accumulation points of $e_n(x)$ is precisely the set $\mathcal{S}$. The result above describes how the $e_n(x)$ are asymptotically distributed on $\mathcal{S}$, see Figure 1.

Remark 1.6. Let $\mathcal{M}_1(M)$ denote the space of probability measures on a measure space M. The sequence $e_n : [0,1] \to \mathcal{M}_1([0,1])$ induces the pushforward sequence $e_{n*} : \mathcal{M}_1([0,1]) \to \mathcal{M}_1(\mathcal{M}_1([0,1]))$, so, the theorem above is equivalent to $\lim_{n\to\infty} e_{n*}\lambda = \omega$ for every absolutely continuous probability measure λ where ω is the distribution of $\nu_{Z_{\alpha,\bar{p}}}$.

Since $(e_{n*}\lambda)(\mathcal{E}) = \lambda(e_n \in \mathcal{E})$, we can view Theorem 1.5 as a statement about the limit of $e_{n*}\lambda$ for all absolutely continuous measures λ.

Since $e_{n*}\lambda$ is a probability measure on the convex set $\mathcal{M}_1([0,1])$, we can consider its expectation $\mathbb{E}(e_{n*}\lambda) = \int_{\mathcal{M}_1([0,1])} \omega \, de_{n*}\lambda \in \mathcal{M}_1([0,1])$, which can be interpreted as a probability measure on $[0,1]$ satisfying $\mathbb{E}(e_{n*}\lambda)(E) = \int \omega(E) \, de_{n*}\lambda(\omega)$.

We have the following consequence of Theorem B:

Corollary 1.7. *Let λ be an absolutely continuous probability measure on $[0,1]$. Then*

(a) $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j\lambda = \int e_n\,d\lambda = \mathbb{E}(e_{n*}\lambda)$ *for all* $n \geq 1$.

(b) The common limit of the sequences of measures in (a) is given by $\lim_{n\to\infty}\mathbb{E}(e_{n*}\lambda) = \nu_{\bar{p}}$.

Beyond two neutral fixed points. Our results generalise to any finite number of neutral fixed points $\xi_1,\dots,\xi_d$. The proof of Theorem 1.2 requires no modifications. The treatment of the arcsine law in [Tha02; TZ06] assumes $d = 2$ but this restriction is removed by Sera & Yano [SY19] using a different method. We use [SY19] to generalise Theorem 1.5. The generalisations of Theorem 1.2 and 1.5 are stated in Section 2 as Theorems C and B respectively.

The final main result in this paper is Theorem A which shows that for general $d \geq 1$ and $\alpha \in (0,1)$, the set of limit points of the sequence of empirical measures e_n is almost surely equal to the $(d-1)$-dimensional simplex

$$\mathcal{S} := \{p_1\delta_{\xi_1} + \cdots + p_d\delta_{\xi_d} : p_1,\dots,p_d \geq 0,\ p_1 + \cdots + p_d = 1\}.$$

Previous arguments in [ATZ05; CL24] exploited connectedness of the set of limit points and were effective for $d \leq 2$, $\alpha \in (0,1]$. Our method uses instead the full support of the arcsine law in [SY19] combined with Theorem B. However, this full support property fails for $\alpha = 1$ which means that Theorem A remains open for $d \geq 3$, $\alpha = 1$.

The remainder of the paper is organised as follows. In Section 2, we present an abstract setup which allows for any finite number of neutral fixed points, and we state our main results Theorem A, B and C. Also, we state and prove Corollaries 2.8 and 2.9 which generalise Corollaries 1.7 and 1.4 respectively. Theorems A, B and C are proved in Section 3. In Section 4, we verify that our abstract setup includes the classes of intermittent maps in [CLM23; Tha80].

Notation. We use “big O” and $\lesssim$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \lesssim b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq Cb_n$ for all $n \geq n_0$. As usual, $a_n = o(b_n)$ means that $a_n/b_n \to 0$ and $a_n \sim b_n$ means that $a_n/b_n \to 1$.

2. Abstract statement of results

2.1. **Maps with Gibbs-Markov first return maps.** In this section we recall some definitions and basic properties of maps with Gibbs-Markov first return maps needed to state our main assumptions.

Let X be a compact metric space with Borel probability measure m and let $f : X \to X$ be a nonsingular measurable map. We suppose that f is ergodic and conservative. (A standard reference for this material is [Aar97, Ch 1].)

We say that the map f together with a countable partition $\mathcal{P}$ of X into positive measure subsets is a ***Markov map*** if the restriction of f to each partition element is a measurable bijection onto a union of partition elements. The map f is ***topologically mixing*** if for every $a, b \in \mathcal{P}$ there exists an $N \geq 1$ such that $f^n a \cap b \neq \emptyset$ for every $n \geq N$.

Let $Y \subset X$ be a union of partition elements. We define the ***first return map*** $F : Y \to Y$ by setting $F(y) := f^{\tau(y)}y$ where $\tau(y) := \inf\{n \geq 1 : f^n y \in Y\}$. The map F is ***Gibbs-Markov*** if there exists a refinement $\mathcal{P}_Y$ of the partition $\mathcal{P}$ for which F is again a Markov map and the following additional properties hold:

- **finite images**: $\mathrm{Card}\{Fa : a \in \mathcal{P}_Y\} < \infty$;

- **bounded distortion**: there exists $\theta \in (0,1)$ and $C > 0$ such that the function $\log \frac{dm}{dm\circ F}$ is d_θ-Lipschitz on elements of $\mathcal{P}_Y$ where d_θ is the metric $d_\theta(x,y) := \theta^{s(x,y)}$ and $s(x,y) := \inf\{n \geq 0 : F^n x, F^n y$ lie in different elements of $\mathcal{P}_Y\}$.

We denote by $\mathcal{B}_\theta(Y)$ the space of real-valued functions which are Lipschitz continuous on Y with respect to the metric d_θ. We recall the following classical result about the existence of invariant measures for maps with Gibbs-Markov first return maps.

Lemma 2.1. *Suppose that $f : X \to X$ is a conservative ergodic Markov map with Gibbs-Markov first return map $F = f^\tau : Y \to Y$. Then f preserves a unique (up to scaling) absolutely continuous σ-finite measure μ, and $\mu(X) < \infty$ if and only if $\tau \in L^1(m)$.*

Moreover, $\mu(Y) < \infty$ and the density $h = d\mu/dm$ satisfies $h1_Y$, $h^{-1}1_Y \in \mathcal{B}_\theta(Y)$.

Proof. Standard references include [AD01; ADU93] and [Aar97, Chapter 4]. □

2.2. **Assumptions and notations.** Let (X, m) be a metric space with Borel probability measure m. Throughout, we suppose that $f : X \to X$ is is a conservative ergodic Markov map with Gibbs-Markov first return map $F = f^\tau : Y \to Y$, and moreover that f and F are topologically mixing. Let μ be the f-invariant absolutely continuous σ-finite measure in Lemma 2.1. We assume that $\mu(X) = \infty$.

H1: There exists fixed points $\xi_1, \dots, \xi_d \in X$ such that

$$\mu\big(X \setminus \{B_\varepsilon(\xi_1) \cup \cdots \cup B_\varepsilon(\xi_d)\}\big) < \infty \qquad \text{for all } \varepsilon > 0 \text{ sufficiently small.}$$

This assumption implies that neighbourhoods of the fixed points $\xi_1, \dots, \xi_d$ carry the infinite part of the mass.

The next assumption ensures that the inducing set Y *dynamically separates* $\xi_1, \dots, \xi_d$ and that the excursion times from Y to neighbourhoods of the fixed points have certain tail distributions.

H2: There exist constants $\alpha \in (0,1]$, $c_1, \dots, c_d > 0$ and a measurable partition $X_1, \dots, X_d$ of $X \setminus Y$ such that

(a) $\xi_k \in \operatorname{Int} X_k$, for $k = 1, \dots, d$;

(b) For $k \neq \ell$, orbits cannot pass from X_k to X_ℓ without first entering Y. Equivalently,

$$\{\tau = n\} = \bigcup_{k=1,\dots,d} \{\tau^{(k)} = n-1\} \quad \text{for } 1 \leq n < \infty$$

where $\tau^{(k)}(y) := \operatorname{Card}\{n \leq \tau(y) : f^n y \in X_k\}$, for $y \in Y$, $k = 1, \dots, d$;

(c) $\mu(\tau^{(k)} > n) \sim c_k n^{-\alpha}$ as $n \to \infty$ for $k = 1, \dots, d$.

Note that

$$\mu(\tau > n) \sim c_\tau n^{-\alpha} \quad \text{where} \quad c_\tau := c_1 + \cdots + c_d > 0. \tag{2.1}$$

Remark 2.2. We expect that the results in this paper remain valid when the constants c_k in H2(c) are replaced by $c_k \ell(n)$ where $\ell(n)$ is slowly varying. However, the proof of Theorem C requires further calculations.

Our third assumption is somewhat technical to state but is very mild. Let $\mathcal{P}_r$ denote the r'th refinement of the partition $\mathcal{P}$ under f^r and let $\mathcal{P}_r^*$ consist of $r+1$ cylinders $a = [a_0, \dots, a_r] \in \mathcal{P}_r$ where $a_0, \dots, a_{r-1} \not\subset Y$ and $a_r \subset Y$. Given $\rho \in L^1(X, m)$ and $r \geq 0$, define

$$Q_r^\rho : Y \to \mathbb{R}, \qquad Q_r^\rho := \sum_{a \in \mathcal{P}_r^*} 1_{f^r a} \cdot \frac{\rho}{Jf^r} \circ f^r|_a^{-1} \tag{2.2}$$

where $Jf^r := \frac{dm \circ f^r}{dm}$ denotes the Jacobian of f^r with respect to m.

> **H3**: There is a dense subset $K(X) \subset L^1(X, m)$ such that $Q_r^\rho \in \mathcal{B}_\theta(Y)$ for all $\rho \in K(X)$ and $r \geq 0$.

Remark 2.3. Assumption H3 is automatic if f has finitely many branches and there exists $\eta > 0$ such that f is $C^{1+\eta}$ on $f^{-r}Y$ for each fixed $r \geq 0$, which is the case in the applications considered in this paper. For such maps, we can take $K(X) = C^\eta(X)$.

Our final assumption is the following "smooth tail" estimate:

> **H4**: If $\alpha \in (0, 1/2]$, then $\mu(\tau = n) = O(n^{-(1+\alpha)})$.

2.3. **Almost sure behaviour of the empirical measures.** We let $\mathcal{M}_1(X)$ denote the space of Borel probability measures on X. Define the map $e_n : X \to \mathcal{M}_1(X)$ which maps each point $x \in X$ to its n^{th} empirical measure $e_n(x)$ as given in (1.1).

Our first result concerns the almost sure behaviour of the empirical measures e_n. For $x \in X$, we let $\mathcal{L}(x) \subset \mathcal{M}_1(X)$ denote the set of weak-$*$ accumulation points of $e_n(x)$. Let $\mathcal{S}_0$ be the simplex

$$\mathcal{S}_0 := \{p \in [0,1]^d : p_1 + \cdots + p_d = 1\}.$$

For $p \in \mathcal{S}_0$, let $\nu_p := p_1\delta_{\xi_1} + \cdots + p_d\delta_{\xi_d}$ be the corresponding convex combination of Dirac masses at the fixed points, and define

$$\mathcal{S} := \{\nu_p : p \in \mathcal{S}_0\},$$

to be the set of all such convex combinations. With this notation in place we can state our first result.

Theorem A. *Suppose that H1–H2 hold and that $\alpha \in (0, 1)$. Then*

$$\mathcal{L}(x) = \mathcal{S} \quad \textit{for a.e. } x \in X.$$

In particular, when $d \geq 2$ there are no physical measures.

Remark 2.4. The inclusion $\mathcal{L}(x) \subset \mathcal{S}$ is elementary, see Lemma 3.2 below, so it is the reverse inclusion that is of interest. The case $d = 1$ is trivial, and the case $d = 2$ is well-understood. Indeed, there are techniques [ATZ05; CL24] for showing that $\delta_{\xi_k} \in \mathcal{L}(x)$ a.e. for each k and Theorem A then follows for $d = 2$ by connectedness of $\mathcal{L}(x)$. This approach includes the case $\alpha = 1$.

For $d \geq 3$, Theorem A is completely new to the best of our knowledge. Our proof involves a different technique which works for all d but only in the range $\alpha \in (0, 1)$. We expect that Theorem A holds also for $d \geq 3$, $\alpha = 1$, but new ideas seem to be required.

2.4. **Distributional convergence of the empirical measures.** Our second result concerns strong distributional convergence of the sequence e_n of empirical measures.

Notation: Suppose that Z_n is a sequence of measurable functions on X taking values in some Borel space M and that Z is a random variable with distribution $\omega \in \mathcal{M}_1(M)$ also taking values in M (but not necessarily defined on X). Given $\lambda \in \mathcal{M}_1(X)$, we write $Z_n \to_\lambda Z$ if $\lim_{n\to\infty} Z_{n*}\lambda = \omega$.

Now let $\mathcal{M}_1^{\mathrm{ac}}(X)$ denote the set of Borel probability measures on X which are absolutely continuous with respect to m. We say that Z_n *converges strongly in distribution* to Z, and write $Z_n \to_d Z$, if $Z_n \to_\lambda Z$ for all $\lambda \in \mathcal{M}_1^{\mathrm{ac}}(X)$.

Fix ε so that the neighbourhoods $B_\varepsilon(\xi_k)$, $k = 1, \dots, d$ are disjoint. As was the case for the Thaler maps in Theorem 1.5, we first consider strong distributional convergence for occupation times $S_n = (S_n^1, \dots, S_n^d) : X \to [0,1]^d$ defined by

$$S_n^k := \sum_{j=0}^{n-1} 1_{B_\varepsilon(\xi_k)} \circ f^j, \quad \text{for } k = 1, \dots, d,\ n \geq 1. \tag{2.3}$$

Following [SY19; Ser20], we consider a multiray generalisation of the classical arcsine law. For $\alpha \in (0,1)$, $p \in \mathcal{S}_0$, let $\zeta_1, \dots, \zeta_d$ be independent $[0,\infty)$-valued random variables defined on a common probability space with one-sided α-stable distribution characterised for $k = 1, \dots, d$ by

$$\mathbb{E}\exp(-t\zeta_k) = \exp(-t^\alpha p_k), \quad t > 0. \tag{2.4}$$

Define the $\mathcal{S}_0$-valued random variable

$$Z_{\alpha,p} := \frac{1}{\zeta_1 + \cdots + \zeta_d}(\zeta_1, \dots, \zeta_d). \tag{2.5}$$

When $\alpha = 1$, we define $Z_{1,p} \equiv p$.

Remark 2.5. In the special case that $d = 2$, one has that $Z_{\alpha,p}^{(2)} = 1 - Z_{\alpha,p}^{(1)}$ and for $\alpha \in (0,1)$ the distribution of $Z_{\alpha,p}^{(1)}$ admits the continuous density in (1.2) (identifying p with p_1).

Remark 2.6. The first moment of the multiray arcsine law $\mathbb{E}Z_{\alpha,p}$ is given by $\mathbb{E}Z_{\alpha,p} = p$. This can be verified using the double Laplace formula [SY19, Proposition 2.6],

$$\int_0^\infty e^{-qt}\mathbb{E}\Big(\exp\Big\{-t\sum_{k=1}^d \lambda_k\zeta_k\Big\}\Big)\,dt = \frac{\sum_{k=1}^d p_k(q+\lambda_k)^{\alpha-1}}{\sum_{k=1}^d p_k(q+\lambda_k)^\alpha}, \quad \lambda = (\lambda_1, \dots, \lambda_d) \in [0,\infty)^d,\ q > 0,$$

as derived in [Yan17, Proposition 3.6]. Differentiating w.r.t. λ_k and setting $\lambda = 0$ yields $\mathbb{E}Z_{\alpha,p}^{(k)} = p_k$.

We can now recall the arcsine law of [SY19; Ser20]. Recall from (2.1) that $c_\tau = \sum_{k=1}^d c_k$. Set

$$\bar{p} = (\bar{p}_1, \dots, \bar{p}_d) := c_\tau^{-1}(c_1, \dots, c_d) \in \mathcal{S}_0.$$

Theorem 2.7. *Suppose that H1–H2 hold. Then* $\frac{1}{n}S_n \to_d Z_{\alpha,\bar{p}}$.

Proof. It suffices to check that H1–H2 imply [Ser20, Assumptions 2.1–2.3] as the result then follows from [Ser20, Corollary 4.2 and Theorem 3.3]. Let $F : Y \to Y$ be the Gibbs-Markov first return map in H2. Notice that H2(b) yields [Ser20, Assumption 2.1], and then H2(c) together with [Ser20, Lemma 2.4] gives [Ser20, Assumption 2.2]. Finally, as F is topologically mixing it is exponentially continued fraction mixing (see for example [Aar97, Section 4]) which immediately implies [Ser20, Assumption 2.3]. □

To each $\mathcal{S}_0$-valued random variable Z, we can associate the $\mathcal{S}$-valued random variable $\nu_Z = Z_1\delta_{\xi_1} + \cdots + Z_d\delta_{\xi_d}$. Notice that $\mathbb{E}\nu_Z = \sum_{k=1}^d \mathbb{E}Z_k\delta_{\xi_k} = \sum_{k=1}^d (\mathbb{E}Z)_k\delta_{\xi_k} = \nu_{\mathbb{E}Z}$. In particular, $\mathbb{E}\nu_{Z_{\alpha,p}} = \nu_p$.

Theorem B. *Suppose that H1–H2 hold. Then* $e_n \to_d \nu_{Z_{\alpha,\bar p}}$. *Equivalently,* $\lim_{n\to\infty} e_{n*}\lambda = \omega$ *for all* $\lambda \in \mathcal{M}_1^{\rm ac}(X)$ *where* $\omega \in \mathcal{M}_1(\mathcal{S})$ *is the distribution of* $\nu_{Z_{\alpha,\bar p}}$.

As in the introduction, we obtain the following consequence:

Corollary 2.8. *Let* $\lambda \in \mathcal{M}_1^{\rm ac}(X)$. *Then*

(a) $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j\lambda = \int e_n\,d\lambda = \mathbb{E}(e_{n*}\lambda)$ *for all* $n \ge 1$.

(b) The common limit of the sequences of measures in (a) is given by $\lim_{n\to\infty}\mathbb{E}(e_{n*}\lambda) = \nu_{\bar p}$.

Proof. Let $i : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ be the identity map. Then

$$\mathbb{E}(e_{n*}\lambda) = \int_{\mathcal{M}_1(X)} \omega\, d(e_{n*}\lambda)(\omega) = \int_{\mathcal{M}_1(X)} i(\omega)\, d(e_{n*}\lambda)(\omega) = \int_X i\circ e_n(x)\, d\lambda(x) = \int_X e_n\, d\lambda.$$

Next,

$$\Big(\int_X \delta_{f^j x}\, d\lambda(x)\Big)(E) = \int_X \delta_x(E)\, df_*^j\lambda(x) = \int_X 1_E\, df_*^j\lambda = (f_*^j\lambda)(E),$$

so $\int_X \delta_{f^j x}\, d\lambda(x) = f_*^j\lambda$. Hence

$$\int_X e_n\, d\lambda = \frac{1}{n}\sum_{j=0}^{n-1}\int_X \delta_{f^j x}\, d\lambda(x). = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\lambda.$$

(b) Since $\mathbb{E}$ acts continuously from $\mathcal{M}_1(\mathcal{M}_1(X))$ to $\mathcal{M}_1(X)$, it follows from Theorem B that $\lim_{n\to\infty}\mathbb{E}(e_{n*}\lambda) = \mathbb{E}\nu_{Z_{\alpha,\bar p}} = \nu_{\bar p}$. □

2.5. **Existence and uniqueness of natural measures.** Our third result concerns weak-$*$ convergence of pushforwards $f_*^n\lambda$ of absolutely continuous probability measures $\lambda \in \mathcal{M}_1^{\rm ac}(X)$.

Theorem C. *Suppose that H1–H4 hold with* $\alpha \in (0,1]$. *Then*

$$\lim_{n\to\infty} f_*^n\lambda = \nu_{\bar p} \quad \textit{for all } \lambda \in \mathcal{M}_1^{\rm ac}(X).$$

As in the introduction, we obtain the following result on decay of correlations:

Corollary 2.9. *Suppose* $\varphi : X \to \mathbb{R}$ *is bounded, measurable, and continuous at each* $\xi_1,\dots,\xi_d$. *Then*

$$\lim_{n\to\infty}\int \psi\cdot\varphi\circ f^n\, dm = \int \psi\, dm \int \varphi\, d\nu_{\bar p} \quad \textit{for all } \psi \in L^1(m).$$

Equivalently, $\lim_{n\to\infty}\int \psi\cdot\varphi\circ f^n\, d\mu = \int\psi\, d\mu\int\varphi\, d\nu_{\bar p}$ *for all* $\psi \in L^1(\mu)$.

Proof. We begin by proving that the first limit holds.

By Theorem C, $f_*^n\lambda \to \nu_{\bar p}$ weak-*, so by definition $\int\varphi\, df_*^n\lambda \to \int\varphi\, d\nu_{\bar p}$ for all continuous $\varphi : X \to \mathbb{R}$. By [Bil95, Theorem 25.7], $\int\varphi\, df_*^n\lambda \to \int\varphi\, d\nu_{\bar p}$ for all bounded measurable functions $\varphi : X \to \mathbb{R}$ that are continuous except on a set of $\nu_{\bar p}$-measure zero, which is precisely the class of observables in the statement of the corollary.

Fix such a φ. Let $\psi \in L^1(m)$ with $\int \psi\, dm \neq 0$ and write $\psi = \rho \int \psi\, dm$ where $\rho \in L^1(m)$ and $\int \rho\, dm = 1$. Let $d\lambda = \rho\, dm$. Then

$$(\textstyle\int \psi\, dm)^{-1} \int \psi \cdot \varphi \circ f^n\, dm = \int \rho \cdot \varphi \circ f^n\, dm = \int \varphi \circ f^n\, d\lambda = \int \varphi\, df_*^n \lambda \to \int \varphi\, d\nu_{\bar{p}}.$$

The case $\int \psi\, dm = 0$ is dealt with by approximating, concluding the proof of the first limit.

If $\psi \in L^1(\mu)$, then $\psi h \in L^1(m)$ and the second limit follows from the first. Similarly, the first limit follows from the second. □

3. Proofs

In this section, we prove the three main theorems in this paper. It turns out to be convenient to prove them in the order B, A, C.

3.1. **Ergodic theorem**.

In this subsection, we recall the ergodic theorem for infinite measure systems and derive a consequence for the systems studied in this paper.

Lemma 3.1. *Suppose that H1 holds. If $\varphi \in L^1(\mu)$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j = 0$ almost everywhere.*

Proof. This is an easy consequence of the Hopf ratio ergodic theorem, see for example [Aar97, Exercise 2.2.1]. □

Lemma 3.2. *Suppose that H1 holds. Then $\mathcal{L}(x) \subset \mathcal{S}$ for almost every $x \in X$.*

Proof. We provide the details for completeness (cf. [ATZ05]).

For $\varepsilon > 0$, define $X^{(\varepsilon)} := X \setminus \{B_\varepsilon(\xi_1) \cup \cdots \cup B_\varepsilon(\xi_d)\}$. By H1, $\mu(X^{(\varepsilon)}) < \infty$ for all $\varepsilon > 0$. Choose $\psi : X \to [0,1]$ continuous and supported in $X^{(\varepsilon/2)}$ such that $\psi|X^{(\varepsilon)} \equiv 1$. In particular, $\int_X \psi\, d\mu < \infty$ so, by Lemma 3.1, there exists $X' \subset X$ with $\mu(X \setminus X') = 0$ such that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j x) = 0$ for all $x \in X'$.

Suppose that $x \in X'$ and $\omega \in \mathcal{L}(x)$, and choose a subsequence n_i such that $e_{n_i}(x) \to \omega$. Then $\frac{1}{n_i} \sum_{j=0}^{n_i-1} \psi(f^j x) \to \int_X \psi\, d\omega$, yielding $\omega(X^{(\varepsilon)}) \leq \int_X \psi\, d\omega = 0$. Hence, elements of $\mathcal{L}(x)$ are supported in $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_d)$ for all $x \in X'$. Since $\varepsilon > 0$ can be chosen arbitrarily small, the result follows. □

Corollary 3.3. *Suppose that $e_{n_k} \to_m Z$ for some $\mathcal{M}_1(X)$-valued random variable Z and some subsequence n_k. Then Z takes values in $\mathcal{S}$.*

Proof. Let $\psi : \mathcal{M}_1(X) \to \mathbb{R}$ be continuous and supported in $\mathcal{M}_1(X) \setminus \mathcal{S}$. By the dominated convergence theorem,

$$\int_X \psi(e_{n_k}(x))\, dm(x) \to \mathbb{E}\, \psi(Z).$$

But $\psi(e_{n_k}) \to 0$ a.e. by Lemma 3.2. Applying the dominated convergence theorem once more, $\mathbb{E}\, \psi(Z) = 0$, and the result follows. □

Remark 3.4. Corollary 3.3 shows that the only possible distributional limit points of the sequence e_n are random variables of the form ν_Z where Z is an $\mathcal{S}_0$-valued random variable.

3.2. **Proof of Theorem B.** Fix $\varepsilon > 0$ such that $B_\varepsilon(\xi_k)$ are disjoint for $k = 1, \dots, d$ and define $S_n : X \to [0,1]^d$ as in (2.3).

Lemma 3.5. *Let Z be a random variable with values in $\mathcal{S}_0$. Then $e_n \to_m \nu_Z$ if and only if $n^{-1} S_n \to_m Z$.*

Proof. Define

$$\pi : M_1(X) \to [0,1]^d, \qquad \pi(\omega) := (\omega(B_\varepsilon(\xi_1)), \dots, \omega(B_\varepsilon(\xi_d))).$$

Note that π restricts to the natural identification $\nu_Z \mapsto Z$ between $\mathcal{S}$ and $\mathcal{S}_0$. Also, π is continuous at elements in $\mathcal{S}$ and satisfies $\pi(e_n) = \frac{1}{n} S_n$.

In particular, if $e_{n_k} \to_m \nu_Z$ for some subsequence n_k, then it is an immediate consequence of the continuity of π at ν_Z and the continuous mapping theorem that $\frac{1}{n} S_{n_k} = \pi(e_{n_k}) \to_m \pi(\nu_Z) = Z$ completing the proof in one direction.

The converse follows by a standard probabilistic argument. Suppose that $\frac{1}{n} S_n \to_m Z$. Let $\widehat{Z}$ be a distributional limit point for e_n, so $e_{n_k} \to_m \widehat{Z}$ for some subsequence n_k. By Remark 3.4, $\widehat{Z} = \nu_A$ for some random variable A with values in $\mathcal{S}_0$. By what we just proved, $\frac{1}{n} S_{n_k} \to_m A$. Hence, $A = Z$, so $\widehat{Z} = \nu_Z$ is the unique distributional limit point for e_n. This means that $e_n \to_m \nu_Z$. □

Proof of Theorem B. By Theorem 2.7, $n^{-1} S_n \to_m Z_{\alpha,\bar{p}}$. Hence by Lemma 3.5, $e_n \to_m \nu_{Z_{\alpha,\bar{p}}}$.

Let d_W denote the *Wasserstein distance* on $\mathcal{M}_1(X)$,

$$d_W(\omega, \omega') := \sup_{\varphi \in \mathrm{Lip}_1} \Big| \int \varphi \, d\omega - \int \varphi \, d\omega' \Big|,$$

where $\mathrm{Lip}_1 = \{\varphi : X \to \mathbb{R} : \mathrm{Lip}\, \varphi + \|\varphi\|_\infty \le 1\}$ and $\mathrm{Lip}\, \varphi$ denotes the smallest Lipschitz constant of $\varphi : X \to \mathbb{R}$. Recall that d_W induces the weak-$*$ topology on $\mathcal{M}_1(X)$. Also, $d_W(e_n \circ f, e_n) \le \frac{2}{n}$, so the functions $e_n : X \to \mathcal{M}_1(X)$ satisfy the "asymptotic invariance" condition $d_W(e_n \circ f, e_n) \to_m 0$. Hence, we may apply [Zwe07, Theorem 1] to deduce from $e_n \to_m \nu_{Z_{\alpha,\bar{p}}}$ that $e_n \to_d \nu_{Z_{\alpha,\bar{p}}}$. □

3.3. **Proof of Theorem A.** We first show that the limiting random variable $\nu_{Z_{\alpha,\bar{p}}}$ in Theorem B has full support in $\mathcal{S}$ when $\alpha \in (0,1)$.

Lemma 3.6. *Let $\alpha \in (0,1)$, $p \in \mathcal{S}_0$ with $p_i > 0$ for all i. Then $\mathbb{P}(\nu_{Z_{\alpha,p}} \in B_\varepsilon(\nu)) > 0$ for all $\varepsilon > 0$, $\nu \in \mathcal{S}$.*

Proof. Let $\zeta_1, \dots, \zeta_d$ be the independent, α-stable random variables that appear in the definition (2.5) of $Z_{\alpha,p}$. Recall [Nol20, Lemma 1.1, Proposition 3.2] that as the ζ_k are non-negative α-stable random variables with Laplace transforms given by (2.4), their distributions are fully supported and continuous on the half line $[0, \infty)$.

For every $q \in \mathcal{S}_0$, $\varepsilon > 0$ the set

$$U := \left\{ x \in [0,\infty)^d : \frac{x_k}{x_1 + \cdots + x_d} \in (q_k - \varepsilon, q_k + \varepsilon), \ \text{for all } k = 1, \dots, d \right\}$$

contains a nonempty open rectangle $\prod_{k=1}^d (a_k, b_k)$. By definition of $Z_{\alpha,p}$ and the independence of the ζ_k,

$$\begin{aligned}\mathbb{P}(Z_{\alpha,p} \in B_\varepsilon(q)) &= \mathbb{P}((\zeta_1, \dots, \zeta_d) \in U) \\ &\geq \mathbb{P}(\zeta_k \in (a_k, b_k) : k = 1, \dots, d) = \prod_{k=1}^d \mathbb{P}(\zeta_k \in (a_k, b_k)) > 0.\end{aligned}$$

It follows that $\mathbb{P}(\nu_{Z_{\alpha,p}} \in B_\varepsilon(\nu_q)) > 0$. □

Proof of Theorem A. By Lemma 3.2, $\mathcal{L}(x) \subset \mathcal{S}$ for almost every x. To prove the converse, let $\nu \in \mathcal{S}$ and consider the function

$$\varphi(x) \coloneqq \liminf_{n\to\infty} d(e_n(x), \nu),$$

where d is any metric metrising the weak-$*$ topology. We will show that $\varphi = 0$ almost everywhere and so $\nu \in \mathcal{L}(x)$ for almost every x. The result then follows since $\nu \in \mathcal{S}$ is arbitrary.

Notice that φ is invariant for f and so, by ergodicity, must be almost everywhere equal to some constant $c \geq 0$. Suppose for contradiction that $c > 0$, and let $0 < \varepsilon < c$. It then follows that $m\{x : e_n(x) \in B_\varepsilon(\nu) \text{ for infinitely many } n\} = 0$.

By Theorem B, $e_n \to_m \nu_{Z_{\alpha,\bar{p}}}$ with $\bar{p}_i > 0$ for all i. By the Portmanteau lemma together with Lemma 3.6,

$$\liminf_{n\to\infty} m(e_n \in B_\varepsilon(\nu)) \geq \mathbb{P}(\nu_{Z_{\alpha,\bar{p}}} \in B_\varepsilon(\nu)) > 0.$$

Hence

$$\begin{aligned} m\{x : e_n(x) \in B_\varepsilon(\nu) \text{ for infinitely many } n\} &= m\Big(\bigcap_{n=1} \bigcup_{\ell\geq n} \{e_\ell \in B_\varepsilon(\nu)\}\Big) \\ &= \lim_{n\to\infty} m\Big(\bigcup_{\ell\geq n} \{e_\ell \in B_\varepsilon(\nu)\}\Big) \\ &\geq \liminf_{n\to\infty} m(e_n \in B_\varepsilon(\nu)) > 0, \end{aligned}$$

where we have used the fact that the sequence of sets $A_n = \bigcup_{\ell\geq n}^\infty \{e_\ell \in B_\varepsilon(\nu)\}$ is decreasing: $A_{n+1} \subset A_n$. This contradicts our assumption that $c > 0$ and so $\varphi(x) = 0$ almost everywhere. □

3.4. **Proof of Theorem C.** Let f satisfy H1–H3. If $\alpha \in (0, \frac{1}{2}]$, then we suppose moreover that $\mu(\tau = n) = O(n^{-(\alpha+1)})$.

We let $L : L^1(X, \mu) \to L^1(X, \mu)$ be the *transfer operator* for f defined by the relation

$$\int_X v \cdot w \circ f \, d\mu = \int_X Lv \cdot w \, d\mu, \quad \text{for all } v \in L^1(X, \mu),\ w \in L^\infty(X).$$

Define for $n \geq 0$,

$$T_n : L^1(Y, \mu|_Y) \to L^1(Y, \mu|_Y), \qquad T_n v = 1_Y L^n(1_Y v).$$

We recall some results from [Gou11; MT12] which describe the asymptotic behaviour of T_n acting on the space $\mathcal{B}_\theta(Y)$ in Section 2.1.

Theorem 3.7. *Let $c_\tau \in (0, \infty)$ be as in (2.1). Then, for every $v \in \mathcal{B}_\theta(Y)$,*

$$\begin{aligned} c_\tau \log n \, T_n v &\to \int_Y v \, d\mu, && \alpha = 1 \\ c_\tau n^{1-\alpha} T_n v &\to \tfrac{1}{\pi} \sin \pi\alpha \int_Y v \, d\mu, && \alpha \in (0, 1) \end{aligned}$$

uniformly on Y as $n \to \infty$.

Proof. By assumption, $F = f^\tau : Y \to Y$ is a Gibbs-Markov map and $\mu(\tau > n) \sim c_\tau n^{-\alpha}$. Also, the underlying conservative ergodic map $f : X \to X$ is topologically mixing.

In the range $\alpha \in (\frac{1}{2}, 1]$ we can apply [MT12, Theorem 2.1, Proposition 11.4].

In the range $\alpha \in (0, \frac{1}{2}]$, we use [Gou11, Theorem 1.4]. The hypotheses in [Gou11] are stated slightly differently than in [MT12]. In the notation of these papers, the essential differences are as follows: (i) There is a stronger assumption on $\|R_n\|$ which holds since $\mu(\tau = n) = O(n^{-(\alpha+1)})$; (ii) There is the requirement that $R(1)$ has no eigenvalues on the unit circle besides 1 which holds since F is topologically mixing (see for example [AD01, Theorem 1.6]). □

Define

$$Y_0 := Y; \quad \text{and} \quad Y_r := f^{-r}Y \setminus \{\cup_{i=0}^{r-1} f^{-i}Y\}, \quad \text{for } r \geq 1.$$

Recall that $h = \frac{d\mu}{dm}$ denotes the density of the invariant measure μ. Let $\lambda \in \mathcal{M}_1^{\mathrm{ac}}(X)$ with density $\rho = \frac{d\lambda}{dm}$. Let $K(X)$ be the dense subset of $L^1(X, m)$ in H3.

Proposition 3.8. *If $\rho \in K(X)$, then $L^r(\rho h^{-1} 1_{Y_r}) \in \mathcal{B}_\theta(Y)$ for all $r \geq 0$.*

Proof. Note first that $L^r(\rho h^{-1} 1_{Y_r})$ is supported in Y by definition of Y_r. Let $M : L^1(X) \to L^1(X)$ denote the transfer operator for the reference measure m, so $L = h^{-1} M h$ and M has the pointwise formula

$$Mv = \sum_{a \in \mathcal{P}} 1_{fa} \cdot \frac{v}{Jf} \circ f|_a^{-1}.$$

Hence, recalling (2.2),

$$\begin{aligned} L^r(\rho h^{-1} 1_{Y_r}) = h^{-1} M^r(\rho 1_{Y_r}) &= h^{-1} \sum_{a \in \mathcal{P}_r} 1_{f^r a} \cdot 1_{Y_r} \circ f^r|_a^{-1} \cdot \frac{\rho}{Jf^r} \circ f^r|_a^{-1} \\ &= h^{-1} \sum_{a \in \mathcal{P}_r^*} 1_{f^r a} \cdot \frac{\rho}{Jf^r} \circ f^r|_a^{-1} = h^{-1} Q_r^\rho. \end{aligned}$$

By H3, $Q_r^\rho \in \mathcal{B}_\theta(Y)$, while $h^{-1} 1_Y \in \mathcal{B}_\theta(Y)$ by Lemma 2.1. Hence $h^{-1} Q_r^\rho \in \mathcal{B}_\theta(Y)$. □

For $r \geq 0$, define $\rho_r = \rho h^{-1} 1_{\bigcup_{i=0}^r f^{-i}Y}$. Notice that

$$\rho_r = \sum_{j=0}^r \rho h^{-1} 1_{Y_j}. \tag{3.1}$$

Corollary 3.9. *For all $\rho \in K(X)$ and $r \geq 0$,*

$$\begin{aligned} c_\tau \log n \, L^n \rho_r &\to \int_X \rho_r \, d\mu, && \alpha = 1 \\ c_\tau n^{1-\alpha} L^n \rho_r &\to \tfrac{1}{\pi} \sin \pi\alpha \int_X \rho_r \, d\mu, && \alpha \in (0, 1) \end{aligned}$$

uniformly on Y as $n \to \infty$.

Proof. We give the details in the case $\alpha = 1$. The case $\alpha \in (0, 1)$ is similar.

For $n \geq r \geq 0$, equation (3.1) gives

$$1_Y L^n \rho_r = \sum_{j=0}^r 1_Y L^n(\rho h^{-1} 1_{Y_j}) = \sum_{j=0}^r T_{n-j} L^j(\rho h^{-1} 1_{Y_j}).$$

Using again (3.1) and the fact that $\int_X \rho h^{-1} 1_{Y_j}\, d\mu = \int_Y L^j(\rho h^{-1} 1_{Y_j})\, d\mu$, we obtain that on Y

$$\begin{aligned}
&c_\tau \log n\, L^n \rho_r - \int_X \rho_r\, d\mu \\
&= \sum_{j=0}^{r} \frac{\log n}{\log(n-j)} \Big\{ c_\tau \log(n-j) T_{n-j} L^j(\rho h^{-1} 1_{Y_j}) - \int_Y L^j(\rho h^{-1} 1_{Y_j})\, d\mu \Big\} \\
&\quad + \sum_{j=0}^{r} \Big\{ \frac{\log n}{\log(n-j)} - 1 \Big\} \int_X \rho h^{-1} 1_{Y_j}\, d\mu.
\end{aligned}$$

The result follows by Theorem 3.7 and Proposition 3.8. □

Now, let $X_1, \dots, X_d$ be the partition of $X \setminus Y$ in H2.

Lemma 3.10. *For all $\rho \in K(X)$, $n > r \geq 0$, $k = 1, \dots, d$,*

$$\int_X 1_{X_k}\, df_*^n(\lambda|_{\bigcup_{i=0}^r f^{-i}Y}) = \sum_{j=1}^{n} \int_{\{\tau^{(k)} \geq j\}} 1_Y L^{n-j} \rho_r\, d\mu.$$

Proof. Let $E_r = \bigcup_{i=0}^r f^{-i}Y$. Suppose that $x \in E_r$ and that $f^n x \in X_k$ for some $k = 1, \dots, d$. Then x must have made its last return to Y at some time $n-j$ for some $1 \leq j \leq n$. Moreover, by H2(b), $f^m x \in X_k$ for $n-j+1 \leq m \leq n$. Hence,

$$\begin{aligned}
\{x \in E_r : f^n x \in X_k\} &= \bigcup_{j=1}^{n} \{x \in E_r : f^{n-j}x \in Y \text{ and } f^m x \in X_k,\ n-j+1 \leq m \leq n\} \\
&= \bigcup_{j=1}^{n} \{x \in E_r : f^{n-j}x \in Y \text{ and } \tau^{(k)} \circ f^{n-j} \geq j\}.
\end{aligned}$$

As this is a disjoint union,

$$\begin{aligned}
\int_X 1_{X_k}\, df_*^n(\lambda|_{E_r}) &= \int_X 1_{X_k} \circ f^n \cdot 1_{E_r} \cdot \rho \cdot h^{-1}\, d\mu \\
&= \sum_{j=1}^{n} \int_X (1_{\{\tau^{(k)} \geq j\}} 1_Y) \circ f^{n-j} \cdot 1_{E_r} \cdot \rho \cdot h^{-1}\, d\mu \\
&= \sum_{j=1}^{n} \int_{\{\tau^{(k)} > j-1\}} 1_Y L^{n-j} \rho_r\, d\mu.
\end{aligned}$$

□

Proposition 3.11. *For all $\rho \in K(X)$, $r \geq 0$, $k = 1, \dots, d$,*

$$\lim_{n\to\infty} \int_X 1_{X_k} df_*^n(\lambda|_{\bigcup_{i=0}^r f^{-i}Y}) = \bar{p}_k \lambda\left(\bigcup_{i=0}^{r} f^{-i}Y\right).$$

Proof. Again, we set $E_r = \bigcup_{i=0}^r f^{-i}Y$. Note that $\int \rho_r\, d\mu = \int_{E_r} \rho h^{-1}\, d\mu = \mathrm{Leb}(E_r)$.

First, we consider the case $\alpha \in (0,1)$. Set

$$\varepsilon_n := \sup_Y \left| L^n \rho_r - \frac{d_\alpha \lambda(E_r)}{n^{1-\alpha}} \right|,$$

where $d_\alpha = \frac{\sin\pi\alpha}{\pi c_\tau}$. By Corollary 3.9, $\varepsilon_n = o(n^{\alpha-1})$, Hence, by Lemma 3.10,

$$\begin{aligned}
\int 1_{X_k}\, df_*^n(\lambda|_{E_r}) &= \sum_{j=1}^{n}\int_{\{\tau^{(k)}\ge j\}} 1_Y L^{n-j}\rho_r\, d\mu \\
&= \sum_{j=1}^{n-1}\int_{\{\tau^{(k)}\ge j\}} 1_Y L^{n-j}\rho_r\, d\mu + O(\mu(\tau^{(k)}\ge n)) \\
&= d_\alpha\lambda(E_r)\sum_{j=1}^{n-1}\frac{\mu(\tau^{(k)}\ge j)}{(n-j)^{1-\alpha}} + O\Big(\sum_{j=1}^{n-1}\frac{\varepsilon_{n-j}\mu(\tau^{(k)}\ge j)}{(n-j)^{1-\alpha}}\Big) + O(n^{-\alpha}). \qquad (3.2)
\end{aligned}$$

As $\mu(\tau^{(k)}\ge j)\sim c_k j^{-\alpha}$ we can conclude from Lemma A.1 and Lemma A.2 that

$$\sum_{j=1}^{n-1}\frac{\mu(\tau^{(k)}\ge j)}{(n-j)^{1-\alpha}} \to \frac{c_k}{\alpha}\frac{\pi}{\sin\pi\alpha} \quad\text{and}\quad \sum_{j=1}^{n-1}\frac{\varepsilon_{n-j}\mu(\tau^{(k)}\ge j)}{(n-j)^{1-\alpha}} \to 0. \qquad (3.3)$$

Combining (3.2) and (3.3) we obtain

$$\int_{X_k} df_*^n(\lambda|_{E_r}) = \frac{c_k}{c_\tau}\lambda(E_r) + o(1) = \bar p_k\lambda(E_r) + o(1),$$

concluding the result in the case that $\alpha\in(0,1)$.

When $\alpha=1$, we proceed in the same manner as before. Set

$$\varepsilon_n := \sup_Y\Big|L^n\rho_r - \frac{\lambda(E_r)}{c_\tau\log n}\Big|$$

and note by Corollary 3.9 that $\varepsilon_n = o(1/\log n)$ Hence, by Lemma 3.10,

$$\begin{aligned}
\int 1_{X_k}\, df_*^n(\lambda|_{E_r}) &= \sum_{j=1}^{n}\int_{\{\tau^{(k)}\ge j\}} 1_Y L^{n-j}\rho_r\, d\mu \\
&= \sum_{j=1}^{n-2}\int_{\{\tau^{(k)}\ge j\}} 1_Y L^{n-j}\rho_r\, d\mu + O(\mu(\tau^{(k)}\ge n-1)) \\
&= \frac{\lambda(E_r)}{c_\tau}\sum_{j=1}^{n-2}\frac{\mu(\tau^{(k)}\ge j)}{\log(n-j)} + O\Big(\sum_{j=1}^{n-2}\frac{\varepsilon_{n-j}\mu(\tau^{(k)}\ge j)}{\log(n-j)}\Big) + O(n^{-1}). \qquad (3.4)
\end{aligned}$$

Using Lemma A.3 and Lemma A.4 we conclude that

$$\sum_{j=1}^{n-2}\frac{\mu(\tau^{(k)}\ge j)}{\log(n-j)} \to c_k \quad\text{and}\quad \sum_{j=1}^{n-2}\frac{\varepsilon_{n-j}\mu(\tau^{(k)}\ge j)}{\log(n-j)} \to 0. \qquad (3.5)$$

Combining (3.4) and (3.5) we obtain $\int_{X_k} df_*^n(\lambda|_{E_r}) = \bar p_k\lambda(E_r) + o(1)$, concluding the proof. □

Proof of Theorem C. Since $K(X)$ is dense in $L^1(X,m)$, the conclusion of Proposition 3.11 holds for general $\lambda\in\mathcal{M}_1^{\rm ac}(X)$.

Let $\omega = \lim_{i\to\infty} f_*^{n_i}\lambda$ be a subsequential limit of $f_*^n\lambda$. By Lemma 3.2, $\omega=\nu_p$ for some $p\in\mathcal{S}_0$. Since $\nu_p(\partial X_k)=0$,

$$\lim_{i\to\infty} f_*^{n_i}\lambda(X_k) = \nu_p(X_k) = p_k.$$

by the Portmanteau Lemma. Also, setting $E_r = \bigcup_{i=0}^r f^{-i}Y$, it follows from Proposition 3.11 that

$$\lim_{i\to\infty} f_*^{n_i}(\lambda|_{E_r})(X_k) = \bar p_k\lambda(E_r).$$

Hence, $p_k\ge \bar p_k\lambda(E_r)$. Letting $r\to\infty$, we obtain that $p_k\ge\bar p_k$ for $k=1,\dots,d$ and so $p=\bar p$.

We have shown that $\nu_{\bar{p}}$ is the unique subsequential limit of $f_*^n\lambda$. By compactness of $\mathcal{M}_1(X)$, it follows that $f_*^n\lambda = \nu_{\bar{p}}$. □

4. Examples

In this section, we apply our main results to intermittent maps. Throughout, we write $|E| = \mathrm{Leb}(E)$ for measurable subsets $E \subset [0,1]$.

4.1. **Thaler maps.** We define $\mathcal{T}$ to be the class of interval maps $f : [0,1] \to [0,1]$ studied in [Tha80; Tha83] which satisfy the following:

T1: The exists $0 = \xi_1 < \xi_2 < \cdots < \xi_d = 1$, $d \geq 2$, with $f\xi_k = \xi_k$ and $f'(\xi_k) = 1$ for $k = 1, \dots, d$;

T2: There exist subintervals $I_1, \dots, I_d$, $d \geq 2$, with $\xi_k \in \mathrm{Int}\, I_k$, such that $\bigcup_{k=1}^d I_k = [0,1]$ and such that the restriction $f|_{I_k}$ extends to a C^2 diffeomorphism $f_k : \bar{I}_k \to [0,1]$;

T3: $f'(x) > 1$ for all $x \notin \{\xi_1, \dots, \xi_d\}$ and there exists an $\varepsilon > 0$ so that f is concave (resp. convex) on the interval $(\xi_k - \varepsilon, \xi_k) \cap I_k$ (resp. $(\xi_k, \xi_k + \varepsilon) \cap I_k$);

T4: There exist $\alpha \in (0,1]$ and $b_1, \dots, b_d > 0$ such that for every $k = 1, \dots, d$,

$$|fx - x| \sim b_k |x - \xi_k|^{1+1/\alpha} \quad \text{as } x \to \xi_k.$$

Remark 4.1. When $d = 2$, conditions T1–T4 reduce to conditions 1–4 given in the Introduction.

Thaler maps $f \in \mathcal{T}$ can be shown to lie in the abstract setting of Section 2 and hence Theorems A, B and C apply to these examples. The verification of the hypotheses in Section 2 is essentially contained in [Ser20, Proof of Theorem 4.6] and the references therein. For completeness we recall the main steps of this argument here. Different inducing schemes are needed in the cases $d = 2$ and $d \geq 3$, and the case $d = 2$ will be treated in the more general setting considered in Section 4.2, so we focus here on the case $d \geq 3$.[1]

By [Tha80; Tha83], $f \in \mathcal{T}$ is conservative and ergodic with a unique (up to scaling) invariant absolutely continuous measure μ; moreover $\mu([0,1]) = \infty$. Since the branches $f_k : \bar{I}_k \to [0,1]$ are onto, it is immediate that f is a topologically mixing Markov map.

Set

$$X_k := I_k \cap f^{-1} I_k, \quad k = 1, \dots, d; \quad \text{and} \quad Y := [0,1] \setminus (X_1 \cup \cdots \cup X_k).$$

It is immediate from the definitions that H2(a,b) hold. Below, we show:

Proposition 4.2. *The first return map $F : Y \to Y$ is a topologically mixing Gibbs-Markov map.*

We denote the density of μ by h. By [Tha80], it follows from Lemma 2.1 and Proposition 4.2 that h is bounded on compact subsets of $[0,1] \setminus \{\xi_1, \dots, \xi_d\}$. Hence, H1 is satisfied. Also H3 holds by Remark 2.3. Hence it remains to prove Proposition 4.2, to verify H2(c) and H4.

First, we describe the inducing scheme. We begin by setting $Y_k := I_k \setminus X_k$. Then $Y = Y_1 \dot{\cup} \cdots \dot{\cup} Y_d$. Let $g_k := f_k^{-1} : [0,1] \to \bar{I}_k$ denote the inverse branch of f on $\bar{I}_k$. Set $X_{k,n} := g_k^n Y_k$ for $n \geq 1$, $k = 1, \dots, d$. Notice that $\{X_{k,n},\ n \geq 1\}$ is a partition of X_k and that the maps $f : X_{k,n+1} \to X_{k,n}$, $f : X_{k,1} \to Y_k$ are bijections. For $j \neq k$, set

$$Y_{j,k,1} := g_j Y_k \quad \text{and} \quad Y_{j,k,n} := g_j X_{k,n-1},\ n \geq 2.$$

[1] There seem to be some typos in [Ser20, Equation (4.7)] where x_i should be x_j and f_j should be f_i.

Then $\{Y_{j,k,n} : j \neq k,\, n \geq 1\}$ is a partition of Y_j for each j and hence

$$\mathcal{P}_Y := \{Y_{j,k,n} : j \neq k,\, n \geq 1\}$$

is a partition of Y.

Let $F = f^\tau : Y \to Y$ be the first return map to Y. Then $\tau = n$ on $\bigcup_{j\neq k} Y_{j,k,n}$ and $F = f^n : Y_{j,k,n} \to Y_k$ is a bijection for all $j \neq k$ and all n.

Proof of Proposition 4.2. By construction, F is Markov with respect to the partition $\mathcal{P}_Y$. As $FY_{j,k,n} = Y_k$ it is clear that F has finite images. Moreover, as $FY_k = \bigcup_{\ell\neq k} Y_\ell$ we see that $F^3 Y_{j,k,n} = Y$ and so F is topologically mixing.

For $0 \leq m \leq n-1$ and $x, y \in [0,1]$,

$$\log \frac{(f^{n-m})'(f^m x)}{(f^{n-m})'(f^m y)} = \sum_{i=m}^{n-1} \log \frac{f'(f^i x)}{f'(f^i y)} = \sum_{i=m}^{n-1} \frac{|f''(z_i)|}{f'(z_i)} |f^i x - f^i y| \leq |f''|_\infty \sum_{i=m}^{n-1} |f^i x - f^i y|, \tag{4.1}$$

where $z_i \in [f^i x, f^i y]$ is chosen by the mean value theorem. In particular, for $x, y \in Y_{j,k,n}$,

$$\log \frac{(f^{n-m})'(f^m x)}{(f^{n-m})'(f^m y)} \leq |f''|_\infty \left(|Y_{j,k,n}| + \sum_{i=1}^{\infty} |X_{k,i}| \right) \leq |f''|_\infty.$$

Hence for $i = 0, \ldots, n-1$,

$$\frac{|f^i x - f^i y|}{|X_{k,n-i}|} \leq e^{2|f''|_\infty} \frac{|f^{n-i} f^i x - f^{n-i} f^i y|}{|f^{n-i} X_{k,n-i}|} = e^{2|f''|_\infty} \frac{|Fx - Fy|}{|Y_k|}, \tag{4.2}$$

Inserting (4.2) into (4.1) with $m = 0$, we obtain that

$$\log \left| \frac{F'x}{F'y} \right| \leq C |Fx - Fy| \tag{4.3}$$

where $C = |f''|_\infty e^{2|f''|_\infty} \max_k |Y_k|^{-1}$.

Let $\lambda = \inf_Y f' > 1$ and set $\theta = \lambda^{-1}$. If $s(x,y) = n$, then $1 \geq |F^n x - F^n y| \geq \lambda^{n-1} |Fx - Fy|$. Combining this inequality with (4.3), we obtain $\log \left| \frac{F'x}{F'y} \right| \leq C\theta^{n-1} = C\theta^{-1} d_\theta(x,y)$, which concludes the proof. □

Let $e_k = 1$ for $k = 1, d$ and $e_k = 2$ for $2 \leq k \leq d-1$. Define

$$c_k := e_k b_k^{-\alpha} \alpha^\alpha \sum_{j=1,\ldots,d,\, j\neq k} h(g_j \xi_k) g_j'(\xi_k).$$

In particular, for $d = 2$, we have

$$c_1 = b_1^{-\alpha} \alpha^\alpha h(g_2 0) g_2'(0), \quad c_2 = b_2^{-\alpha} \alpha^\alpha h(g_1 1) g_1'(1). \tag{4.4}$$

The remaining ingredients, namely H2(c) and H4, follow from the next result.

Lemma 4.3. *$\mu(\tau^{(k)} = n) \sim \alpha c_k n^{-(1+\alpha)}$ as $n \to \infty$ for $k = 1, \ldots, d$.*

Proof. Note that $\{\tau^{(k)} = n\} = \bigcup_{j=1,\ldots,d,\, j\neq k} Y_{j,k,n-1}$.

Fix $j, k \in \{1, \ldots, d\}$, $j \neq k$. For $k \neq 1, d$, the sets $X_{k,n}$ have two connected components $X_{k,n}^\pm$ which lie to the left or right of ξ_k and both accumulate at ξ_k as $n \to \infty$. Accordingly, define $Y_{j,k,n}^\pm := g_j X_{k,n-1}^\pm$. It suffices to show that $\mu(Y_{j,k,n}^\pm)$ for $k \neq 1, d$ and $\mu(Y_{j,k,n})$ for $k = 1, d$ have the asymptotic

$$b_k^{-\alpha} \alpha^{1+\alpha} h(g_j \xi_k) g_j'(\xi_k) n^{-(1+\alpha)} \quad \text{as } n \to \infty.$$

We give the details for $\mu(Y^+_{j,k,n})$, $k \neq 1, d$, the other cases being similar.

By [Tha83, Lemma 4], h is continuous on $[0,1] \setminus \{\xi_1, \dots, \xi_d\}$ (in the standard topology). Hence,

$$\mu(Y^+_{j,k,n}) - h(g_j\xi_k)|Y^+_{j,k,n}| = \int_{Y^+_{j,k,n}} (h - h(g_j\xi_k))\, d\,\mathrm{Leb} \le |Y^+_{j,k,n}| \sup_{Y^+_{j,k,n}} |h - h(g_j\xi_k)| \tag{4.5}$$

It follows that $\mu(Y^+_{j,k,n}) \sim h(g_j\xi_k)|Y^+_{j,k,n}|$. By the mean value theorem, there exists $z_n \in X^+_{k,n}$ so that $|Y^+_{j,k,n}| = g_j'(z_n)|X^+_{k,n-1}| \sim g_j'(\xi_k)|X^+_{k,n-1}|$. Combining these last two estimates,

$$\mu(Y^+_{j,k,n}) \sim h(g_j\xi_k)g_j'(\xi_k)|X^+_{k,n-1}|. \tag{4.6}$$

It remains to estimate $|X^+_{k,n}|$. We recall the following standard calculation.

Sublemma 4.4 (see for example [Aar97, Lemma 4.8.6]). *Suppose that $T : [0, C] \to [0, \infty)$ is such that $Tx > x$ for each $x \in (0, C]$ and $Tx \sim x + bx^{1+p}$ as $x \to 0$. Let $z_n = Tz_{n+1}$. Then*

$$z_n \sim (pbn)^{-1/p} \quad \textit{and} \quad z_n - z_{n+1} \sim b^{-1/p}(pn)^{-(1+1/p)} \quad \textit{as } n \to \infty$$

Proof. It follows from the assumptions that the sequence z_n is strictly decreasing with $z_n \to 0$. We then calculate

$$z_n^{-p} \sim z_{n+1}^{-p}(1 + bz_{n+1}^p)^{-p} = z_{n+1}^{-p}(1 - pbz_{n+1}^p + o(z_{n+1}^{2p})) = z_{n+1}^{-p} - pb + o(1)$$

as $n \to \infty$. Summing over n, this yields $z_n^{-p} = pbn + o(n)$ which gives the estimate for z_n. Finally, $z_n - z_{n+1} = Tz_{n+1} - z_{n+1} \sim bz_{n+1}^{1+p} \sim b^{-1/p}(pn)^{-(1+1/p)}$. ☐

Recalling T4, the estimate for $|X^+_{k,n}|$ reduces after a change of coordinates to the situation in Sublemma 4.4 with $p = 1/\alpha$ and $b = b_k$. Hence $|X^+_{k,n}| \sim b_k^{-\alpha}(n/\alpha)^{-(1+\alpha)}$. Combining this with (4.6), we obtain

$$\mu(Y^+_{j,k,n}) \sim h(g_j\xi_k)g_j'(\xi_k)b_k^{-\alpha}(\alpha/n)^{1+\alpha}$$

as required. ☐

4.2. **Intermittent maps with critical points and/or singularities.** We now consider a class of intermittent interval maps with two branches that possibly admit a critical point and/or a singularity at the discontinuity. This class of maps include the maps in $\mathcal{T}$ with $d = 2$ and the maps described in [CLM23]. We define $\mathcal{F}$ to be set of maps $f : [-1, 1] \to [-1, 1]$ which satisfy the following conditions.

F0: There exist a $c \in (-1, 1)$ such that the restrictions the restrictions $f_- := f : (-1, c) \to (-1, 1)$ and $f_+ : (c, 1) \to (-1, 1)$ are C^2 orientation preserving diffeomorphisms with no fixed points.

Remark 4.5. For notational simplicity we will assume that $c = 0$. Notice also that we do not assume that the $f_\pm$ extend to C^2 functions on the closure of $I_\pm$.

F1: There exist $\ell_+, \ell_- > 0$ and $k_+, k_- > 0$ such that

$$f(x) = \begin{cases} x + b_-(1+x)^{1+\ell_-} + o((1+x)^{1+\ell_-}), & \text{for } x \in U_{-1} \\ 1 - a_-|x|^{k_-}, & \text{for } x \in U_{0-} \\ -1 + a_+|x|^{k_+}, & \text{for } x \in U_{0+} \\ x - b_+(1-x)^{1+\ell_+} + o((1-x)^{1+\ell_+}) & \text{for } x \in U_{+1} \end{cases} \tag{4.7}$$

whenever $k_+, k_- \neq 1$ for some $a_\pm, b_\pm > 0$, and some neighbourhoods U_{-1}, U_0 of $-1, 0$ in $[-1, 0]$ and some neighbourhoods U_{0+}, U_{+1} of $0, 1$ in $[0, 1]$. If $k_+ = 1$ and/or $k_- = 1$, then we replace the corresponding lines in (4.7) with the assumption that $f'(0-) = a_- > 1$ and/or $f'(0+) = a_+ > 1$ respectively.

Remark 4.6. It will be convenient to assume that $fU_{0\pm} \subset U_{\pm 1}$. Notice that this assumption posses no restriction on F1 as $U_{0\pm}$ can be taken to be arbitrarily small.

Remark 4.7. This definition is more general than the one in [CLM23] as F1 stipulates only an asymptotic behaviour near the fixed points. However, it does not include the maps in [MS22] due to the restriction mentioned in Remark 2.2. We expect that our results hold also for the maps in [MS22].

Suppose that f satisfies F0 and let $\gamma_\pm \in I_\pm$ be the two points of period 2 for f. Define the intervals

$$Y := [\gamma_-, \gamma_+], \quad X_{\pm,n} := f_\pm^{-n} Y, \quad \text{and} \quad Y^\pm_{\pm,n+1} := f_\pm^{-1} X_{\mp,n}.$$

By definition $f : Y_{\pm,n+1} \to X_{\mp,n}$ and $f : X_{\mp,n} \to Y_{\mp,n-1}$ are bijections and so $\mathcal{P} := \{Y_{\pm,k}, X_{\pm,n} : n \geq 1, k \geq 2\}$ forms a Markov partition for f. Our final condition will ensure that our maps have good expansion and distortion properties.

F2: f is convex (resp. concave) on U_{-1} (resp. U_1) and moreover

(1) If f is not C^2 on $\overline{U}_{-1}$ (resp. $\overline{U}_{+1}$), then $f''(x) \lesssim (1+x)^{\ell_- - 1}$ (resp. $|f''(x)| \lesssim (1-x)^{\ell_+ - 1}$).

(2) If $k_\pm \neq 1$, then there exists a $\lambda > 1$ such that $(f^n)'(x) > \lambda$ for each $x \in Y_{\pm,n}$ and each $Y_{\pm,n} \not\subset U_{0\pm}$.

Remark 4.8. Notice that F2.1 is only assumed when the map is not C^2 at the fixed points and that F2.2 is only an assumption about the map outside of the neighbourhoods $U_{0\pm}, U_{\pm 1}$ and is trivially satisfied if $f'(x) > 1$ for each $x \notin \{-1, 1\}$.

We let $\alpha_+ := 1/\ell_+ k_-$, $\alpha_- := 1/\ell_- k_+$ and define

$$c_1 := h(0) a_-^{-1/k_-} (\ell_+ b_+)^{-1/\alpha_-} \alpha_-^2, \quad \text{and} \quad c_2 := h(0) a_+^{-1/k_+} (\ell_- b_-)^{-1/\alpha_+} \alpha_+^2.$$

We will assume that $\alpha_+ = \alpha_- = \alpha \in (0,1)$.

Theorem 4.9. *Suppose that $f \in \mathcal{F}$ with $\alpha \in (0,1)$. Then H1–H4 hold.*

Throughout this section we fix

$$Y := [\gamma_-, \gamma_+], \quad X_- := [0, \gamma_-], \quad X_+ := [\gamma_+, 0],$$

and let $\tau : Y \to \mathbb{N}$ denote the first return time to Y and $\tau^{(\pm)}(x) := \mathrm{Card}\{n \leq \tau(x) : f^n(x) \in X_\pm\}$. Let F be the first return map $F : Y \to Y$. By construction

Lemma 4.10. $F|_{Y_{\pm,n}} = f^n$ *and the interval Y dynamically separates X_-, X_+.*

In the remainder of this section we will show that f satisfies assumptions H1–H3 and thus conclude the proof of Theorem 4.9. Notice that $X_{\pm,n}$ and $Y_{\pm,n}$ here play the same roles as $\Delta_n^\pm$ and $\delta_n^\pm$ respectively in [CLM23]. The next Lemma shows that the asymptotic behaviour of the sizes of these partition elements remains exactly the same as in [CLM23].

Lemma 4.11.

$$|X_{\pm,n}| \sim b_\pm^{-1/\ell_\pm} (\ell_\pm n)^{-1-1/\ell_\pm} \quad \textit{and} \quad \mathrm{Leb}(\tau^{(\pm)} = n) = |Y_{\pm,n}| \sim a_\pm^{-1/k_\pm} (\ell_\mp b_\mp)^{-\alpha_\pm} \alpha_\pm n^{-1-\alpha_\pm}.$$

Proof. We will only explicitly prove the estimates for $Y_{+,n}$ and $X_{-,n}$ as the other estimates follow in the same way. Let $\gamma_n := f_-^{-n}\gamma$ so that $X_{-,n} = [\gamma_n, \gamma_{n-1}]$. As in the previous section we can apply Sublemma 4.4 to the sequence $z_n := 1 + \gamma_n$ with $b = b_-$ and $p = \ell_-$ to obtain

$$z_n \sim (b_-\ell_- n)^{-1/\ell_-}, \quad \text{and} \quad |X_{\pm,n}| \sim b_\pm^{-1/\ell_\pm}(\ell_\pm n)^{-1-1/\ell_\pm}. \tag{4.8}$$

Now, notice that as $y \downarrow 0$ we have by definition $f_+^{-1}y = \left(\frac{y+1}{a_+}\right)^{1/k_+}$. It follows that $f_+^{-1}\gamma_n = \left(\frac{1+\gamma_n}{a_+}\right)^{1/k_+}$. From (4.8) we obtain $f_+^{-1}\gamma_n \sim a_+^{-1/k_+}(\ell_- b_- n)^{-1/(\ell_- k_+)}$ which yields the claimed asymptotics for $|Y_{\pm,n}|$. □

Having established Lemma 4.11 the argument that F is Gibbs-Markov follows essentially verbatim from [CLM23] replacing the roles of $\delta_n^\pm$ with $Y_{\pm,n}$ and $\Delta_n^\pm$ with $Y_n^\pm$. For completeness we include the main steps of this argument.

Proposition 4.12. *There exists $\lambda > 1$ such that $F'(y) > \lambda$ for all $y \in Y$.*

Proof. We follow the proof of [CLM23, Proposition 3.6.]. We only consider the case $y \in Y_{+,n}$ as the case that $y \in Y_{-,n}$ is the same. Define the function $\phi := f_+ \circ f_- \circ f_+$ and notice that $\phi : Y_{+,n+1} \to Y_{+,n}$ bijectively.

Sublemma 4.13. *Let $y \in Y_{+,n}$ where n is such that $Y_{\pm,n} \subset U_{0-}$, then* $\dfrac{f'(y)f'(fy)}{f'(\phi(y))} > 1$.

Proof. If $k_+ \in (0,1)$ then f' is decreasing on U_{0+} and so, as $y < \phi(y)$, we have that $f'(y)/f'(\phi(y)) > 1$. By construction $f(y) \in U_1$ for every $y \in U_{0+}$, so $f'(fy) > 1$ and we are finished.

Assume now that $k_+ > 1$ and to ease notation set $k = k_+, a = a_+, b = b_-, \ell = \ell_-$. Setting $x = fy$ we recall that $x \in U_{-1}$ by Remark 4.6 and using the convexity of f on U_{-1} we obtain that $fx \le f(-1) + f'(x)(x+1) = -1 + f'(x)ay^k$ which in turn implies that $\phi(y) \le f'(x)^{1/k}y$. As $y < \phi(y)$ and as $(f'(y))/(f'(\phi(y))) = (y/\phi(y))^{k-1}$,

$$\frac{f'(y)}{f'(\phi(y))}f'(fy) \ge \left(\frac{\phi(y)}{y}\right)\left(\frac{f'(x)^{1/k}y}{y}\right)^{-k} f'(x) \ge 1.$$

□

Let $m_+ = \min\{m : Y_{+,m} \subset U_{0+}\}$. Condition F2 implies that $F'y \ge \lambda$ for all $y \in Y_{+,m}$ whenever $m \le m_+$. The Sublemma above allows us to conclude

$$(f^{n+1})'(y) = f'(y) \cdot f'(fy) \cdots f'(f^m y) = \frac{f'(y) \cdot f'(fy)}{f'(\phi(y))}(f^m)'(\phi(y)) \ge (f^n)'(\phi(y)),$$

and so $F'y > F(\phi(y))$. Proceeding inductively we obtain $F'(y) \ge F'(\phi^{n+1-m^+}(y)) > \lambda$. □

Lemma 4.14. *There exists a $C > 0$ and a $\theta \in (0,1)$ such that*

$$\left|\log\frac{F'(x)}{F'(y)}\right| \le C\theta^{s(x,y)}.$$

Proof. We only consider $x, y \in Y_{+,n}$ as the argument for $x, y \in Y_{-,n}$ is the same. Calculating as in (4.1) one finds

$$\left|\log\frac{(f^{n-j})'(f^jx)}{(f^{n-j})'(f^jy)}\right| \le \left|\frac{f''u_k}{f'u_k}\right||Y_{+,n}| + \sum_{k=1}^{n-1}\left|\frac{f''u_k}{f'u_k}\right||X_{-,n-k}| \tag{4.9}$$

Notice the $f''(u_0)/f'(u_0) \lesssim u_0^{-1} \lesssim (f^{-1}\gamma_n)^{-1} \lesssim (1+\gamma_n)^{-1/k_+}$ and (4.8) gives that $(1+\gamma_n^-)^{1/k_+} \lesssim n^{-\alpha_-}$. So, as $|Y_{+,n}| \lesssim n^{-1-\alpha_-}$ we find that the first term in (4.9) is uniformly bounded in n. If f is C^2 on $\overline{U}_{-1}$ the second term in (4.9) is summable and whence uniformly bounded. Otherwise, by assumption F2, (4.8) and Lemma 4.11 we have that $f''(u_k)|Y_{+,n}| \lesssim (1+\gamma_{n-k}^-)^{\ell_- -1}|Y_{+,n-k}| \lesssim (n-k)^{-1+1/\ell_-}(n-k)^{-1-1/\ell_-} \lesssim (n-k)^{-2}$ which is summable. Thus, (4.9) is uniformly bounded and we can calculate as in (4.2) to find $\log|F'x/F'y| \le C|Fx-Fy|$. Thus, proceeding in the same way as in the proof of Proposition 4.2 one concludes the result with $\theta := \tilde{\lambda}^{-1}$. □

Proof of Theorem 4.9. Lemma 4.12 and Proposition 4.14 imply that $F: Y \to Y$ is Gibbs-Markov, which yields H1. Remark 2.3 yields H3 and so it only remains to show H2 and H4. By Lemma 2.1 F preserves an absolutely continuous invariant measure μ with density h. We claim that h is continuous on Y. Given the validity of this claim we can apply the same argument as in (4.5) to conclude from Lemma 4.11 that

$$\mu(\tau^{\pm}=n) = \mu(Y_{\pm,n}) \sim h(0)|Y_{\pm,n}| \sim h(0)a_{\pm}^{-1/k_{\pm}}(\ell_{\mp}b_{\mp})^{-\alpha_{\pm}}\alpha_{\pm}n^{-1-\alpha_{\pm}}.$$

Together with Lemma 4.10, this concludes H2 and H4.

Let $\mathcal{P}_Y^{(n)}$ denote the n^{th} refined partition of F and let $a \in \mathcal{P}_Y^{(n)}$. From the uniform expansion of F and the mean-value theorem we know that for $j \le n$, $|F^jx - F^jy| \le \lambda^{-(n-j)}|F^nx - F^ny|$ for every $x, y \in a$. Moreover, recalling that $\log|F'x/F'y| \le C|Fx-Fy|$ for $x, y \in Y$ it follows that for any $x, y \in a$

$$\log\left|\frac{(F^n)'x}{(F^n)'y}\right| \le \sum_{j=0}^{n-1}\left|\frac{(F^j)'x}{(F^j)'y}\right| \le \frac{C}{\lambda-1}|F^nx - F^ny|, \tag{4.10}$$

and so $|(F^n)'x/(F^n)'y) - 1| \lesssim |F^nx - F^ny|$. Moreover, (4.10) implies $1/[(F^n)'y] \lesssim |a|$ for all $y \in a$. Using the standard pointwise formula for the transfer operator L_Y for F, we obtain

$$\begin{aligned}|L_Y^n1(x) - L_Y^n1(y)| &= \left|\sum_{\tilde{x}\in F^{-n}x,\,\tilde{y}\in F^{-n}y}\frac{1}{(F^n)'\tilde{x}} - \frac{1}{(F^n)'\tilde{y}}\right| \\ &\le \sum_{\tilde{x}\in F^{-n}x,\,\tilde{y}\in F^{-n}}\left|\frac{(F^n)'\tilde{y}}{(F^n)'\tilde{x}} - 1\right|\left|\frac{1}{(F^n)'\tilde{y}}\right| \\ &\lesssim \sum_{a\in\mathcal{P}_Y^{(n)}}|x-y||a| \lesssim |x-y|.\end{aligned}$$

Similarly, one can check that L_Y^n is uniformly bounded. The Arzelà-Ascoli Theorem then yields that $\frac{1}{n}\sum_{j=1}^{n-1}L_Y^j1$ has a subsequence which is uniformly convergent to some Lipschitz $\tilde{h}: Y \to \mathbb{R}$. As $\tilde{h}$ is necessarily the density of an invariant measure it follows that $h1_Y = \tilde{h}$ yielding that h is Lipschitz (and hence continuous). This establishes our claim and concludes the proof. □

Appendix A. Series estimates

In this appendix, we carry out some calculations required in Section 3.4.

Lemma A.1. *Let* $\alpha \in (0,1)$. *Then* $\lim_{n\to\infty}\sum_{j=1}^{n-1}(n-j)^{\alpha-1}j^{-\alpha} = \frac{\pi}{\sin\alpha\pi}$.

Proof. The function $(n-x)^{\alpha-1}x^{-\alpha}$ has one critical point at $x=\alpha n$. Hence, by approximating integrals by Riemann sums on the intervals $[0,\alpha n]$ and $[\alpha n, n]$,

$$\sum_{j=1}^{n-1}(n-j)^{\alpha-1}j^{-\alpha}=\int_0^n (n-x)^{\alpha-1}x^{-\alpha}\,dx+O(n^{-\alpha})+O(n^{\alpha-1}).$$

Using standard properties of Beta and Gamma functions, we obtain

$$\begin{aligned}\int_0^n (n-x)^{\alpha-1}x^{-\alpha}\,dx&=\int_0^1 (1-x)^{\alpha-1}x^{-\alpha}\,dx=\mathrm{B}(\alpha,1-\alpha)=\Gamma(\alpha)\Gamma(1-\alpha)/\Gamma(1)\\&=\Gamma(\alpha)\Gamma(1-\alpha)=\tfrac{\pi}{\sin\alpha\pi},\end{aligned}$$

as required. □

Lemma A.2. *Let* $\alpha\in(0,1)$. *Suppose that* $g:\mathbb{N}\to\mathbb{R}$ *satisfies* $\lim_{n\to\infty}g(n)=0$. *Then*

$$\lim_{n\to\infty}\sum_{j=1}^{n-1}j^{-\alpha}g(j)\,(n-j)^{\alpha-1}=0$$

and $\lim_{n\to\infty}\sum_{j=1}^{n-1}(n-j)^{\alpha-1}g(n-j)\,j^{-\alpha}=0$.

Proof. Set $C=\max_n|g(n)|$. Let $\varepsilon\in(0,\frac12)$. Then

$$\begin{aligned}\sum_{j=1}^{n-1}j^{-\alpha}g(j)\,(n-j)^{\alpha-1}&=\sum_{1\le j<\varepsilon n}j^{-\alpha}g(j)(n-j)^{\alpha-1}+\sum_{\varepsilon n\le j<n}j^{-\alpha}g(j)(n-j)^{\alpha-1}\\&\le C2^{1-\alpha}n^{\alpha-1}\sum_{1\le j<\varepsilon n}j^{-\alpha}+(\varepsilon n)^{-\alpha}\max_{k\ge\varepsilon n}g(k)\sum_{\varepsilon n\le j<n}(n-j)^{\alpha-1}\\&\lesssim n^{\alpha-1}(1+(\varepsilon n)^{1-\alpha})+(\varepsilon n)^{-\alpha}\max_{k\ge\varepsilon n}g(k)n^{\alpha}\\&\lesssim n^{\alpha-1}+\varepsilon^{1-\alpha}+\varepsilon^{-\alpha}\max_{k\ge\varepsilon n}g(k).\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\sum_{j=1}^{n-1}j^{-\alpha}g(j)(n-j)^{\alpha-1}\lesssim\varepsilon^{1-\alpha}.$$

The first limit follows since ε is arbitrarily small. Replacing α by $1-\alpha$, we obtain the second limit. □

Lemma A.3. *Suppose that* $g(x)=o(1)$ *as* $x\to\infty$. *Then* $\sum_{j=1}^{n}\frac{g(j)}{j}=o(\log n)$.

Proof. For $\varepsilon>0$, choose $n_0\ge1$ such that $|g(x)|<\varepsilon$ for $x>n_0$. Then

$$\sum_{j=1}^{n}\frac{g(j)}{j}\le\sum_{j=1}^{n_0}\frac{g(j)}{j}+\varepsilon\sum_{j=n_0}^{n}\frac{1}{j}\le\sum_{j=1}^{n_0}\frac{g(j)}{j}+\varepsilon\log n.$$

Hence $\limsup_{n\to\infty}(\log n)^{-1}\sum_{j=1}^{n}\frac{g(j)}{j}\le\varepsilon$. □

Lemma A.4. *Let* $g_1,g_2:[0,\infty)\to\mathbb{R}$ *with* $\lim_{x\to\infty}g_i(x)=1$ *for* $i=1,2$. *Then*

$$\lim_{n\to\infty}\sum_{j=1}^{n-2}\frac{g_1(j)}{j}\frac{g_2(n-j)}{\log(n-j)}=1.$$

Proof. Write

$$\sum_{j=1}^{n-2} \frac{g_1(j)}{j} \frac{g_2(n-j)}{\log(n-j)} = S_1 + S_2$$

where

$$S_1 = \sum_{1\le j\le n/2} \frac{g_1(j)}{j} \frac{g_2(n-j)}{\log(n-j)}, \qquad S_2 = \sum_{n/2<j\le n-2} \frac{g_1(j)}{j} \frac{g_2(n-j)}{\log(n-j)}.$$

Now, $|S_2| \le |g_1|_\infty |g_2|_\infty \frac{2}{n} \sum_{2\le j\le n} \frac{1}{\log j}$. But

$$\sum_{2\le j\le n} \frac{1}{\log j} \lesssim \frac{n}{\log n} + \sum_{n/\log n\le j\le n} \frac{1}{\log j} \le \frac{n}{\log n} + \frac{1}{\log(n/\log n)} n \lesssim \frac{n}{\log n},$$

so $|S_2| \lesssim \frac{1}{\log n}$.

By Lemma A.3, $\sum_{j=1}^{n} \frac{g_1(j)}{j} \sim \log n$. Hence,

$$S_1 \le \sup_{k\ge n/2} g_2(k) \frac{1}{\log n/2} \sum_{1\le j\le n/2} \frac{g_1(j)}{j} \sim \frac{1}{\log n/2} \log n/2 = 1.$$

Similarly,

$$S_1 \ge \inf_{k\ge n/2} g_2(k) \frac{1}{\log n} \sum_{1\le j\le n/2} \frac{g_1(j)}{j} \sim \frac{1}{\log n} \log n/2 \sim 1.$$

Hence $S_1 + S_2 \sim S_1 \sim 1$ as required. □

Acknowledgment. We are very grateful to Stefano Luzzatto for important discussions that led to this paper and for the hospitality of ICTP where part of this research was carried out.

References

[Aar97] Jon Aaronson. *An Introduction to Infinite Ergodic Theory.* Vol. 50. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997, pp. xii+284. doi: 10.1090/surv/050.

[AD01] Jon Aaronson and Manfred Denker. "Local Limit Theorems for Partial Sums of Stationary Sequences Generated by Gibbs-Markov Maps". In: *Stochastics and Dynamics* 1.2 (2001), pp. 193–237. doi: 10.1142/S0219493701000114.

[ADU93] Jon Aaronson, Manfred Denker, and Mariusz Urbański. "Ergodic Theory for Markov Fibred Systems and Parabolic Rational Maps". In: *Transactions of the American Mathematical Society* 337.2 (1993), pp. 495–548. doi: 10.2307/2154231.

[ATZ05] Jon Aaronson, Maximilian Thaler, and Roland Zweimüller. "Occupation Times of Sets of Infinite Measure for Ergodic Transformations". In: *Ergodic Theory and Dynamical Systems* 25.4 (2005), pp. 959–976. doi: 10.1017/S0143385704001051.

[Bil95] Patrick Billingsley. *Probability and Measure.* 3rd ed. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1995, pp. xiv+593.

[BB03] Michael Blank and Leonid Bunimovich. "Multicomponent Dynamical Systems: SRB Measures and Phase Transitions". In: *Nonlinearity* 16.1 (2003), pp. 387–401. doi: 10.1088/0951-7715/16/1/322.

[CE12] Eleonora Catsigeras and Heber Enrich. "Equilibrium States and SRB-like Measures of C^1-Expanding Maps of the Circle". In: *Portugaliae Mathematica. Nova Série* 69.3 (2012), pp. 193–212. doi: 10.4171/PM/1913.

[CL24] Douglas Coates and Stefano Luzzatto. "Persistent Non-Statistical Dynamics in One-Dimensional Maps". In: *Communications in Mathematical Physics* 405.4 (2024), p. 102. doi: 10.1007/s00220-024-04957-0.

[CLM23] Douglas Coates, Stefano Luzzatto, and Mubarak Muhammad. "Doubly Intermittent Full Branch Maps with Critical Points and Singularities". In: *Communications in Mathematical Physics* 402.2 (2023), pp. 1845–1878. doi: 10.1007/s00220-023-04766-x.

[CYZ20] Sylvain Crovisier, Dawei Yang, and Jinhua Zhang. "Empirical Measures of Partially Hyperbolic Attractors". In: *Communications in Mathematical Physics* 375.1 (2020), pp. 725–764. doi: 10.1007/s00220-019-03668-1.

[Gou11] Sébastien Gouëzel. "Correlation Asymptotics from Large Deviations in Dynamical Systems with Infinite Measure". In: *Colloquium Mathematicum* 125.2 (2011), pp. 193–212. doi: 10.4064/cm125-2-5.

[JT05] Esa Järvenpää and Tapani Tolonen. "Relations between Natural and Observable Measures". In: *Nonlinearity* 18.2 (2005), pp. 897–912. doi: 10.1088/0951-7715/18/2/021.

[Kel04] Gerhard Keller. "Completely Mixing Maps without Limit Measure". In: *Colloquium Mathematicum* 100.1 (2004), pp. 73–76. doi: 10.4064/cm100-1-6.

[Lam58] John Lamperti. "An Occupation Time Theorem for a Class of Stochastic Processes". In: *Transactions of the American Mathematical Society* 88 (1958), pp. 380–387. doi: 10.2307/1993222.

[Lév39] Paul Lévy. "Sur certains processus stochastiques homogènes". In: *Compositio Mathematica* 7 (1939), pp. 283–339.

[MT12] Ian Melbourne and Dalia Terhesiu. "Operator Renewal Theory and Mixing Rates for Dynamical Systems with Infinite Measure". In: *Inventiones Mathematicae* 189.1 (2012), pp. 61–110. doi: 10.1007/s00222-011-0361-4.

[Mis05] Michał Misiurewicz. "Ergodic Natural Measures". In: *Algebraic and Topological Dynamics. Proceedings of the Conference, Bonn, Germany, May 1–July 31, 2004.* Providence, RI: American Mathematical Society (AMS), 2005, pp. 1–6.

[MS22] Muhammad Mubarak and Tanja I. Schindler. *Doubly Intermittent Maps with Critical Points, Unbounded Derivatives and Regularly Varying Tail.* Nov. 2022. doi: 10.48550/arXiv.2211.15648. eprint: 2211.15648 (math).

[Nol20] John P. Nolan. *Univariate Stable Distributions: Models for Heavy Tailed Data.* Springer Series in Operations Research and Financial Engineering. Springer, Cham, 2020, pp. xv+333. doi: 10.1007/978-3-030-52915-4.

[PM80] Yves Pomeau and Paul Manneville. "Intermittent Transition to Turbulence in Dissipative Dynamical Systems". In: *Communications in Mathematical Physics* 74.2 (1980/06/01, 1980), pp. 189–197. doi: 10.1007/BF01197757.

[Ser20] Toru Sera. "Functional Limit Theorem for Occupation Time Processes of Intermittent Maps". In: *Nonlinearity* 33.3 (2020), pp. 1183–1217. doi: 10.1088/1361-6544/ab5ceb.

[SY19] Toru Sera and Kouji Yano. "Multiray Generalization of the Arcsine Laws for Occupation Times of Infinite Ergodic Transformations". In: *Transactions of the American Mathematical Society* 372.5 (2019), pp. 3191–3209. doi: 10.1090/tran/7755.

[Tal20] Amin Talebi. *Statistical Instability and Non-Statistical Dynamics.* Dec. 2020. doi: 10.48550/arXiv.2012.14462. arXiv: 2012.14462 [math].

[TZ06] M. Thaler and R. Zweimüller. "Distributional Limit Theorems in Infinite Ergodic Theory". In: *Probability Theory and Related Fields* 135.1 (2006), pp. 15–52. doi: 10.1007/s00440-005-0454-3.

[Tha80] Maximilian Thaler. "Estimates of the Invariant Densities of Endomorphisms with Indifferent Fixed Points". In: *Israel Journal of Mathematics* 37.4 (1980), pp. 303–314. doi: 10.1007/BF02788928.

[Tha83] Maximilian Thaler. "Transformations on $[0,1]$ with Infinite Invariant Measures". In: *Israel Journal of Mathematics* 46.1-2 (1983), pp. 67–96. doi: 10.1007/BF02760623.

[Tha02] Maximilian Thaler. “A Limit Theorem for Sojourns near Indifferent Fixed Points of One-Dimensional Maps”. In: *Ergodic Theory and Dynamical Systems* 22.4 (2002), pp. 1289–1312. doi: 10.1017/S0143385702000573.

[Yan17] Yuko Yano. “On the joint law of the occupation times for a diffusion process on multiray”. In: *J. Theoret. Probab.* 30.2 (2017), pp. 490–509. doi: 10.1007/s10959-015-0654-4.

[Zwe02] Roland Zweimüller. “Exact \(\mathcal C^ınfty\) Covering Maps of the Circle without (Weak) Limit Measure”. In: *Colloquium Mathematicum* 93.2 (2002), pp. 295–302. doi: 10.4064/cm93-2-9.

[Zwe07] Roland Zweimüller. “Mixing Limit Theorems for Ergodic Transformations”. In: *Journal of Theoretical Probability* 20.4 (2007), pp. 1059–1071. doi: 10.1007/s10959-007-0085-y.

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Brazil

Email address: dcoates@icmc.usp.br

URL: douglascoates.github.io

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Email address: i.melbourne@warwick.ac.uk

URL: https://homepages.warwick.ac.uk/~maslaq/research

Department of Mathematical Sciences, Sharif University of Technology, Iran,
Institute for Research in Fundamental Sciences (IPM), Iran

Email address: amin.talebi@sharif.edu