Correction to: A note on diffusion limits of chaotic skew-product flows

Ian Melbourne and Andrew Stuart
Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
19 April 2015

Abstract
This fixes a gap in the averaging argument in our paper: A note on diffusion limits of chaotic skew product flows. Nonlinearity (2011) 1361–1367, and moreover shows that the large deviation estimate assumed there is redundant.

Recall from [2] that 
\[ Z(\epsilon)(t) = \int_0^t g(x(\epsilon)(s),y(\epsilon)(s)) \, ds \]
where 
\[ g(x,y) = f(x,y) - F(x). \]
In [2, Section 3], it is argued that 
\[ Z(\epsilon) \to 0 \text{ in } L^1(C([0,T],\mathbb{R}^d);\mu), \]
but the proof is incorrect. Specifically, the proof introduces a random variable \( J_n \) (see below) that depends on \( x(\epsilon)(n\epsilon^{3/2}) \) and \( y(1)(s) \), and derives an estimate for \( \mathbb{E}|J_n| \). This estimate takes into account the randomness of \( y(1)(s) \) but overlooks the randomness of \( x(\epsilon)(n\epsilon^{3/2}) \).

In this note, we correct the argument in [2]. Moreover, in contrast to [2], our proof does not require any large deviation estimates. Hence the weak invariance principle is a sufficient (as well as necessary) condition for the main result in [2].

**Lemma 1** \( Z(\epsilon) \to 0 \text{ in } L^1(C([0,T],\mathbb{R}^d);\mu) \) as \( \epsilon \to 0 \) for each \( T > 0 \).

**Proof** Following the calculation in [2, Section 3] with \( \delta = \epsilon^{3/2} \), we obtain

\[
\max_{[0,T]} |Z(\epsilon)| = I_1 + I_2 + O(\epsilon^{3/2}) = I_2 + O(\epsilon^{1/2}) = \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} |J_n| + O(\epsilon^{1/2}), \tag{1}
\]

where 
\[ J_n = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(x(\epsilon)(n\epsilon^{3/2}),y(1)(s)) \, ds. \]

(The intermediate expressions \( I_1 \) and \( I_2 \) are defined in [2] but the formulas are not required here.)

For \( u \in \mathbb{R}^d \) fixed, we define
\[
\tilde{J}_n(u) = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(u,y(1)(s)) \, ds = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} A_u \circ \phi_s \, ds, \quad A_u(y) = g(u,y).
\]
Note that $\tilde{J}_n(u) = \tilde{J}_0(u) \circ \phi_n^{-1/2}$, and so $\mathbb{E}|\tilde{J}_n(u)| = \mathbb{E}|\tilde{J}_0(u)|$. By the ergodic theorem, $\mathbb{E}|\tilde{J}_0(u)| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for each $u$.

Let $Q > 0$ and write $Z^{(\varepsilon)} = Z^{(\varepsilon)}_{Q,1} + Z^{(\varepsilon)}_{Q,2}$ where

$$
Z^{(\varepsilon)}_{Q,1}(t) = Z^{(\varepsilon)}(t)1_{B_\varepsilon(Q)}, \quad Z^{(\varepsilon)}_{Q,2}(t) = Z^{(\varepsilon)}(t)1_{B_\varepsilon(Q)^c}, \quad B_\varepsilon(Q) = \{\max_{[0,T]} |x^{(\varepsilon)}| \leq Q\}.
$$

For any $a > 0$, there exists a finite subset $S \subset \mathbb{R}^d$ such that $\text{dist}(x, S) \leq a/(2 \text{Lip } f)$ for any $x$ with $|x| \leq Q$. Then for all $n \geq 0$, $\varepsilon > 0$,

$$
1_{B_\varepsilon(Q)}|J_n| \leq \sum_{u \in S} |\tilde{J}_n(u)| + a.
$$

Hence by (1),

$$
\mathbb{E}_{[0,T]} \max_{[0,T]} |Z^{(\varepsilon)}_{Q,1}| \leq \varepsilon^{3/2} \sum_{n=0}^{[T\varepsilon^{-3/2}] - 1} \sum_{u \in S} \mathbb{E}|\tilde{J}_n(u)| + Ta + O(\varepsilon^{1/2})
$$

$$
= \varepsilon^{3/2} \sum_{n=0}^{[T\varepsilon^{-3/2}] - 1} \sum_{u \in S} \mathbb{E}|\tilde{J}_0(u)| + Ta + O(\varepsilon^{1/2}) \leq T \sum_{u \in S} \mathbb{E}|\tilde{J}_0(u)| + Ta + O(\varepsilon^{1/2}).
$$

Since $a > 0$ is arbitrary, we obtain for each fixed $Q$ that $\max_{[0,T]} |Z^{(\varepsilon)}_{Q,1}| \rightarrow 0$ in $L^1$, and hence in probability, as $\varepsilon \rightarrow 0$.

Next, since $x^{(\varepsilon)} - W^{(\varepsilon)}$ is bounded on $[0,T]$, for $Q$ sufficiently large

$$
\mu\{|Z^{(\varepsilon)}_{Q,2}| > 0\} \leq \mu\{|x^{(\varepsilon)}| \geq Q\} \leq \mu\{|W^{(\varepsilon)}| \geq Q/2\}.
$$

Fix $c > 0$. Increasing $Q$ if necessary, we can arrange that $\mu\{|x^{(\varepsilon)}| \geq Q/2\} < c/4$. By the continuous mapping theorem, $\max_{[0,T]} |W^{(\varepsilon)}| \rightarrow_d \max_{[0,T]} |\sqrt{\Sigma}W|$. Hence there exists $\varepsilon_0 > 0$ such that $\mu\{|W^{(\varepsilon)}| \geq Q/2\} < c/2$ for all $\varepsilon \in (0, \varepsilon_0)$. For such $\varepsilon$,

$$
\mu\{|Z^{(\varepsilon)}_{Q,2}| > 0\} < c/2.
$$

Shrinking $\varepsilon_0$ if necessary, we also have that $\mu\{|Z^{(\varepsilon)}_{Q,1}| > c/2\} < c/2$. Hence $\mu\{|Z^{(\varepsilon)}| > c\} < c$, and so $\max_{[0,T]} |Z^{(\varepsilon)}| \rightarrow 0$ in probability. Finally, since $|\max_{[0,T]} |Z^{(\varepsilon)}||_\infty \leq 2|f||_\infty T$, it follows from the bounded convergence theorem that $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \max_{t \in [0,T]} |Z^{(\varepsilon)}(t)| = 0$ as required.

**Remark 2** The subsequent paper [1] contains the same error (see [1, Appendix A]). The gap is fixed in identical manner to above, and the large deviation assumptions throughout [1] are again unnecessary.
References
