Symmetric ω-limit sets for smooth Γ-equivariant dynamical systems with Γ^0 abelian *

Ian Melbourne †
Department of Mathematics
University of Houston
Houston, Texas 77204-3476
USA

Ian Stewart ‡
Mathematics Institute
University of Warwick
Coventry CV4 7AL
England

July 9, 1997

Abstract

The symmetry groups of attractors for smooth equivariant dynamical systems have been classified when the underlying group of symmetries Γ is finite. The problems that arise when Γ is compact but infinite are of a completely different nature. We investigate the case when the connected component of the identity Γ^0 is abelian and show that under fairly mild assumptions on the dynamics, it is typically the case that the symmetry of an ω-limit set contains the continuous symmetries Γ^0. Here, typicality is interpreted in both a topological and probabilistic sense (genericity and prevalence).

1 Introduction

Symmetric dynamical systems are common in models of natural or technological systems. For symmetric dynamical systems, attention has recently been focused

*Appeared: *Nonlinearity* **10** (1997) 1551–1567

†Supported in part by NSF Grant DMS-9403624, by ONR Grant N00014-94-1-0317 and by the CNRS

‡Supported in part by a grant from the EPSRC (United Kingdom) and an EU laboratory twinning grant under the SCIENCE programme. Research carried out under the auspices of the European Bifurcation Theory Group.

upon the question 'what is the symmetry group of an ω-limit set?' We here consider only the case of discrete dynamics.

More precisely, let Γ be a compact Lie group acting on $\mathbb{R}^n$. As usual, we may identify Γ with a closed subgroup of the group $\mathbf{O}(n)$ of $n \times n$ orthogonal matrices, so that the action is simply matrix multiplication of vectors in $\mathbb{R}^n$. Recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is Γ-*equivariant* if $f(\gamma x) = \gamma f(x)$ for every $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$.

Suppose that A is an ω-limit set for the Γ-equivariant map f. We define the *symmetry group of* A to be the subgroup

$$\Sigma_A = \{\gamma \in \Gamma,\ \gamma A = A\}.$$

Since A is a closed subset, Σ_A is a closed subgroup of Γ. The subgroup Σ_A has a physical interpretation as *symmetry on average*, see Dellnitz *et al.* [10, 28].

A converse question arises: which closed subgroups of Γ can be realized as the symmetry group of such an ω-limit set? This has a simple answer for two 'classical' cases. If $A = \{x\}$ is a fixed point for f, then Σ_A is just the isotropy subgroup $\Sigma_x = \{\gamma \in \Gamma,\ \gamma x = x\}$ of the point x. If A is a periodic orbit, then Σ_A contains the isotropy subgroup of the points in A and is a cyclic extension of this isotropy subgroup.

An important observation (both for theory and applications) is that for more complicated ω-limit sets the subgroup Σ_A may be much larger than the isotropy subgroup of any individual point in A. Recent results of [28, 4, 15] yield a good understanding of the case when Γ is finite. In particular, for 'most' actions of Γ (such as the high-dimensional or infinite-dimensional actions that arise in applications) Σ_A can be any subgroup of Γ. Furthermore, each subgroup of Γ can be realized by an Axiom A attractor [15] and thus occurs in a structurally stable manner.

The situation is quite different when Γ is not finite. Now we can hope to perturb f along continuous group orbits to show that certain subgroups of Γ arise as the symmetry group of an ω-limit set only in degenerate situations. Hence we are interested in classifying those subgroups that arise typically.

These problems are well-understood for relative periodic orbits, [13, 23]. Write $A = \omega(x_0)$ where $x_0 \in \mathbb{R}^n$. We say that A is a *relative periodic orbit* if A is contained in the union of finitely many Γ-orbits in $\mathbb{R}^n$. That is, if we pass to the orbit space $\mathbb{R}^n/\Gamma$ then A becomes a periodic orbit. We give three examples, assuming throughout that f is smooth and that A contains points of trivial isotropy.

Examples when A is a relative periodic orbit[13, 23]

1. If $\Gamma = \mathbf{SO}(2)$ then generically $\Sigma_A = \mathbf{SO}(2)$.

2. If $\Gamma = \mathbf{O}(2)$ then generically $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbb{D}_1$ (a two element subgroup of $\mathbf{O}(2)$ generated by a reflection).

3. If $\Gamma = \mathbf{O}(3)$ then generically $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{SO}(2) \oplus \mathbb{Z}_2^c$. (Here $\mathbb{Z}_2^c$ is the two element subgroup generated by $-I$, where I is the identity.)

We are interested in obtaining similar results when A is an arbitrary ω-limit set. As before, we restrict to the case when f is smooth and A contains at least one point with trivial isotropy. For the examples above, the following results are currently known.

Examples when A is a general ω-limit set

$1'$ If $\Gamma = \mathbf{SO}(2)$ then generically $\Sigma_A = \mathbf{SO}(2)$.

$2'$ If $\Gamma = \mathbf{O}(2)$ and the dynamics in A is 'mildly irregular' then generically $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{O}(2)$.

$3'$ If $\Gamma = \mathbf{O}(3)$ and the dynamics in A is 'very irregular' then generically $\Sigma_A = \mathbf{SO}(3)$ or $\Sigma_A = \mathbf{O}(3)$.

Define Γ^0 to be the connected component of the identity in Γ. In the second set of examples, the conclusion is that generically Σ_A contains Γ^0. The results in $1'$ and $2'$ are obtained in this paper; here Γ^0 is abelian. Of course, for $2'$ we must explain what we mean by 'mildly irregular'. The result in $3'$ is proved in [3] (see also [11]) and 'very irregular' includes the cases of hyperbolic dynamics. Stronger results concerning the lifting of ergodicity, mixing and so on, can be found in [16]. Other relevant results in the smooth context in the ergodic theory literature include [6, 7, 30].

Smoothness, irregularity of dynamics and typicality There are various issues such as smooth versus nonsmooth dynamical systems, regular versus irregular dynamics, and topological versus measure-theoretic 'typicality' that are intertwined and require clarification.

The questions addressed in this paper are well-understood for mappings that are measurable [24, 31] or continuous [14]: generically $\Gamma^0 \subset \Sigma_A$. This includes the case when $A = \omega(x_0)$ is a relative periodic orbit provided $x_0 \notin A$. Hence, the expected behavior of smooth and nonsmooth dynamical systems is quite different when there is a relative periodic orbit, and it is the smooth context that is required for applications.

Comparison of the two sets of examples above for smooth dynamical systems indicates that the typical symmetry of A depends on the dynamics on A. Moreover, an example in [12] shows that for 'intermediate' situations, such as irrational rotations on tori, the 'generic' outcome and the 'probable' outcome for Σ_A may differ. In this paper, our irregularity assumptions are such that $\Gamma^0 \subset \Sigma_A$ holds typically both in the topological sense (genericity) and in the measure-theoretic sense (prevalence [20]).

In this paper we develop a method for attacking the case when Γ^0 is abelian. Our results are particularly complete when the whole of Γ is abelian (including Example $1'$ above), as we describe in the remainder of the introduction.

Suppose that $\Gamma \subset \mathbf{O}(n)$ is an abelian compact Lie group and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant mapping with ω-limit set $A = \omega(x_0)$. A Γ-*cocycle* is a map $\phi : \mathbb{R}^n \to \Gamma^0$. We form a perturbation of f by defining $f_\phi(x) = \phi(x)f(x)$. Note that f_ϕ is automatically Γ-equivariant (since Γ is abelian) and has the same dynamics as f up to displacements along the continuous part of the group. Let A_ϕ denote the ω-limit set of x_0 under f_ϕ.

Let $\mathcal{Z}_k$ denote the space of compactly supported C^k Γ-cocycles.

Theorem 1.1 *For each nonnegative integer k, there is a residual and prevalent subset $Z \subset \mathcal{Z}_k$ such that Σ_{A_ϕ} contains Γ^0 for each $\phi \in Z$.*

The definitions of residual and prevalent are given in §3.2. Roughly speaking, the prevalence property means that Σ_{A_ϕ} contains Γ^0 for 'almost every' C^k Γ-cocycle ϕ.

As indicated in [26], Theorem 1.1 reduces to showing that if $\{s_k\}$ is an unbounded sequence of real numbers then the set $\{s_k\theta \bmod 1\}$ is dense in $[0, 1)$ for almost every $\theta \in [0, 1)$. In the special case when $s_k = 2^k$ we have the orbit $\{2^k\theta\}$ of an initial condition θ under the expanding circle map $g : S^1 \to S^1$ that doubles angles. It is a well known fact from dynamical systems theory that almost every point in S^1 has a dense orbit under g. Hence it is natural to use dynamical systems methods for general sequences $\{s_k\}$ and to consider the *expanding sequence* of maps defined by $g_k(\theta) = s_k\theta$.

In §2 we consider such expanding sequences and their generalization to sequences of $d \times d$ matrices with real entries applied to $\theta \in [0, 1)^d$. In particular, we prove that such sequences are mixing when a suitable expansivity condition is satisfied. This result generalizes (and relies on) a theorem of Berend and Bergelson [5] formulated in the case when the matrices have integer entries. (The theorem of [5] in turn generalizes the classical result which states that expanding endomorphisms of tori are mixing.)

In §3, we give a precise statement and proof of Theorem 1.1. In §4 we obtain fairly powerful results when Γ^0 is abelian but Γ is not necessarily abelian. Finally, in §5 we generalize our results to include ω-limit sets lying in fixed-point subspaces corresponding to isotropy subgroups of Γ.

2 Expanding sequences on the torus

Throughout this section, λ denotes Lebesgue (or Haar) measure on $[0,1)^d$ or $\mathbf{T}^d$. Consider the expanding map on the circle $S^1 = \mathbf{T}^1$ defined by $g(\theta) = 2\theta$. The map g preserves Haar measure and it is well known that g is ergodic. A consequence is that λ-almost every point has a dense orbit in S^1 under iteration by g. Said differently, the sequence $\{2^k\theta\}$ is dense in S^1 for almost every $\theta \in S^1$.

Now suppose that $\{n_k\}$ is an arbitrary sequence of integers. We can ask the question: is the sequence $\{n_k\theta\}$ dense in S^1 for almost every θ? It is clearly necessary that the sequence $\{n_k\}$ is unbounded and this turns out also to be sufficient.

Even more generally, we consider an arbitrary sequence of real numbers $\{s_k\}$. The sequence $\{s_k\theta\}$ is no longer well defined but we show that $\{s_k\theta \bmod 1\}$ is dense in $[0,1)$ for almost every $\theta \in [0,1)$ if and only if $\{s_k\}$ is unbounded. We also consider analogous questions on higher-dimensional tori (or $[0,1)^d$).

In later sections of this paper, we shall draw heavily on the results of this section. However, much of the time the considerably simpler one-dimensional case will suffice. This case is covered in §2.1 and the higher-dimensional case in §2.2.

2.1 Expanding sequences on the circle

Definition 2.1 Suppose that $\{s_k\}$ is a sequence of real numbers and $|s_k| \to \infty$. Then the sequence of mappings $g_k : [0,1) \to [0,1)$ defined by $g_k(\theta) = s_k\theta \bmod 1$ is an *expanding sequence on* $[0,1)$.

Theorem 2.2 *Suppose that* $\{g_k\}$ *is an expanding sequence on* $[0,1)$. *Then* $\lambda(E \cap g_k^{-1}(F)) \to \lambda(E)\lambda(F)$ *as* $k \to \infty$ *for all measurable sets* $E, F \subset [0,1)$.

Proof This result is a special case of Theorem 2.6 below and can also be proved by a direct computation. ∎

Theorem 2.2 states that expanding sequences on $[0,1)$ are strong mixing, where the usual notion of strong mixing for maps is extended in the obvious way. Namely, if μ is a probability measure on X then a sequence of measurable maps $g_k : X \to X$ is *strong mixing* if

$$\mu(E \cap g_k^{-1}(F)) \to \mu(E)\mu(F)$$

as $k \to \infty$ for all measurable sets $E, F \subset X$. (We do not require that the maps g_k are measure preserving.)

Some (but not all, see [5]) of the standard properties of strong mixing maps go through for strong mixing sequences. Two that we require are the following.

Proposition 2.3 *Suppose that the sequences $g_k : X \to X$ and $h_k : Y \to Y$ are strong mixing with respect to measures μ_X and μ_Y. Then*

(a) If the support of μ_X is the whole of X, then the orbit $\{g_k(x)\}$ is dense in X for μ_X-almost every $x \in X$.

(b) The cartesian product $g_k \times h_k : X \times Y \to X \times Y$ is strong mixing (with respect to the product measure $\mu_X \times \mu_Y$).

Proof As in the case of strong mixing maps, these properties for strong mixing sequences follow directly from the definitions. ∎

Corollary 2.4 *Suppose that $\{s_k\}$ is a sequence of real numbers. For each $\theta \in [0,1)$ consider the orbit $\{s_k\theta\}$ computed* mod1. *This orbit is dense in $[0,1)$ for λ-almost every $\theta \in [0,1)$ if and only if $\{s_k\}$ is unbounded.*

Proof If $\{s_k\}$ is bounded, then the orbit $\{s_k\theta\}$ is bounded away from $1/2$ for θ small enough and in particular is not dense. Conversely, if $\{s_k\}$ is unbounded, we can pass without loss to a subsequence so that $|s_k| \to \infty$. The resulting sequence of maps is expanding and hence strong mixing by Theorem 2.2. The required conclusion follows from Proposition 2.3(a). ∎

Thanks to Proposition 2.3(b) we have an immediate extension to higher-dimensional tori.

Corollary 2.5 *Suppose that $\{s_k\}$ is a sequence of real numbers. For each $\theta \in [0,1)^d$ consider the orbit $\{s_k\theta\}$ where each of the d components is computed* mod 1. *This orbit is dense in $[0,1)^d$ for λ-almost every $\theta \in [0,1)^d$ if and only if $\{s_k\}$ is unbounded.*

2.2 Expanding sequences on higher-dimensional tori

Suppose that $g : \mathbf{T}^d \to \mathbf{T}^d$ is a continuous homomorphism of the torus. Then it is well known, see for example [25], that g can be represented uniquely as a linear map $\tilde{g}$ on $\mathbb{R}^d$ with integer coefficients. More precisely, $\pi\tilde{g} = g\pi$ where $\pi : \mathbb{R}^d \to \mathbf{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the canonical projection. The map g preserves Haar measure and is ergodic (and strong mixing) if and only if no eigenvalue of $\tilde{g}$ is a root of unity. If each eigenvalue has absolute value greater than one, then the map g is said to be expanding.

Berend and Bergelson [5] considered generalizations of this result to sequences of continuous homomorphisms $g_k : \mathbf{T}^d \to \mathbf{T}^d$. In particular, they proved that if the sequence is *expanding* in the sense that $\tilde{g}_k^{-1} \to 0$ as $k \to \infty$, then $\{g_k\}$ is strong mixing.

Now suppose that $\tilde{g}$ is a nonsingular linear map on $\mathbb{R}^d$ with arbitrary real entries. Then $\tilde{g}$ induces a *pseudo-homomorphism* $g : [0,1)^d \to [0,1)^d$ defined by $g(\theta) = \tilde{g}(\theta) \bmod \mathbb{Z}^d$. As before, we say that the sequence $\{g_k\}$ is expanding if $\tilde{g}_k^{-1} \to 0$. We have the following generalization of the theorem of Berend and Bergelson.

Theorem 2.6 *Expanding sequences of pseudo-homomorphisms $g_k : [0,1)^d \to [0,1)^d$ are strong mixing.*

Proof Define a sequence of homomorphisms $h_k : \mathbf{T}^d \to \mathbf{T}^d$ by setting $\tilde{h}_k : \mathbb{R}^d \to \mathbb{R}^d$ to be the linear map whose entries are the integer parts of the corresponding entries of $\tilde{g}_k$. Let $\| \ \|$ denote the operator norm with respect to the Euclidean norm on $\mathbb{R}^d$. Then $\|\tilde{g}_k - \tilde{h}_k\| \leq 1$ and it follows that $\|I - \tilde{g}_k^{-1}\tilde{h}_k\| \leq \|\tilde{g}_k^{-1}\| \to 0$. In particular, we have that $\tilde{h}_k$ is invertible, that $\tilde{h}_k^{-1}\tilde{g}_k \to I$, and that $\tilde{h}_k^{-1} \to 0$. The theorem of Berend and Bergelson guarantees that h_k is strong mixing.

Suppressing subscript k's, we compute that

$$\begin{aligned}
\lambda(E \cap g^{-1}F) &= \sum_{n \in \mathbb{Z}^d} \lambda(E \cap \tilde{g}^{-1}(F+n)) \\
&= |\det \tilde{h}^{-1}\tilde{g}|^{-1} \sum_{n \in \mathbb{Z}^d} \lambda(\tilde{h}^{-1}\tilde{g}E \cap \tilde{h}^{-1}(F+n)) \\
&= |\det \tilde{h}^{-1}\tilde{g}|^{-1} \lambda(\tilde{h}^{-1}\tilde{g}E \cap h^{-1}F).
\end{aligned}$$

(We temporarily extended the measure λ to Lebesgue measure on $\mathbb{R}^d$.) Now, $\det \tilde{h}^{-1}\tilde{g} \to 1$ as $\tilde{g}^{-1} \to 0$. In addition, we have

$$\begin{aligned}
|\lambda(\tilde{h}^{-1}\tilde{g}E \cap h^{-1}F) - \lambda(E)\lambda(F)| \leq{} & |\lambda(\tilde{h}^{-1}\tilde{g}E \cap h^{-1}F) - \lambda(E \cap h^{-1}F)| \\
& + |\lambda(E \cap h^{-1}F) - \lambda(E)\lambda(F)|.
\end{aligned}$$

The second term in the right-hand-side converges to zero since h_k is strong mixing. It remains to show that the first term converges to zero. But

$$|\lambda(\tilde{h}^{-1}\tilde{g}E \cap h^{-1}F) - \lambda(E \cap h^{-1}F)| \leq \lambda(\tilde{h}^{-1}\tilde{g}E \triangle E),$$

which clearly converges to zero if E is compact and hence does so for all measurable sets $E \subset [0,1)^d$. We conclude that $\lambda(E \cap g^{-1}F) \to \lambda(E)\lambda(F)$ as $\tilde{g}^{-1} \to 0$. ∎

The following corollary is sufficient for the applications in the later sections.

Corollary 2.7 *Suppose that $\{S_k\}$ is a sequence of (eventually) invertible linear operators on $\mathbb{R}^d$. For each $\theta \in [0,1)^d$ consider the orbit $\{S_k\theta\}$ where the entries are computed* mod1. *This orbit is dense in $[0,1)^d$ for λ-almost every $\theta \in [0,1)$ if and only if the sequence $\|S_k^{-1}\|^{-1}$ is unbounded.*

Proof Mimic the proof of Corollary 2.4. ∎

3 The case Γ abelian

In this section we prove Theorem 1.1 by following the ideas in [26], but making them completely precise. In so doing we introduce a general method for addressing issues concerning the subgroups that typically arise as symmetry groups of ω-limit sets.

In §3.1, we restate the main result. In particular, we define an appropriate class of perturbations — smooth Γ-cocycles — and we define typicality in terms of sets of cocycles that are both residual and prevalent in the set of all cocycles. These notions of residuality (or genericity) and prevalence are recalled in §3.2. It is also shown that for the problem at hand, genericity is a consequence of prevalence. The proof is given in §3.3.

3.1 Statement of the result

We begin by specifying the requisite class of perturbations. Cocycle extensions are considered extensively in the ergodic theory literature, see for example [24, 31]. For the moment, we shall not make any assumptions on the group Γ.

A Γ-*cocycle* is a map $\phi : \mathbb{R}^n \to \Gamma^0$ satisfying the equivariance condition

$$\phi(\gamma x) = \gamma\phi(x)\gamma^{-1}, \tag{3.1}$$

for all $\gamma \in \Gamma$. If k is a non-negative integer or $k = \infty$ we let $\mathcal{Z}_k$ denote the space of compactly supported C^k Γ-cocycles. Note that $\mathcal{Z}_k$ is a group under pointwise multiplication and is abelian if Γ^0 is abelian. Moreover, $\mathcal{Z}_k$ equipped with the C^k uniform topology has the structure of a complete metric space and a topological group.

Now suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k Γ-equivariant map and that $x_0 \in \mathbb{R}^n$. Let $A = \omega(x_0)$ denote the ω-limit set of x_0 under f. For each cocycle $\phi \in \mathcal{Z}_k$ define $f_\phi(x) = \phi(x)f(x)$ and let A_ϕ denote the ω-limit set of x_0 under f_ϕ. The mapping $f_\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called the *extension of f by the cocycle ϕ* and is Γ-equivariant by condition (3.1).

When ϕ is near the identity we think of f_ϕ as a small perturbation of f, in which case A_ϕ is the perturbed ω-limit set. Then we are interested in the symmetry group Σ_{A_ϕ} of A_ϕ.

Remark 3.1 (a) Extension by a cocycle does not change the underlying dynamics corresponding to f, only the movement along group orbits. Indeed, if we quotient out by the connected component Γ^0 and pass to the mapping on the orbit space defined by $\tilde{f} : \mathbb{R}^n/\Gamma^0 \to \mathbb{R}^n/\Gamma^0$ then $\tilde{f}_\phi = \tilde{f}$ for all $\phi \in \mathcal{Z}_k$.
(b) We consider cocycles with values in Γ^0 rather than the whole of Γ since we are interested only in the effects of small perturbations.

Theorem 3.2 *Let $\Gamma \subset \mathbf{O}(n)$ be an abelian compact Lie group, and let $A = \omega(x_0)$ be an ω-limit set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Define*

$$Z = \{\phi \in \mathcal{Z}_k : \Gamma^0 \subset \Sigma_{A_\phi}\}. \tag{3.2}$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

Remark 3.3 (a) The statement and proof of Theorem 3.2 are valid more generally for any compact Lie group Γ where Γ^0 is central in Γ, that is elements of Γ^0 commute with all elements in Γ. (Note that this condition implies that Γ^0 is a torus.) In particular, Γ may be any group of the form $\mathbf{T} \times H$ where $\mathbf{T}$ is a torus and H is finite.
(b) Theorem 3.2 has an ergodic-theoretic analogue (except that prevalence is not mentioned — the notion had not been defined at the time): see Jones and Parry [22]. The results in later sections, in particular Theorem 4.13, are not to our knowledge to be found in the ergodic theory literature.
(c) The definitions of residual and prevalent sets are recalled in §3.2, where we also show that it is sufficient to prove prevalence of the set Z in (3.2).

3.2 Residual and prevalent sets

Suppose that X is a complete metric space and that $R \subset X$ is a subset. The set R is said to be *residual* in X if there are countably many open dense subsets $U_i \subset X$ such that $\bigcap_i U_i \subset R$. Residual subsets are dense in X (every complete metric space is a Baire space), and in particular they are nonempty.

Often a property is said to hold *generically* in X if it holds on a residual subset. The implicit suggestion is that residual sets are in some sense large. However it is well known that even in $\mathbb{R}$ there are residual sets of Lebesgue measure zero. In finite dimensions it is reasonable to require large sets to have large measure (ideally their complements should have measure zero). In order to make sense of such a requirement in infinite dimensions, Hunt, Sauer and Yorke [20] introduce the notion of prevalence. Although formulated primarily for vector spaces, this notion also applies to abelian topological groups [21], and we recall the definition of prevalence in this context (see also [9]). It is also possible to define prevalence in a reasonable way for nonabelian topological groups, see Mycielski [29].

Suppose that in addition to being a complete metric space, X is also an abelian topological group (whose group operations are continuous with respect to the topology induced by the metric).

Definition 3.4 A Borel subset $R \subset X$ is *prevalent* if there is a compactly supported probability measure μ defined on the Borel sets in X such that

$$\mu(y \in X, x + y \in R) = 1, \text{ for all } x \in X.$$

A general subset $R \subset X$ is prevalent if it contains a prevalent Borel set.

Remark 3.5 Hunt *et al.* introduce additional terminology where the complement $X - R$ of the prevalent set R is said to be *shy* and the measure μ is *transverse* to $X - R$. We do not require these notions here.

For convenience we recall some basic properties of prevalence, see Hunt *et al.* [20] for details.

Proposition 3.6 ([20]) *Suppose that X has the structure of a complete metric space and an abelian topological group.*

(a) If R is prevalent then so is every translate $R + x$.

(b) Prevalent sets are dense.

(c) A countable intersection of prevalent sets is prevalent.

(d) If X is a compact group, then the prevalent subsets are precisely those of full Haar measure.

The only property of prevalent sets that we shall make explicit use of in this paper is (b).

Proposition 3.7 ([14]) *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group, and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous Γ-equivariant map with ω-limit set $A = \omega(x_0)$. Then Z is a countable intersection of open sets.*

Proof Let $W = \bigcup_{\gamma\in\Gamma^0} \gamma A$. Then it is clear that $A_\phi \subset W$ for all $\phi \in \mathcal{Z}_k$. Hence Z consists of those cocycles for which $A_\phi = W$. Let $\{W_n\}$ be a countable base for W and set

$$Z_{m,n} = \bigcup_{j\geq m} \{\phi \in \mathcal{Z}_k,\ f_\phi^j(x_0) \in W_n\}.$$

Then each of the $Z_{m,n}$ is open and $Z = \bigcap_{m,n} Z_{m,n}$. ∎

Corollary 3.8 *Under the assumptions of Proposition 3.7,*

(a) Z is residual in $\mathcal{Z}_k$ if and only if Z is dense.

(b) If Z is prevalent in $\mathcal{Z}_k$ then Z is residual.

Proof As mentioned previously, residual sets are dense. Conversely, if Z is dense then Proposition 3.7 implies that Z is a countable intersection of open dense sets, and hence is residual, proving (a). If Γ^0 is abelian then part (b) is immediate by Proposition 3.6(b). Even if Γ^0 is not abelian, we can appeal to the results of [29]. ∎

Remark 3.9 We have seen that prevalence of Z automatically implies that Z is residual. The converse is not true: Eliasson [12] considers an example where $\Gamma = \mathbf{SO}(3)$ and Z is residual but not prevalent. Indeed in a probabilistic sense Z is rather small (though not of measure zero) with $\Sigma_A = \mathbf{SO}(2)$ preferred to $\Sigma_A = \mathbf{SO}(3)$. The ω-limit set A in [12] is a torus with a constant irrational flow (where the constant satisfies a Diophantine condition).

Our main interest in this paper is in the case Γ^0 abelian, but it seems highly plausible that the result of [12] is valid also when $\Gamma = \mathbf{O}(2)$ with $\mathbb{D}_1 \subset \mathbf{O}(2)$ playing the role of $\mathbf{SO}(2) \subset \mathbf{SO}(3)$.

Remark 3.10 Prevalence and genericity are often opposing notions, and it is questionable how useful either notion is on its own. However, either notion implies density and this is a standard use of the theory of residual sets. Given the remarks in [20] it is amusing to observe that we use the fact that prevalent sets are dense to show that the set Z defined in (3.2) is residual.

This observation has some nontrivial content when we prove genericity in the C^∞ topology. The perturbations that we use are C^k-small for k finite but are not C^∞ small. We have no direct proof that $Z \subset \mathcal{Z}_\infty$ is residual. Rather, we prove prevalence, deduce density and apply Corollary 3.8.

3.3 Proof of Theorem 3.2

A key step in the proof of Theorem 3.2 is to consider a class $\mathcal{C}$ of 'almost constant' cocycles in $\mathcal{Z}_k$. In the terminology of Hunt *et al.* [20], these cocycles form a *probe* for the prevalence of the set Z.

Let $U, V \subset \mathbb{R}^n$ be Γ-invariant open subsets with $A \cap V \neq \emptyset$ and $\overline{V} \subset U$. For $\gamma \in \Gamma^0$, define ϕ_γ to have support in U and to take the value γ on V. Given a suitable smoothing on $U - V$, this defines uniquely a cocycle $\phi_\gamma \in \mathcal{Z}_\infty$.

To construct the smoothing, we make two arbitrary choices. Let $L(\Gamma)$ denote the Lie algebra of Γ^0.

(i) Choose a Γ-invariant C^∞ bump function $b : \mathbb{R}^n \to [0,1]$ with support in U and such that $b|_V \equiv 1$.

(ii) Choose neighborhoods $N \subset \Gamma^0$, $N' \subset L(\Gamma)$ with $1 \in N$, $0 \in N'$, such that the restricted map $\exp : N' \to N$ is a diffeomorphism.

For each $\gamma \in N$ we define the cocycle ϕ_γ by

$$\phi_\gamma(x) = \exp(b(x)\exp^{-1}(\gamma)) = \gamma^{b(x)}.$$

The resulting collection $\mathcal{C}$ of almost constant cocycles is in one-to-one correspondence with N. Let λ denote Haar measure on Γ^0 restricted to N and normalized so that $\lambda(N) = 1$. In the obvious way, λ transforms to a probability measure on $\mathcal{Z}_k$ supported on N.

For $\gamma \in N$, define $f_\gamma = \phi_\gamma f$ to be the extension of f by the almost constant cocycle with value γ and let A_γ be the ω-limit set of x_0 under f_γ. Note that $f_\gamma(x) = \gamma^{b(x)} f(x)$ where $0 \leq b(x) \leq 1$.

Lemma 3.11 *Let $\Gamma \subset \mathbf{O}(n)$ be a abelian compact Lie group, and let $A = \omega(x_0)$ be an ω-limit set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Then $\Gamma^0 \subset \Sigma_{A_\gamma}$ for λ-almost every $\gamma \in N$.*

Proof Let $W = \bigcup_{\gamma\in\Gamma^0} \gamma A$ and observe that $A_\gamma \subset W$ for every $\gamma \in \Gamma^0$. Hence, $\Gamma^0 \subset \Sigma_{A_\gamma}$ if and only if $A_\gamma = W$. We show that $W \subset A_\gamma$ for almost every $\gamma \in \Gamma^0$.

Let $\{y_i\}$ be a countable dense subset of A. We claim that there is for each i a subset $G_i \subset \Gamma^0$ with $\lambda(G_i) = 1$, such that $\Gamma^0 y_i \subset A_\gamma$ for $\gamma \in G_i$. Let $G = \bigcap G_i$. Then $\lambda(G) = 1$ and $\bigcup \Gamma^0 y_i \subset A_\gamma$ for $\gamma \in G$. Since the ω-limit set A_γ is closed, we have

$$W = \overline{\bigcup \Gamma^0 y_i} \subset A_\gamma$$

for all $\gamma \in G$, as required.

It remains to prove the claim. Since $y \in \omega(x_0)$, there is a strictly increasing sequence $\{n_k\}$ of integers such that $f^{n_k}(x_0) \to y$. For $\gamma \in N$, we have $f_\gamma^{n_k}(x_0) = \gamma^{s_k} f^{n_k}(x_0)$ where $s_k = b(x_0) + b(f(x_0)) + \cdots + b(f^{n_k - 1}(x_0))$. The sequence $\{s_k\}$ is unbounded since $b(f^n(x_0)) = 1$ infinitely often (whenever $f^n(x_0) \subset V$). Let $R(\gamma)$ denote the set of limit points of the sequence $\{\gamma^{s_k}\}$. Then clearly $R(\gamma)y \subset A_\gamma$. Translating Corollary 2.5 from additive to multiplicative notation, we have that $R(\gamma) = \Gamma^0$ for almost every $\gamma \in \Gamma^0$, as required. ∎

Proof of Theorem 3.2 By Corollary 3.8 it is enough to prove prevalence of the set Z defined by (3.2). Recall that we defined the probability measure λ on $\mathcal{C} \subset \mathcal{Z}_k$. By the definition of prevalence, it is sufficient to prove that

$$\lambda(\{\psi \in \mathcal{Z}_k : \phi + \psi \in Z\}) = 1$$

for each $\phi \in \mathcal{Z}_k$. Equivalently, for each $\phi \in \mathcal{Z}_k$,

$$\lambda(\{\phi_\gamma \in \mathcal{C} : \phi + \phi_\gamma \in Z\}) = 1. \tag{3.3}$$

Let f and A satisfy the hypotheses of Theorem 3.2 (and Lemma 3.11) and suppose that $\phi \in \mathcal{Z}_k$ is given. Set $f' = f_\phi$, $A' = A_\phi$ and observe that f' and A' also satisfy these hypotheses. Hence by Lemma 3.11, $\Gamma^0 \subset \Sigma_{A'_\gamma}$ for λ-almost every $\gamma \in N$. But $A'_\gamma = A_{\phi+\phi_\gamma}$, so

$$\lambda(\{\gamma \in N : \Gamma^0 \subset \Sigma_{A_{\phi+\phi_\gamma}}\}) = 1.$$

This is equivalent to equation (3.3), and the theorem is proved. ∎

4 The case Γ^0 abelian

In this section, we generalize Theorem 3.2 to the case when only Γ^0 is assumed to be abelian. As mentioned in the introduction, it is necessary to take into account the underlying dynamics. In addition, many of our results have simpler statements when the ω-limit set A is assumed to be topologically transitive, that is, $A = \omega(z)$ for some $z \in A$.

In §4.1 we describe some simple extensions of our main theorems that do not require assumptions on the underlying dynamics. Then, in the main part of this section, we focus on the group $\Gamma = \mathbf{O}(2)$. Passing to the $\mathbf{SO}(2)$-orbit space yields a $\mathbb{Z}_2$-equivariant dynamical system. In preparation, we consider in §4.2 such systems as defining a '$\mathbb{Z}_2$-symmetric deterministic game'. We describe what we mean for such a game to be *completely ruinous* and argue that this is a weak condition on the 'chaoticity' of the dynamics. In §4.3 we prove that if the associated game is completely ruinous then we obtain the required conclusion that typically $\mathbf{SO}(2) \subset \Sigma_A$. Finally, in §4.4 we derive an analogous result for arbitrary groups Γ with Γ^0 abelian.

4.1 Simple extensions of Theorem 3.2

We observed in Remark 3.3(a) that Theorem 3.2 is valid for compact Lie groups Γ such that Γ^0 is central in Γ. Equivalently, $C(\Gamma^0) = \Gamma$ where $C(\Gamma^0)$ denotes the centralizer of Γ^0 in Γ. In this section, we consider the problems that arise when the centralizer condition fails.

Proposition 4.1 *Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group with Γ^0 abelian, and let $A = \omega(x_0)$ be an ω-limit set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Assume that there is a point $z \in A$ such that $\gamma z \notin A$ for all $\gamma \in \Gamma - C(\Gamma^0)$. Define*

$$Z = \{\phi \in \mathcal{Z}_k : \Gamma^0 \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

Proof Take z as in the statement of the proposition and let $U \subset \mathbb{R}^n$ be a Γ-invariant open subset containing Γz. Let U^0 denote the connected component of U that contains z. Then U^0 is Γ^0-invariant and we can choose U small enough so that $\gamma U^0 \cap U^0 = \emptyset$ for all $\gamma \in \Gamma - \Gamma^0$. Let $U' = \bigcup_{\gamma \in C(\Gamma^0)} \gamma U^0$. Then it follows from the hypothesis on z that, for U small enough, $f^n(x_0) \notin U - U'$ for all $n \geq 0$. Moreover, since cocycles take values only in Γ^0 we have

$$f_\phi^n(x_0) \notin U - U' \text{ for all } \phi \in \mathcal{Z}_k,\ n \geq 0. \tag{4.1}$$

Now proceed as in §3 to define a class of almost constant cocycles taking values $\gamma \in \Gamma^0$ on U_0. (As before, we must smooth the cocycle by a bump function b and restrict the values of γ to a small neighborhood N of the identity.) The only difference is that the cocycle restricted to U^0 determines the cocycle on $U - U^0$. This causes no problem by (4.1). ∎

The statement of Proposition 4.1 is simpler for topologically transitive sets, thanks to the following proposition.

Proposition 4.2 *Suppose that $A = \omega(z)$ where $z \in A$. Then $\sigma \in \Sigma_A$ if and only if $\sigma z \in A$.*

Corollary 4.3 *Let Γ be compact with Γ^0 abelian and suppose that A is a topologically transitive set satisfying $\Sigma_A \subset C(\Gamma^0)$. Then typically (in the senses of genericity and prevalence) $\Gamma^0 \subset \Sigma_A$.*

Remark 4.4 It follows from the corollary that if A is topologically transitive and $\Sigma_A \subset \Gamma^0$, then typically $\Sigma_A = \Gamma^0$. In particular, if Γ^0 is nontrivial, then typically $\Sigma_A \neq \mathbf{1}$.

4.2 Deterministic $\mathbb{Z}_2$-symmetric games

Let $\mathbb{Z}_2$ act linearly on $\mathbb{R}^n$ and denote the nontrivial element of $\mathbb{Z}_2$ by ρ. Suppose that $g : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth $\mathbb{Z}_2$-equivariant map with a $\mathbb{Z}_2$-symmetric ω-limit set $\Lambda = \omega(x_0)$. Assume that there is a point $z \in \Lambda$ with trivial isotropy. Using the initial condition x_0 we construct a deterministic game as follows.

Let $U_+ \subset \mathbb{R}^n$ be an open neighborhood of z such that $U_- = \rho U_+$ does not intersect U_+. Choose a smaller open neighborhood V_+ of z such that $\overline{V_+} \subset U_+$. Define a smooth bump function $b : \mathbb{R}^n \to [0,1]$ supported in U_+ such that $b|_{V_+} \equiv 1$. Then we define $B : \mathbb{R}^n \to [-1,1]$ supported in $U = U_+ \cup U_-$ such that

$$B(x) = b(x), \quad B(\rho x) = -b(x), \tag{4.2}$$

for all $x \in U_+$. Set $a_i = B(g^i(x_0))$ and define $s_n = \sum_{i=1}^n a_i$.

Definition 4.5 The dynamical system g is *ruinous* if the sequence $\{s_n\}$ is unbounded (for some choice of bump function b).

Remark 4.6 The 'deterministic' game' that we have described is reminiscent of a simple symmetric random walk. For example, suppose that a player gambles repeatedly on the outcome of tosses of a coin. In particular, suppose that the coin is fair and that the player always chooses heads. Let $a_i = 1$ if the i'th toss is a head and $a_i = -1$ if the i'th toss is a tail. Then s_n defined as before represents the players cumulative profit or loss after the n'th toss. It is well-known that for each integer L, it is certain (probability one) that $s_n = L$ infinitely often. In particular, the player will eventually run out of money no matter the size of the initial funds.

A ruinous $\mathbb{Z}_2$-symmetric game is a deterministic version of a simple symmetric random walk. The $\mathbb{Z}_2$-symmetry, as preserved by condition (4.2), ensures that the game is 'fair'. The hypothesis that the game is 'ruinous' implies that $s_n \geq L$ infinitely often (this is no longer a probabilistic statement). As shown in the two examples below, a $\mathbb{Z}_2$-symmetric deterministic game is often, but not always, ruinous.

Example 1 Suppose that Λ is a period two point, $\Lambda = \{z, \rho z\}$. After a transient, we have $s_n = (-1)^n$ and the game is not ruinous.

Example 2 Suppose that $z \in \Lambda$ is point of trivial isotropy and that $g(z) = z$. Then the dynamical system is ruinous, as can be seen by choosing the bump function b so that $z \in V_+$. Since $z \in \omega(x_0)$, the iterates of x_0 under g lie in V_+ for arbitrarily long periods of time. Hence, there are arbitrary large blocks in which $a_n = 1$ and this is enough to imply that $\{s_n\}$ is unbounded. (There are also arbitrarily large blocks in which $a_n = -1$ but this does not affect the argument.)

In Theorem 4.8 below, we generalize Example 2 considerably. The upshot is that many $\mathbb{Z}_2$-symmetric systems are ruinous.

Proposition 4.7 *Let $\{a_n\}$ be a sequence of real numbers with sequence of partial sums $\{s_n\}$. Then $\{s_n\}$ is unbounded if and only if for any $L > 0$ there exist integers $N' > N \geq 1$ such that*

$$\left|\textstyle\sum_{i=N+1}^{N'} a_n\right| \geq L. \tag{4.3}$$

Proof It is clear that the given condition is necessary for $\{s_n\}$ to be unbounded. To prove sufficiency, it is enough to show that for each n_0 there exists n_1 such that $|s_{n_1}| - |s_{n_0}| \geq 1$.

Let $L = 2(|s_{n_0}| + 1)$, with corresponding integers $N' \geq N \geq 1$. Without loss of generality we can assume that $\sum_{i=N+1}^{N'} a_n \geq L$. If $s_N \leq -|s_{n_0}| - 1$ then $|s_N| \geq |s_{n_0}| + 1$ and we simply take $n_1 = N$.

The other case to consider is when $s_N > -|s_{n_0}| - 1$. Set $n_1 = N'$ and compute that

$$s_{n_1} = s_{N'} = s_N + L > -(|s_{n_0}| + 1) + 2(|s_{n_0}| + 1) = |s_{n_0}| + 1,$$

and again we obtain the required inequality. ∎

Theorem 4.8 *Let $\Lambda = \omega(x_0)$ be an ω-limit set and let $x_1 \in \Lambda$. Suppose that $\omega(x_1)$ is ruinous. Then Λ is ruinous.*

Proof Choose $z \in \omega(x_1)$ and bump function b so that the sequence $\{a_n\}$ obtained from considering the iterates $g^n(x_1)$ has unbounded partial sums. In particular, condition (4.3) is satisfied.

Let $\{a'_n\}$ denote the corresponding sequence for x_0. Since $x_1 \in \omega(x_0)$, finite segments of arbitrarily length of the sequence $\{a_n\}$ are present also in $\{a'_n\}$ (up to any specified accuracy). Hence, condition (4.3) is also satisfied for $\{a'_n\}$. Now apply Proposition 4.7. ∎

Corollary 4.9 *Suppose that Λ_0 is a closed dynamically-invariant subspace of Λ and that Λ_0 is not $\mathbb{Z}_2$-symmetric. Then Λ is ruinous.*

Now we complicate the situation somewhat. Let W be an open subset of $\mathbb{R}^n$ that intersects Λ. We say that the system is *ruinous with respect to* W if the subsequence $\{s_n : g^n(x_0) \in W\}$ is unbounded. In terms of the game, this means that the player keeps track of the score s_n as before but interprets this quantity as a profit or loss only at times n when $g^n(x_0) \in W$. We say that the system is *completely ruinous* if it is ruinous with respect to all open sets W that intersect Λ.

We feel that it is counterintuitive that in a ruinous system it is possible to choose W so that the profits balance the losses (within some bound) every time the trajectory $g^n(x_0)$ happens to lie in W. At the very least it would appear to be a rather strong restriction on the dynamics. Unfortunately, we have few results to substantiate our intuition (short of assuming that there is a symbolic dynamics). Thus we pose the following problem.

Open Problem Find reasonable conditions on a ruinous dynamical system Λ such that the system is completely ruinous. (The bump function b may be modified if necessary.)

Associate to each $y \in \Lambda$ a strictly increasing sequence $\{n_k(y)\}$ with $g^{n_k(y)}(x_0) \to y$. Let $s_k(y) = s_{n_k(y)}$. Then $\{s_k(y)\}$ is a subsequence of $\{s_n\}$. Let

$$Y = \{y \in \Lambda : \{n_k(y)\} \text{ can be chosen so that } \{s_k(y)\} \text{ is unbounded}\}.$$

Proposition 4.10 *With the above notation,*

(a) Y is closed.

(b) $g(Y) \subset Y$.

(c) If Y contains a point y with $\omega(y) = \Lambda$ then $Y = \Lambda$.

(d) If Λ is compact then Λ is ruinous if and only if $Y \neq \emptyset$.

(e) Λ is completely ruinous if and only if $Y = \Lambda$.

Proof It is clear that Y is closed. Suppose that $y \in Y$. Then we can choose an increasing sequence $\{n_k\}$ such that $\{s_k\}$ is unbounded and $g^{n_k}(x_0) \to y$. Let $n'_k = n_k + 1$ so that $g^{n'_k}(x_0) \to g(y)$. Then $|s'_k - s_k| \leq 1$ for all k, so that $\{s'_k\}$ is unbounded and $g(y) \in Y$ proving part (b). Part (c) is immediate from parts (a) and (b).

Next we prove part (d). If Y is nonempty, then $\{s_n\}$ has an unbounded subsequence and hence is itself unbounded. Conversely, if $\{s_n\}$ is unbounded, we can pass to a monotone unbounded subsequence $\{s_{n_j}\}$. By compactness of Λ we may pass if necessary to a further (necessarily unbounded) subsequence so that $g^{n_j}(x_0) \to y \in \Lambda$. By construction $y \in Y$.

Finally, it follows from part (a) that $Y = \Lambda$ if and only if Y is dense in Λ. But this is precisely the condition that Λ is completely ruinous. ∎

Recall that Λ is *minimal* if $\omega(x) = \Lambda$ for every $x \in \Lambda$.

Proposition 4.11 *If Λ is minimal and compact, then Λ is ruinous if and only if Λ is completely ruinous.*

Proof Since Λ is compact and ruinous, Proposition 4.10(d) implies that Y is nonempty. Choose $y \in Y$. By part (c) of the proposition, $\omega(y) \subset Y$. But Λ is minimal, so $\omega(y) = \Lambda \subset Y$. ∎

4.3 The group $\Gamma = \mathbf{O}(2)$

Remark 4.4 implies that if $\Gamma = \mathbf{O}(2)$ and A is topologically transitive then either $\Sigma_A = \mathbf{SO}(2)$ or Σ_A contains (some conjugate of) $\mathbb{D}_1$, that is, Σ_A is (conjugate to) $\mathbb{D}_k$ for some k. In particular the cyclic subgroups $\mathbb{Z}_k \subset \mathbf{SO}(2)$ are not expected to be realized as the symmetry group of A. Note also that both the subgroups $\mathbf{SO}(2)$ and $\mathbb{D}_1$ are realized by relative periodic orbits (example 2 in the introduction). In this subsection, we show that if A contains a point z with isotropy subgroup $\Sigma_z = \mathbf{1}$ and if the dynamics in A is sufficiently complicated, then typically $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{O}(2)$. (In §5 we relax the condition that some z has $\Sigma_z = \mathbf{1}$.)

To describe our results, it is convenient to quotient out the action of $\Gamma^0 = \mathbf{SO}(2)$. Passing to the orbit space $X = \mathbb{R}^n/\mathbf{SO}(2)$, we have a map $g : X \to X$ with an ω-limit set $\Lambda = \omega(x_0)$. The group $\Gamma/\Gamma^0 \cong \mathbb{Z}_2$ acts on the orbit space, and g is equivariant with respect to this action. Denote the nontrivial element of $\mathbb{Z}_2$ by ρ. Our assumption on A implies that Λ contains a point z with trivial isotropy (inside of $\mathbb{Z}_2$).

Since the orbit space X is singular, there are technical problems in talking about smooth maps on X. To avoid these problems define R_0 to consist of those points $x \in \mathbb{R}^n$ with $\Sigma_x \cap \mathbf{SO}(2) = \mathbf{1}$. Since R_0 is nonempty (by the assumption on A) it follows that R_0 is open and dense in $\mathbb{R}^n$. Moreover, $\mathbf{SO}(2)$ acts fixed-point freely on R_0 and the orbit space $X_0 = R_0/\mathbf{SO}(2)$ is a manifold. We shall need to speak about smoothness of a map on X only inside of X_0. Thus the terminology that Λ is ruinous or completely ruinous as defined in §4.2 makes sense inside the orbit space X. We shall say that A is completely ruinous if Λ is.

Theorem 4.12 *Suppose that $\Gamma = \mathbf{O}(2)$ acts on $\mathbb{R}^n$, and let $A = \omega(x_0)$ be an ω-limit set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that A contains a point of trivial isotropy and that A is completely ruinous. Define*

$$Z = \{\phi \in \mathcal{Z}_k : \mathbf{SO}(2) \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

Proof Choose an open subset $U = U_+ \cup U_- \subset X_0$ and a smooth bump function b supported on U_+ satisfying the conditions for A to be completely ruinous. The set U lifts to an $\mathbf{O}(2)$-invariant set $U' \subset \mathbb{R}^n$. Also the extended bump function B lifts uniquely to a $\mathbf{SO}(2)$-invariant C^∞ bump function on $\mathbb{R}^n$ with support inside U'. Choose a neighborhood N of the identity in $\mathbf{SO}(2)$ as in §3. Then we can define for each $\gamma \in N$ an almost constant cocycle ϕ with $\phi(x) = \gamma^{B(x)}$. The identity

$$\rho\gamma\rho^{-1} = \gamma^{-1} \text{ for all } \gamma \in \Gamma,$$

together with condition (4.2) imply that ϕ satisfies the equivariance condition (3.1).

Let f_γ denote the corresponding cocycle extension of f. Then $f_\gamma(x) = \gamma^{B(x)} f(x_0)$ and so $f_\gamma^n(x_0) = \gamma^{s_n} f^n(x_0)$. Just as in §3, it suffices to prove that for each $y \in A$, $\mathbf{SO}(2)y \subset A_\gamma$ for almost every $\gamma \in N$. Let $\{n_k\}$ be an increasing sequence of integers with $f^{n_k}(x_0) \to y$. Then $f_\gamma^{n_k}(x_0) = \gamma^{s_k} f^{n_k}(x_0)$ where $s_k = s_{n_k}$.

It is sufficient to prove that $\{\gamma^{s_k}\}$ is dense in $\mathbf{SO}(2)$ for almost every $\gamma \in N$. Since A is completely ruinous we can assume that $\{s_k\}$ is unbounded. As before, the problem transforms into a expanding sequence on the circle, and the theorem then follows from Corollary 2.5. ∎

4.4 The general case Γ^0 abelian

In this subsection, we extend our results for the group $\Gamma = \mathbf{O}(2)$ to the general case of a compact Lie group $\Gamma \subset \mathbf{O}(n)$ with Γ^0 abelian. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant map with an ω-limit set A containing a point with trivial isotropy. As before we quotient out the continuous symmetries Γ^0 passing to the orbit space $X = \mathbb{R}^n/\Gamma^0$. Hence we reduce to a H-equivariant map $g : X \to X$ where

$H = \Gamma/\Gamma^0$ is a finite group. The map g has an ω-limit set $\Lambda = \omega(x_0)$ containing points of trivial isotropy.

It will be convenient to use abelian notation in Γ^0 from the outset. Write $H = \{\rho_1, \ldots, \rho_\ell\}$ where $\rho_1 = 1$. The action of ρ_j on Γ^0 by conjugation defines an automorphism of the torus. This automorphism can be represented by an $r \times r$ matrix B_j with integer entries where $r = \dim \Gamma^0$. We note that B_1 is the identity matrix.

Choose $z \in \Lambda$ with $\Sigma_z = \mathbf{1}$. Let U_1 and V_1 be open sets with $z \in V_1$, $\overline{V_1} \subset U_1$. Provided U_1 is small enough, $\rho_j U_1 \cap U_1 = \emptyset$ for $j = 2, \ldots, \ell$. Set $U_j = \rho_j U_1$, $V_j = \rho_j V_1$ and $U = U_1 \cup \cdots \cup U_\ell$.

Next choose a C^∞ bump function $b : U_1 \to [0, 1]$ with support in U_1 and such that $b|_{V_1} \equiv 1$. Then we can define a smooth bounded matrix-valued map B on X with support in U by setting $B(\rho_j x) = b(x)B_j$ for $x \in U_1$ and $j = 1, \ldots \ell$. (When $\Gamma = \mathbf{O}(2)$, we have $H = \mathbb{Z}_2$, $B_1 = 1$, $B_2 = -1$.)

Define the sequence $\{A_n\}$ of $r \times r$ matrices

$$A_n = B(g^n(x_0)),\ n \geq 0.$$

Let $\{S_n\}$ be the sequence of partial sums corresponding to the sequence $\{A_n\}$. We say that A (and Λ) is *ruinous* if b can be chosen so that the sequence $\{\|S_n^{-1}\|^{-1}\}$ is unbounded. From now on, things are much the same as in §4.3. We define the subset $Y \subset \Lambda$ just as before and we say that A is *completely ruinous* if $Y = \Lambda$.

Theorem 4.13 *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group with Γ^0 abelian, and let $A = \omega(x_0)$ be an ω-limit set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that A contains a point of trivial isotropy and that A is completely ruinous. Define*

$$Z = \{\phi \in \mathcal{Z}_k : \Gamma^0 \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

Proof The proof is almost identical to that of Theorem 4.12 up to the point where we reduce to a expanding sequence on the torus Γ^0, so we just sketch the details. The notation differs from what we used previously, in part because we now use additive notation in Γ^0.

The bump function B lifts to a smooth matrix-valued map on $\mathbb{R}^n$. We can choose a neighborhood N of the identity in Γ^0 small enough so that for each $\gamma \in N$ the almost constant cocycle $\phi(x) = B(x)(\gamma)$ is well-defined. By construction, ϕ satisfies the equivariance condition (3.1) and lies in $\mathcal{Z}_\infty$.

Let f_γ denote the corresponding cocycle extension of f. Then $f_\gamma^n(x_0) = S_n(\gamma)f^n(x_0)$. Suppose that $y \in A$ and let $\{n_k\}$ be an increasing sequence of integers with $f^{n_k}(x_0) \to y$. Then $f_\gamma^{n_k}(x_0) = S_k(\gamma)f^{n_k}(x_0)$ where $S_k(\gamma) = S_{n_k}(\gamma)$. Since A is completely ruinous, we can assume that the corresponding subsequence $\{\|S_k^{-1}\|^{-1}\}$ is unbounded. In the usual way, the theorem follows from Theorem 2.6. ∎

5 ω-limit sets in fixed-point subspaces

Throughout §4 we worked with ω-limit sets that contain points with trivial isotropy. This condition is violated if an ω-limit set $A = \omega(x_0)$ lies in the fixed-point subspace $\mathrm{Fix}(T)$ of a nontrivial subgroup $T \subset \Gamma$. In this section, we generalize our results to include the case when $A \subset \mathrm{Fix}(T)$.

Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group acting on $\mathbb{R}^n$. We do not assume that Γ^0 is abelian. Define the subgroup consisting of those elements of Γ that fix A pointwise:

$$T_A = \{\gamma \in \Gamma : \gamma x = x \text{ for all } x \in A\}.$$

Then $A \subset \mathrm{Fix}(T_A)$. An elementary computation [26] shows that T_A is a normal subgroup of Σ_A. Thus $T_A \subset \Sigma_A \subset N(T_A)$.

One subtlety that we must deal with is the fact that T_A is not necessarily preserved under cocycle extensions. Observe that $\gamma A \subset \gamma\mathrm{Fix}(T_A) = \mathrm{Fix}(\gamma T_A \gamma^{-1})$ for all $\gamma \in \Gamma$. It is clear when $x_0 \notin A$ that for each $\gamma \in \Gamma^0$ we can choose a cocycle ϕ so that $A_\phi = \gamma A$. At best, only the conjugacy class of T_A is preserved, and questions about Σ_A make sense only in terms of conjugacy classes of Γ.

However, it is not even true that the conjugacy class of T_A is preserved under cocycle extensions. Indeed, in the continuous category the results of Field *et al.* [14] yield the following result.

Theorem 5.1 ([14]) *Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group and f a Γ- equivariant homeomorphism. Suppose that $A = \omega(x_0)$ where x_0 has trivial isotropy and does not lie on a relative periodic orbit. Define*

$$Z = \{\phi \in \mathcal{Z}_0 : T_{A_\phi} = \mathbf{1} \text{ and } \Gamma^0 \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual subset of $\mathcal{Z}_0$.

We have already pointed out that this result gives the 'wrong answer' when A is a relative periodic orbit. Similarly, caution is required when $T_A \neq \mathbf{1}$. Indeed, under reasonable hypotheses T_A is stable (up to conjugacy in Γ) under smooth cocycle perturbations. This is formalized in the following definition.

Definition 5.2 An ω-limit set A is *rigid* if T_{A_ϕ} is conjugate to T_A in Γ for all $\phi \in \mathcal{Z}_1$.

Remark 5.3 An important class of rigid ω-limit sets is provided by sets that are Γ-hyperbolic (where for each $x \in A$ there is a hyperbolic splitting of those directions in $T_x\mathbb{R}^n$ that are transverse to $\Gamma^0 x$). Suppose that A is Γ-hyperbolic and set $A' = \bigcup_{\gamma\in\Gamma} \gamma A$. As for the special case of normally hyperbolic relative periodic orbits, there is an 'asymptotic phase' property, so that if $\omega(x) \subset A'$ then $\omega(x)$ consists of a single trajectory in A'.

The assumption of Γ-hyperbolicity is unnecessarily strong. Actually, we require only that the group orbit of fixed-point subspaces $\bigcup_{\gamma\in\Gamma} \gamma\mathrm{Fix}(T)$ (or some compact submanifold that contains A') is normally hyperbolic, see [19]. For then a trajectory asymptotic to A' is asymptotic to a subset of $\gamma\mathrm{Fix}(T)$ for a single element $\gamma \in \Gamma$.

It is not clear that the rigidity assumption is valid in the absence of normal hyperbolicity. The resolution of this difficulty is of some importance for applications [27]. For related issues concerning ω-limit sets in fixed-point subspaces, see [1, 2].

If H_1 and H_2 are subgroups of Γ we write $H_1 < H_2$ if $H_1 \subset \gamma H_2 \gamma^{-1}$ for some $\gamma \in \Gamma$. As an immediate consequence of the definition of rigidity, we have that

$$T_A < \Sigma_{A_\phi} < N(T_A) \text{ for all } \phi \in \mathcal{Z}_1. \tag{5.1}$$

For example, suppose that $\Gamma = \mathbf{O}(2)$ and A is a rigid ω-limit set with $T_A = \mathbb{D}_k$ for some $k \geq 1$. The normalizer of $\mathbb{D}_k$ in $\mathbf{O}(2)$ is $\mathbb{D}_{2k}$, and we conclude that up to conjugacy either $\Sigma_A = \mathbb{D}_k$ or $\Sigma_A = \mathbb{D}_{2k}$. This is in contrast to Theorem 5.1 which states that in the C^0 category, generically $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{O}(2)$.

In the remainder of this section, we generalize the results of this paper to take into account the continuous symmetries in $N(T_A)$, or, more precisely, in $N(T_A)/T_A$. Define $\Gamma_A = N(T_A)/T_A$ and $S_A = \Sigma_A/T_A$. Observe that f restricts to a Γ_A-equivariant map $f_A : \mathrm{Fix}(T_A) \to \mathrm{Fix}(T_A)$ and that condition (5.1) reduces to the condition that $S_{A_\phi} < \Gamma_A$.

The ω-limit set A is an invariant subset of $\mathrm{Fix}(T_A)$ for the restricted map f_A. We make the simplifying (but noncrucial) assumption that A is topologically transitive. This assumption ensures that A is an ω-limit set for f_A. We say that $A \subset \mathbb{R}^n$ is *completely ruinous* if $A \subset \mathrm{Fix}(T_A)$ is completely ruinous for the Γ_A-equivariant map f_A.

Theorem 5.4 *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group and let $A = \omega(x_0)$ be a rigid topologically transitive set for the Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$.*

Suppose that ${\Gamma_A}^0$ is abelian and that A contains a point with isotropy T_A and is completely ruinous. Let $k \geq 1$ be a positive integer or $k = \infty$ and define

$$Z = \{\phi \in \mathcal{Z}_k : {\Gamma_A}^0 \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

Proof Choose $x_1 \in A$ so that $\omega(x_1) = A$. Let $W = \bigcup_{\gamma \in {\Gamma_A}^0} \gamma A$. Then for any cocycle $\phi \in \mathcal{Z}_k$, there is a $\gamma \in \Gamma^0$ so that $\gamma x_1 \in A_\phi$. It follows that the ω-limit set of γx_1 under f_ϕ is contained in A_ϕ. Thus it is sufficient to prove that for a residual and prevalent subset of cocycles ϕ, the ω-limit set of x_1 under f_ϕ is equal to W. In particular, without loss of generality, we may assume that $x_0 = x_1 \in A$.

The rigidity assumption implies that $A_\phi \subset W$ for any $\phi \in \mathcal{Z}_1$. Thus it is sufficient to construct a probe consisting of almost constant cocycles with the property that $W \subset A_\phi$. Restrict to $\mathrm{Fix}(T_A)$ and consider almost constant cocycles ϕ' supported in a Γ_A-invariant neighborhood U of z. As usual, these cocycles are in one-to-one correspondence with a small neighborhood of the identity in Γ_A. It follows that $W \subset A_{\phi'}$ for almost every ϕ'.

Since $\Sigma_z = T_A$ we can choose open neighborhoods U small enough so that $U \cap \gamma U = \emptyset$ for all $\gamma \in \Gamma - N(T_A)$. In particular, there is no difficulty with hidden symmetries [17, 18] and each cocycle ϕ' lifts to a smooth Γ-cocycle on $\mathbb{R}^n$. We have shown that these cocycles form a probe for the prevalence of Z and hence Z is prevalent and residual as required. ∎

Example Let $\Gamma = \mathbf{O}(2)$ and suppose that A is a rigid topologically transitive set. We assume that T_A is conjugate to $\mathbb{Z}_k$ for some $k \geq 1$, otherwise Γ_A is finite. When $T_A = \mathbb{Z}_k$, $\Gamma_A \cong \mathbf{O}(2)$. If A is a relative periodic orbit, then typically $S_A = \mathbf{SO}(2)$ or $S_A = \mathbb{D}_1$. In other words $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbb{D}_k$. More generally, if A is topologically transitive then $\Sigma_A = \mathbf{SO}(2)$ or Σ_A contains $\mathbb{D}_k$. By the theorem, if A is completely ruinous then $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{O}(2)$.

Acknowledgments We are grateful to Pete Ashwin, Michael Dellnitz, Mike Field and Matt Nicol for helpful conversations. IM acknowledges the hospitality of the Institut Non-Linéaire de Nice where part of this work was carried out.

References

[1] J. C. Alexander, I. Kan, J. A. Yorke and Z. You. Riddled basins, *Int. J. of Bif. and Chaos* **2** (1992) 795-813.

[2] P. Ashwin, J. Buescu and I. N. Stewart. Bubbling of attractors and synchronisation of oscillators, *Phys. Lett. A* **193** (1994) 126-139.

[3] P. Ashwin, P. Chossat and I. N. Stewart. Transitivity of orbits of maps symmetric under compact Lie groups, *Chaos, Solitons and Fractals* **4** (1994) 621-634.

[4] P. Ashwin and I. Melbourne. Symmetry groups of attractors, *Arch. Rational Mech. Anal.* **126** (1994) 59-78.

[5] D. Berend and V. Bergelson. Ergodic and mixing sequences of transformations, *Ergod. Th. Dynam. Sys* **4** (1984) 353-366.

[6] M. I. Brin. Topological transitivity of one class of dynamic systems and flows of frames on manifolds of negative curvature, *Functional Anal.* **9** (1975) 8-16.

[7] M. I. Brin, J. Feldman and A. Katok. Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents, *Annals of Math.* **113** (1981) 159-179.

[8] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*, Grad. Texts in Math. **98**, Springer, New York 1985.

[9] J. P. R. Christensen. On sets of Haar measure zero in abelian Polish groups, *Israel J. Math.* **13** (1972) 255-260.

[10] M. Dellnitz, M. Golubitsky and I. Melbourne. Mechanisms of symmetry creation, in *Bifurcation and Symmetry* (E. Allgower, K. Böhmer, and M. Golubitsky, eds.) ISNM **104**, Birkhäuser, Basel 1992, 99-109.

[11] M. Dellnitz and I. Melbourne. A note on the shadowing lemma and symmetric periodic points, *Nonlinearity* **8** (1995) 1067-1075.

[12] L. H. Eliasson. Ergodic skew-systems on $T^d \times \mathbf{SO}(3,\mathbb{R})$, preprint, Forschungsinstitut für Mathematik ETH Zürich, 1991.

[13] M. J. Field. Local structure for equivariant dynamics, in *Singularity Theory and its Applications* Part II, (M. Roberts, I. Stewart, eds.), Lecture Notes in Math. **1463**, Springer, Berlin, 1991, 142-166.

[14] M. J. Field, M. Golubitsky and M. Nicol. A note on symmetries of invariant sets with compact group actions, in *Equadiff* **8** Tatra Mountains Math. Publ. 4 (1994) 93-104.

[15] M. J. Field, I. Melbourne and M. Nicol. Symmetric attractors for diffeomorphisms and flows. *Proc. London. Math. Soc.* **72** (1996) 657-696.

[16] M. J. Field and W. Parry. Stable ergodicity of skew extensions by compact Lie groups. Preprint, 1997.

[17] M. Golubitsky, J. E. Marsden and D. G. Schaeffer. Bifurcation problems with hidden symmetries, in: *Partial Differential Equations and Dynamical Systems* (W. E. Fitzgibbon III, ed.) Research Notes in Math. **101**, Pitman, San Francisco 1984, 181-210.

[18] M. Golubitsky, I. N. Stewart and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, vol. 2, Appl. Math. Sci. **69**, Springer, New York 1988.

[19] M. W. Hirsch, C. C. Pugh and M. Shub. *Invariant Manifolds*, Lecture Notes in Math. **583**, Springer, New York 1977.

[20] B. R. Hunt, T. Sauer and J. A. Yorke. Prevalence, a translation-invariant 'almost every' on infinite-dimensional spaces, *Bull. Amer. Math. Soc.* **27** (1992) 217-238.

[21] B. R. Hunt, T. Sauer and J. A. Yorke. Prevalence: an addendum, *Bull. Amer. Math. Soc.* **28** (1993) 306-307.

[22] R. Jones and W. Parry. Compact abelian group extensions of dynamical systems II, *Compositio Math.* **25** (1972) 135-147.

[23] M. Krupa. Bifurcations of relative equilibria, *SIAM J. Appl. Math.* **21** (1990) 1453-1486.

[24] G. W. Mackey. Ergodic theory and virtual groups, *Math. Ann.* **166** (1966) 187-207.

[25] R. Mañé. *Ergodic Theory and Differentiable Dynamics*, Springer, New York 1987.

[26] I. Melbourne. Generalizations of a result on symmetry groups of attractors, in *Pattern Formation: Symmetry Methods and Applications* (J. Chadam and W. Langford, eds.) Fields Institute Communications **5**, AMS, 1996.

[27] I. Melbourne. Instantaneous symmetry and symmetry on average in the Couette-Taylor and Faraday experiments, in *Dynamics, Bifurcations, Symmetry* (P. Chossat, ed.) Kluwer, The Netherlands, 1994, 241-257.

[28] I. Melbourne, M. Dellnitz and M. Golubitsky. The structure of symmetric attractors, *Arch. Rational Mech. Anal.* **123** (1993) 75-98.

[29] J. Mycielski. Unsolved problems on the prevalence of ergodicity, instability and algebraic independence, *Ulam Quarterly* **1** (1992) 30-37.

[30] M. G. Nerurkar. On the construction of smooth ergodic skew-products, *Ergod. Th. Dynam. Sys.* **8** (1988) 311-326.

[31] R. J. Zimmer. Random walks on compact groups and the existence of cocycles, *Israel J. Math.* **26** (1977) 84-90.