THE EQUIVARIANT DARBOUX THEOREM

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Abstract. The classical Darboux Theorem states that symplectic forms are locally constant up to isomorphism, or equivalently that any two symplectic forms are locally isomorphic. We consider the corresponding results for symplectic forms that are invariant under the action of a compact Lie group. In this context, it is still true that symplectic forms are locally constant up to isomorphism but it is not true that any two symplectic forms are locally isomorphic.

1. Introduction

The Darboux theorem plays a fundamental role in the theory of Hamiltonian systems. Roughly speaking, the theorem states that locally all finite-dimensional symplectic manifolds of the same dimension look the same. The equivariant Darboux theorem plays an analogous role for Hamiltonian systems that are equivariant with respect to a symplectic action of a compact Lie group of symmetries, see for example [8], [2].

It will be convenient to divide the Darboux theorem and its equivariant counterpart into two parts. Suppose that X is a finite-dimensional manifold with symplectic form ω. In the absence of symmetry, the following statements are valid:

(a) Locally, there is a change of coordinates so that the transformed symplectic form is constant.

(b) There is a further change of coordinates yielding the 'canonical' symplectic form

$$\sum_{i=1}^{n} dq_i \wedge dp_i.$$

Together statements (a) and (b) imply

(c) Any two symplectic forms on symplectic manifolds of the same dimension are locally isomorphic.

In the literature, either (a) or (c) is called the Darboux Theorem. We shall distinguish these statements by referring to (a) as the 'locally constant' result and (c) as the 'locally isomorphic' result. We note that statement (b) follows from a standard result in linear algebra.

Our aim in this paper is to clarify the corresponding results in the equivariant context where there is a compact group of symmetries present. Although results corresponding to statements (a) and (b) can be found in the literature, nowhere

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are they stated together correctly. We note first that the 'locally constant' result still holds. This is an easy consequence of the Darboux-Weinstein Theorem, see Guillemin and Sternberg [5, Theorem 22.1]. However, the analog of the 'locally isomorphic' result ([5, Theorem 22.2]) is false as is implicit in the work of Montaldi, Roberts and Stewart [8]. In short, statement (a) holds but statements (b) and (c) are invalid. On the other hand it is possible to classify the nonisomorphic symplectic forms that fill the role of the single canonical symplectic form in (b), see [8].

The existence of nonisomorphic symplectic forms is intimately related to the representation of the group of symmetries Γ. Indeed, statement (c) is valid if and only if none of the irreducible representations that appear in the representation of Γ are of complex type (that is, each irreducible representation that occurs is of real or quaternionic type). For example, this is the case for the groups $\mathbf{1}$ (no symmetry), $\mathbf{O}(2)$, $\mathbf{SO}(3)$ and $\mathbf{O}(3)$ which have only real representations, and $\mathbf{SU}(2)$ which has only real and quaternionic representations.

However the nontrivial representations of the circle group $\mathbf{SO}(2)$ are of complex type and so uniqueness fails for nontrivial actions of $\mathbf{SO}(2)$. For example, suppose that $\mathbf{SO}(2)$ is acting in the standard way on $\mathbb{R}^2$ which we identify with $\mathbb{C}$. Then the (real) $\mathbf{SO}(2)$-invariant symplectic forms $\frac{1}{2}(dz \wedge id\bar{z})$ and $-\frac{1}{2}(dz \wedge id\bar{z})$ are not isomorphic (see Section 2 for an explicit verification of this fact).

The existence of nonisomorphic symplectic forms (in particular symplectic forms that are not isomorphic to the usual canonical symplectic form) is of some significance in the local bifurcation theory for equivariant Hamiltonian vector fields. For example, certain symplectic forms force spectral stability of equilibria (cyclospectrality in [8]) and the existence of Liapunov centers (weak cyclospectrality in [7]). Also, the expectation that certain collisions of eigenvalues will be dangerous in the sense of Krein is dependent on the symplectic structure present (see [2]).

In Section 2 we show by direct computation that the locally isomorphic result cannot be valid when there is symmetry. In Section 3 we state the Darboux-Weinstein theorem and deduce from this the locally constant result. Then in Section 4 we describe the nonisomorphic symplectic forms to which a symplectic form may locally be transformed by an equivariant change of coordinates. We illustrate our results by listing the nonisomorphic symplectic forms for an action of $\mathbf{SO}(2)$ on $\mathbb{R}^{10}$.

2. Nonisomorphic symplectic forms on $\mathbb{R}^2$

In this section, we illustrate by explicit calculation the failure of the 'locally isomorphic' result (statement (c)) in the equivariant context. The simplest example is the standard action of the circle group $\mathbf{SO}(2)$ on $\mathbb{R}^2$.

It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$. In these coordinates the standard action of $\mathbf{SO}(2)$ is given by

$$z \to e^{i\theta} z, \ \theta \in \mathbf{SO}(2).$$

A symplectic form ω is $\mathbf{SO}(2)$-invariant if

$$\omega(e^{i\theta} z, e^{i\theta} w) = \omega(z, w),$$

for all $\theta \in \mathbf{SO}(2)$ and $z, w \in \mathbb{R}^2$. It is readily shown that

$$\Im(i\bar{z}w), \quad -\Im(i\bar{z}w),$$

are real $\mathbf{SO}(2)$-invariant symplectic forms on $\mathbb{R}^2$. Moreover they correspond to the putative nonisomorphic symplectic forms mentioned in Section 1.

Two symplectic forms ω_1 and ω_2 on $\mathbb{R}^2$ are $\mathbf{SO}(2)$-isomorphic if there is an invertible linear map $P : \mathbb{R}^2 \to \mathbb{R}^2$ commuting with the action of $\mathbf{SO}(2)$ such that $\omega_1(Pz, Pw) = \omega_2(z, w)$ for all $z, w \in \mathbb{R}^2$. Again, it is an easy computation to show that P is given by

$$Pz = ke^{i\theta}z,$$

for some fixed $\theta \in \mathbf{SO}(2)$ and $k > 0$. Suppose now that $\omega(z, w) = \Im(i\bar{z}w)$. Then

$$\begin{aligned} \omega(Pz, Pw) &= \Im(i\overline{Pz}Pw) \\ &= \Im(i\overline{ke^{i\theta}z}ke^{i\theta}w) \\ &= \Im(ik^2e^{-i\theta}\bar{z}e^{i\theta}w) \\ &= k^2\omega(z, w). \end{aligned}$$

In particular, ω and $-\omega$ are nonisomorphic as required.

3. The Darboux-Weinstein Theorem

In this section we consider the equivariant analogue to the 'locally constant' result (statement (a)) in the classical Darboux theorem.

Definition 1. Suppose that X is a finite-dimensional manifold. A *symplectic form* on X is a closed two-form ω such that for each $x \in X$, $\omega_x : T_xX \times T_xX \to \mathbb{R}$ is nondegenerate, that is if $\omega_x(v, w) = 0$ for all $v \in T_xX$ and some $w \in T_xX$ then $w = 0$.

Suppose that Γ is a compact Lie group acting smoothly on a manifold X and that ω is a symplectic form on X. If $x \in X$ and $\gamma \in \Gamma$ there is an induced linear map $T_x\gamma : T_xX \to T_{\gamma x}X$. We shall abuse notation and refer also to this induced map as γ. Then we say that ω is Γ*-invariant* or that the group action is *symplectic* if $\gamma^*\omega = \omega$ for all $x \in X$, that is

$$\omega_{\gamma x}(\gamma v, \gamma w) = \omega_x(v, w),$$

for all $x \in X$, $v, w \in T_xX$.

If the group Γ acts on manifolds X and Y, then a mapping $f : X \to Y$ is Γ-*equivariant* if $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, $x \in X$. It is clear that if ω is a Γ-invariant symplectic form on Y and that $f : X \to Y$ is a Γ-equivariant diffeomorphism, then $f^*\omega$ is a Γ-symplectic form on X.

We are now in a position to state the Darboux-Weinstein Theorem.

Theorem 1. *Suppose that Γ is a compact Lie group acting on a finite-dimensional manifold X and let ω_0, ω_1 be two Γ-invariant symplectic forms on X. Suppose that Y is a Γ-invariant submanifold of X and that $\omega_0|_Y = \omega_1|_Y$. Then there exists an open Γ-invariant neighborhood U of Y and a Γ-equivariant diffeomorphism $f : U \to X$ such that $f|_Y = \mathrm{Id}_Y$ and $f^*\omega_1 = \omega_0$.*

If we take the submanifold Y to consist of a single point x we obtain the 'locally constant' theorem (cf [1, Theorem 8.1.2] in the nonequivariant context).

Corollary 2. *Suppose that Γ is a compact Lie group acting on a finite-dimensional manifold X and let ω be a Γ-invariant symplectic form on X. Let $x \in X$ be a Γ-invariant point and let ω' denote the constant symplectic form that agrees with ω at x. Then there is an open Γ-invariant neighborhood U of x and a Γ-equivariant diffeomorphism $f : U \to X$ such that $f(x) = x$ and $f^*\omega = \omega'$.*

Observe that since $x \in X$ is Γ-invariant, the action of the group Γ on X induces a linear group action of Γ on T_xX.

4. Canonical symplectic forms

Suppose that ω is a symplectic form on an m-dimensional manifold X, and let $x \in X$. Then ω is locally constant and isomorphic to the *canonical* (and constant) symplectic form

$$\sum_{i=1}^{n} dq_i \wedge dp_i$$

where $n = m/2$.

In this section we describe the canonical symplectic forms in the presence of a compact Lie group Γ. As mentioned in the introduction, it is not necessarily the case that there is a unique canonical symplectic form. However, by Corollary 2 we may assume that the symplectic form is locally constant, and thus reduce the problem to one of listing the possible 'canonical' symplectic forms for the action of Γ on a finite-dimensional vector space.

The classification of Γ-invariant symplectic forms was first stated in Montaldi, Roberts and Stewart [8] and follows from Lie-theoretic results in [6]. The result is also an immediate consequence of the linear-algebraic results in Melbourne and Dellnitz [7]. Of course the Lie-theoretic proof is more direct and intrinsic. Here we simply state the results and refer to [8] and [6], or alternatively [7], for the proofs.

In Subsection 4.1 we recall some basic representation theory, see for example [4]. This allows us to reduce to working with symplectic forms over a real division ring. There are three nonisomorphic division rings: the reals, complexes, and quaternions. Then in Subsection 4.2 we list the canonical symplectic forms over each division ring. It is the complexes that lead to nonisomorphic symplectic forms.

4.1. **Some representation theory.** Suppose that Γ is a compact Lie group acting on a vector space V. Define $\mathrm{Hom}_\Gamma(V)$ to be the vector space of Γ-equivariant real matrices

$$\mathrm{Hom}_\Gamma(V) = \{L : V \to V \text{ linear; } \gamma L = L\gamma \text{ for all } \gamma \in \Gamma\}.$$

A subspace U is said to be Γ-*irreducible* if it is invariant under Γ and has no proper invariant subspaces. If U is an irreducible subspace, then $\mathrm{Hom}_\Gamma(U)$ is a real division ring and hence is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The space V may be written as a direct sum of irreducible subspaces

$$V = U_1 \oplus \cdots \oplus U_k.$$

Group together those U_i on which Γ acts isomorphically to obtain the *isotypic decomposition*

$$V = W_1 \oplus \cdots \oplus W_\ell,$$

where each *isotypic component* W_j is the sum of isomorphic irreducible subspaces. The isotypic decomposition is unique, and moreover each isotypic component is left invariant by matrices in $\mathrm{Hom}_\Gamma(V)$. It follows that

$$\mathrm{Hom}_\Gamma(V) = \mathrm{Hom}_\Gamma(W_1) \oplus \cdots \oplus \mathrm{Hom}_\Gamma(W_\ell).$$

Next suppose that W is an isotypic component. We may write $W = U \oplus \cdots \oplus U = \bigoplus_{i=1}^m U$ where U is irreducible and $\mathrm{Hom}_\Gamma(U) \cong \mathcal{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let

$A \in \mathrm{Hom}_\Gamma(W)$. Then $A = \{A_{jk}\}_{1\le j,k\le m}$ where $A_{jk} : U \to U$. It is easy to check that $A_{jk} \in \mathrm{Hom}_\Gamma(U)$. Since $\mathrm{Hom}_\Gamma(U) \cong \mathcal{D}$ we have shown that

$$\mathrm{Hom}_\Gamma(W) \cong \mathrm{Hom}(\mathcal{D}^m),$$

where $\mathrm{Hom}(\mathcal{D}^m)$ denotes the space of $m \times m$ matrices with entries in $\mathcal{D}$. Often it will be convenient to denote the isotypic component W by $\mathcal{D}^m$. We say that an isotypic component $\mathcal{D}^m$ is *real*, *complex* or *quaternionic* depending on $\mathcal{D}$. Also we define the *dimension* of the isotypic component $\mathcal{D}^m$ to be the integer m. Note that the dimension of the corresponding (real) subspace W is a multiple of m but is in general not equal to m.

A symplectic form on $\mathcal{D}^m$ is a nondegenerate anti-symmetric bilinear map $\omega : \mathcal{D}^m \times \mathcal{D}^m \to \mathbb{R}$. Two symplectic forms ω and ω' are isomorphic (over $\mathcal{D}$) if there is a $\mathcal{D}$-linear map $P : \mathcal{D}^m \to \mathcal{D}^m$ such that $\omega(Pv, Pw) = \omega'(v, w)$ for $v, w \in \mathcal{D}^m$. We have the following result (see [3], [8] and [7]).

Proposition 3. (a) *Suppose that ω is a Γ-invariant symplectic form on V. Let $\omega_i = \omega|_{W_i}$. Then ω_i is a Γ-invariant symplectic form on W_i. Moreover two Γ-symplectic forms ω and ω' on V are Γ-isomorphic if and only if the corresponding summands ω_i and ω_i' are Γ-isomorphic for each i.*

(b) *Suppose that W is an isotypic component for Γ so that $\mathrm{Hom}_\Gamma(W) \cong \mathcal{D}^m$ where $\mathcal{D}$ is a real division ring. Then there is a one-to-one correspondence between (real) Γ-invariant symplectic forms on W and symplectic forms on $\mathcal{D}^m$. Moreover, two Γ-invariant symplectic forms on W are isomorphic if and only if the corresponding symplectic forms on $\mathcal{D}^m$ are isomorphic (over $\mathcal{D}$).*

4.2. **Canonical symplectic forms over real division rings.** Let $W = \mathcal{D}^m$ be an isotypic component of dimension m. In this subsection we list the nonisomorphic symplectic forms on $\mathcal{D}^m$. Using Proposition 3 we can then construct the nonisomorphic Γ-invariant symplectic forms on V and hence the canonical locally constant symplectic forms on a Γ-invariant manifold.

We choose coordinates $x_1, \ldots, x_m$ on $\mathbb{R}^m$, $z_1, \ldots, z_m$ on $\mathbb{C}^m$, and $w_1, \ldots, w_m$ on $\mathbb{H}^m$.

Theorem 4. *Suppose that ω is a symplectic form on $\mathcal{D}^m$. Then ω is isomorphic to precisely one of the following* canonical *symplectic forms.*

$\mathcal{D} = \mathbb{R}$: $\displaystyle\sum_{j=1}^{n} dx_j \wedge dx_{j+n}, \quad m = 2n$ *even.*

$\mathcal{D} = \mathbb{C}$: $\displaystyle\Re\sum_{j=1}^{n} dz_j \wedge dz_{j+n} + \tfrac{1}{2}\rho \sum_{k=2n+1}^{m} dz_k \wedge i d\bar{z}_k, \quad 0 \le n \le m/2,\ \rho = \pm 1.$

$\mathcal{D} = \mathbb{H}$: $\displaystyle\Re\sum_{j=1}^{n} dw_j \wedge dw_{j+n}, \quad m = 2n$ *even,*

$$\Re\sum_{j=1}^{n} dw_j \wedge dw_{j+n} + \tfrac{1}{2} dw_m \wedge i d\bar{w}_m, \quad m = 2n+1 \text{ odd.}$$

Remark 1. (a) It follows from Theorem 4 that the equivariant version of Darboux's theorem described in [5] is incorrect whenever there are complex isotypic components in the representation of Γ. In particular there are $m + 1$

nonisomorphic symplectic forms on each complex isotypic component of dimension m.

(b) Our choices of canonical symplectic forms are somewhat different from those in [8]. It turns out that the analysis of linear Hamiltonian vector fields is slightly simplified when working with the symplectic forms listed here (see [7]).

As an example we consider an action of the group $\mathbf{SO}(2)$ on $\mathbb{R}^{10}$. Identify $\mathbb{R}^{10}$ with $\mathbb{R}^2 \times \mathbb{C}^4$ and choose coordinates $v = (x_1, x_2, z_1, z_2, z_3, z_4)$. The action of $\theta \in \mathbf{SO}(2)$ is given by

$$\theta v = (x_1, x_2, e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, e^{i\theta} z_4).$$

In this case there are two isotypic components, $\mathbb{R}^2$ corresponding to two trivial representations of $\mathbf{SO}(2)$ and $\mathbb{C}^4$ which corresponds to four copies of the standard representation of $\mathbf{SO}(2)$.

Applying the results of Subsection 4.1 we can build the canonical symplectic forms on $\mathbb{R}^{10}$ out of the canonical symplectic forms on $\mathbb{R}^2$ and $\mathbb{C}^4$. There are five canonical symplectic forms on $\mathbb{C}^4$:

$$\begin{aligned}
&\pm\tfrac{1}{2}(dz_1 \wedge i d\bar{z}_1 + dz_2 \wedge i d\bar{z}_2 + dz_3 \wedge i d\bar{z}_3 + dz_4 \wedge i d\bar{z}_4),\\
&\Re\, dz_1 \wedge dz_2 \pm \tfrac{1}{2}(dz_3 \wedge i d\bar{z}_3 + dz_4 \wedge i d\bar{z}_4),\\
&\Re(dz_1 \wedge dz_3 + dz_2 \wedge dz_4).
\end{aligned}$$

Hence there are five canonical symplectic forms on $\mathbb{R}^{10}$ given by the direct sum of the symplectic form $dx_1 \wedge dx_2$ on $\mathbb{R}^2$ and one of the symplectic forms on $\mathbb{C}^4$.

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