On the Validity of the 0-1 Test for Chaos

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18 December, 2008

Abstract

In this paper, we present a theoretical justification of the 0–1 test for chaos. In particular, we show that with probability one, the test yields 0 for periodic and quasiperiodic dynamics, and 1 for sufficiently chaotic dynamics.

1 Introduction

In [5], we introduced a new method of detecting chaos in deterministic dynamical system in the form of a binary test. The method applies directly to the time series data and does not require phase space reconstruction. As explained in [5], with probability one the test gives the output $K = 0$ for quasiperiodic dynamics and $K = 1$ for sufficiently chaotic dynamics.

In [6], we proposed a simplified version of the test that is more effective for systems with a moderate amount of noise. The effectiveness of the new method was demonstrated for higher-dimensional systems in [6] and for experimental data [3].

The main aim of this paper is to put the simplified version of the test on a rigorous footing, going far beyond the results indicated in [5] for the original test. In addition, our analysis of the test leads to a significant improvement which was used in our paper [7] detailing the implementation of the test.

We first recall the simplified form of the test proposed in [6]. Let $f : X \to X$ be a map with invariant ergodic probability measure μ. Let $v : X \to \mathbb{R}$ be a scalar square-integrable observable. Choose $c \in (0, 2\pi)$, $x \in X$, and define

$$p_c(n) = \sum_{j=0}^{n-1} e^{ijc} v(f^j x). \tag{1.1}$$

Next, define the mean-square displacement

$$M_c(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} |p_c(j+n) - p_c(j)|^2. \tag{1.2}$$

Finally, let

$$K_c = \lim_{N\to\infty} \frac{\log M_c(n)}{\log n}. \tag{1.3}$$

The claim in [6], substantiated in this paper, is that typically (i) the limit K_c exists, (ii) $K_c \in \{0,1\}$, and (iii) $K_c = 0$ signifies regular dynamics while $K_c = 1$ signifies chaotic dynamics.

Remark 1.1 (a) The definition of $p_c(n)$ in (1.1) is slightly different from in [3, 6, 7] where $p_c(n) = \sum_{j=0}^{n-1} \cos(jc)v(f^j x)$. In the current paper it is natural to simplify analytic calculations rather than numerical computations, but apart from that the methods are equivalent.

(b) For fixed c, it follows from the ergodic theorem that the limit $M_c(n)$ in (1.2) exists for almost every initial condition x and the limit is independent of x. The common limit is

$$M_c(n) = \int_X |p_c(n)|^2 d\mu = \|p_c(n)\|_2^2.$$

To see this, compute that $p_c(j+n) - p_c(j) = e^{ijc} p_c(n) \circ f^j$, and so $M_c(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} |p_c(n)|^2 \circ f^j$ which converges to the space average $\int_X |p_c(n)|^2 d\mu$ almost everywhere.

(c) Strictly speaking, the limit K_c in (1.3) need not be well-defined. Of course, $K_c^+ = \limsup_{N\to\infty} \log M_c(n)/\log n$ is well-defined, and it follows from Proposition 1.4 that $K_c^+ \in [0,2]$ for all c. (In the case of periodic dynamics, $K_c = 2$ for isolated values of c.)

Example 1.2 Consider the logistic map $f : [0,1] \to [0,1]$ given by $f(x) = ax(1-x)$ for $0 \le a \le 4$. This family of maps is particularly well-understood [12, 1]: we can decompose the parameter interval according to $[0,4] = \mathcal{P} \cup \mathcal{C} \cup \mathcal{N}$ where $\mathcal{N}$ has Lebesgue measure zero and the asymptotic dynamics consists of a periodic attractor (of period $q \ge 1$) for $a \in \mathcal{P}$ and a strongly chaotic attractor consisting of $q \ge 1$ disjoint intervals for $a \in \mathcal{C}$ (satisfying the Collet-Eckman condition).

We obtain the following result:

Proposition 1.3 *Let $v : [0,1] \to \mathbb{R}$ be Hölder.*

(a) If $a \in \mathcal{P}$, then $K_c = 0$ for all $c \neq 2\pi j/q$.

(b) If $a \in \mathcal{C}$, then $K_c = 1$ for all $c \neq 2\pi j/q$ unless v is infinitely degenerate[1].

Hence, the test succeeds with probability one for logistic map dynamics.

Part (a) holds for general periodic dynamics (and all continuous observables). In Section 2, we prove that the test yields $K_c = 0$, for almost all c, for quasiperiodic dynamics, provided we make smoothness assumptions on v. This justifies our claim that $K_c = 0$ for regular dynamics.

The chaotic case is discussed extensively in Section 3. In particular we obtain $K_c = 1$ under various assumptions:

(i) Positivity of power spectra

(ii) Exponential decay of autocorrelations

(iii) Summable decay of autocorrelations plus hyperbolicity

(In fact, (ii) and (iii) are sufficient conditions for (i).)

In many situations, including the logistic map with $\mu \in \mathcal{C}$, it is necessary to consider f^q instead of f, and autocorrelations decay only up to a finite cycle (of length q). As shown in Section 3, criteria (ii) and (iii) generalise to this situation.

Summable decay without hyperbolicity assumptions Without making hyperbolicity assumptions, we have no definitive results when autocorrelations decay subexponentially. However, there is some partial information discussed in Section 4. If the autocorrelation function is summable, then the power spectrum $S(c)$ exists for all $c \in (0, 2\pi)$ by the Wiener-Khintchine Theorem [9], implying that

$$M_c(n) = S(c)n + o(n).$$

Under slightly stronger assumptions on the decay rate

$$M_c(n) = S(c)n + O(1).$$

In the former case, $K_c^+ = \limsup_{n\to\infty} \log M_c(n)/\log n \in [0, 1]$. In the latter case, K_c exists and takes the value 0 or 1 depending on where $S(c) = 0$ or $S(c) > 0$ (but see Remark 3.7).

Again, we obtain similar results if autocorrelations are summable up to a finite cycle.

[1]Lying in a closed subspace of infinite codimension in the space of Hölder functions

Improved diagnostic in the test for chaos The $o(n)$ and $O(1)$ terms above are nonuniform in c but in Section 4 we show that the source of nonuniformity is easily dealt with. Define

$$D_c(n) = M_c(n) - (Ev)^2 \frac{1-\cos nc}{1-\cos c}.$$

Here $Ev = \int_X v\, d\mu$ denotes expectation with respect to μ. Under the above conditions we obtain $D_c(n) = S(c)n + o(n)$ (hence $K_c^+ = \limsup_{n\to\infty} \log M_c(n)/\log n \in [0,1]$) for summable autocorrelation functions and $D_c(n) = S(c)n + O(1)$ (hence K_c takes the values either 0 or 1) under slightly stronger conditions on the decay of the autocorrelation function as before, but the $o(n)$ and $O(1)$ terms are now uniform in c (see Section 4). In [7], we proposed using $D_c(n)$ instead of $M_c(n)$ in the numerical implementation of the 0–1 test, and demonstrated the improved performance of the test.

Nonsummable decay The summability condition in the Wiener-Khintchine Theorem can be weakened considerably. For example, if autocorrelations decay at a square summable rate (including k^{-d} for any $d > \frac{1}{2}$), then the power spectrum exists almost everywhere and so $M_c(n) = S(c)n + o(n)$ for almost every c. (In this generality there is no uniformity in the error term for $D_c(n.)$ This and related results is discussed in Section 5.

Correlation method Our emphasis in this paper is on understanding the properties of the limit K_c as defined in (1.3). However, in [7], we proposed computing K_c as the correlation of the mean-square displacement $M_c(n)$ (or $D_c(n)$) with n. The advantages of this approach were demonstrated in [7]. In Section 6, we verify that the theoretical value of K_c remains 0 for regular dynamics and 1 for chaotic dynamics.

The paper concludes with a discussion section (Section 7). We end the introduction by proving that $K_c^+ \in [0,2]$ as claimed in Remark 1.1(c).

Proposition 1.4 *Let $K_c^+ = \limsup_{n\to\infty} \log M_c(n)/\log n$. If v is not identically zero, then $K^+(c) \in [0,2]$ for all c.*

Proof By definition, $\|p_c(n)\|_2 \le n\|v\|_2$ so that $0 \le M_c(n) \le n^2\|v\|_2^2$. Hence $K_c^+ \le 2$.

To prove the lower bound, we use the fact that $\|v\|_2 > 0$. It suffices to show that $\limsup_{n\to\infty} M_c(n) > 0$ for each fixed c. Observe that $p_c(n+1) = e^{inc} v \circ f^n + p_c(n)$ so that

$$\|p_c(n+1)\|_2 \ge \|e^{inc} v \circ f^n\|_2 - \|p_c(n)\|_2 = \|v\|_2 - \|p_c(n)\|_2.$$

Hence $0 < \|v\|_2 \le \|p_c(n)\|_2 + \|p_c(n+1)\|_2$. It follows that $\|p_c(n)\|_2 \not\to 0$, and so $\limsup_{n\to\infty} M_c(n) > 0$ as required. ∎

2 The case of regular dynamics

Part (a) of Proposition 1.3 is a simple direct calculation. If $f : X \to X$ is a map with a periodic orbit of period q and $v : X \to \mathbb{R}$ is continuous, then we obtain $K_c = 0$ for all $c \neq 2\pi j/q$. (For isolated resonant values $c = 2\pi j/q$ a simple argument using the Fourier series for v shows that typically $p_c(n)$ will grow linearly implying $K_c = 2$.) In the case of quasiperiodic dynamics, we require additional smoothness assumptions on the observable v. The test then succeeds with probability one.

Theorem 2.1 *Suppose that $X = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ and that $f : X \to X$ is given by $f(x) = x + \omega \bmod 1$. If $v : X \to \mathbb{R}$ is C^r with $r > m$, then $K_c = 0$ for almost every $c \in [0, 2\pi]$.*

Proof Recall that $X = \mathbb{T}^m$. Write $v : X \to \mathbb{R}$ as a m-dimensional Fourier series $v(x) = \sum_{\ell \in \mathbb{Z}^m} v_\ell e^{i\ell\cdot x}$ where $v_{-\ell} = \bar{v}_\ell$. Then

$$\begin{aligned} M_c(n) &= \int_X |\sum_{j=0}^{n-1} e^{ijc} v \circ f^j|^2 \, dx = \sum_{p,q=0}^{n-1} e^{i(p-q)c} \int_X v \circ f^p \, v \circ f^q \, dx \\ &= \sum_{k=-(n-1)}^{n-1} (n - |k|) e^{ikc} \int_X v \circ f^{|k|} \, v \, dx = s_1 + \cdots + s_n, \end{aligned} \tag{2.1}$$

where

$$s_m = \sum_{j=-(m-1)}^{m-1} e^{ijc} \int_X v \circ f^{|j|} \, v \, dx.$$

We show that $M_c(n)$ is bounded (as a function of n) for almost all c. Compute (formally) that

$$\begin{aligned} s_n &= \sum_{j=-(n-1)}^{n-1} e^{ijc} \sum_\ell |v_\ell|^2 e^{ij(\ell\cdot\omega)} \\ &= \sum_\ell |v_\ell|^2 (e^{i(c+\ell\cdot\omega)} - 1)^{-1} (e^{in(c+\ell\cdot\omega)} - e^{-i(n-1)(c+\ell\cdot\omega)}). \end{aligned} \tag{2.2}$$

Hence

$$s_1 + \cdots + s_n = \sum_\ell |v_\ell|^2 \frac{1 - \cos n(c + \ell\cdot\omega)}{1 - \cos(c + \ell\cdot\omega)}. \tag{2.3}$$

It remains to show that the series (2.2), (2.3) converge. We may ignore the $\ell = 0$ term in these series (these terms are obviously bounded in n). The smoothness assumption on v implies that $|v_\ell| = O(|\ell|^{-r})$. Let $\epsilon > 0$. For almost every $c \in (0, 2\pi)$ there is a constant $d_0 > 0$ such that

$$\operatorname{dist}|c + \ell \cdot \omega, 2\pi\mathbb{Z}| \geq d_0 |\ell|^{-(m+\epsilon)}, \tag{2.4}$$

for all $\ell \in \mathbb{Z}^m - \{0\}$ (cf. [15]). Hence $|e^{i(c+\ell\cdot\omega)} - 1| \geq d_1|\ell|^{-(m+\epsilon)}$ and so

$$\begin{aligned}\sum_{\ell \in \mathbb{Z}^m - \{0\}} |v_\ell|^2 |e^{i(c+\ell\cdot\omega)} - 1|^{-1} &\leq \sum_{k=1}^{\infty} \sum_{|\ell|=k} k^{-2r} d_1^{-1} k^{m+\epsilon} \leq C \sum_{k=1}^{\infty} k^{m-1} k^{-2r} k^{m+\epsilon} \\ &= C \sum_{k=1}^{\infty} k^{-(1+2(r-m-\epsilon/2))} < \infty,\end{aligned}$$

provided we choose $\epsilon > 0$ so small that $r > m + \epsilon/2$. This shows that (2.2) converges and is bounded independent of n, and similarly for (2.3). ∎

Remark 2.2 The extra smoothness of v is required to circumvent the small divisor problems associated with quasiperiodic dynamics. We also require a Diophantine condition on c, satisfied by almost every $c \in [0, 2\pi]$. However, there is no restriction on ω.

3 The case of chaotic dynamics

It is our intention to show that $K_c = 1$ for all almost all c (and reasonable observables v) for sufficiently chaotic dynamical systems. We proceed along three distinct but related avenues, all of which extend Example 1.2 of the logistic map: (i) positivity of power spectra; (ii) decay of autocorrelation functions; (iii) hyperbolicity of the dynamical system.

Recall that for a square-integrable observable $v : X \to \mathbb{R}$ the *power spectrum* $S : [0, 2\pi] \to [0, \infty)$ is defined (assuming it exists) to be the square of the Fourier amplitudes of $v \circ f^j$ per unit time[2], and is given by

$$S(c) = \lim_{n\to\infty} \frac{1}{n} \int_X |\sum_{j=0}^{n-1} e^{ijc} v \circ f^j|^2 \, d\mu = \lim_{n\to\infty} \frac{1}{n} M_c(n).$$

In other words, $M_c(n) = S(c)n + o(n)$. The following result is immediate:

Proposition 3.1 *Let $c \in [0, 2\pi]$. Suppose that $S(c)$ is well-defined and strictly positive. Then $K_c = 1$.* ∎

[2] Often $e^{ij\omega}$ is replaced by $e^{2\pi ij\omega/n}$ in the literature, but this is just a rescaling of the domain.

In particular, if the power spectrum is well-defined and positive almost everywhere, then we obtain $K_c = 1$ with probability one.

Next, we consider the *autocorrelation function* $\rho : \mathbb{N} \to \mathbb{R}$ given by

$$\rho(k) = \int_X v \circ f^k \, v \, d\mu - \Big(\int_X v \, d\mu \Big)^2 .$$

This is well-defined for all L^2 observables v.

If $\rho(k)$ is summable (i.e. $\sum_{k=0}^{\infty} \rho(k) < \infty$), then it follows from the Wiener-Khintchine theorem [9] that for $c \in (0, 2\pi)$,

$$S(c) = \sum_{k=-\infty}^{\infty} e^{ikc} \rho(|k|).$$

Note that the right-hand-side defines a continuous function on $[0, 2\pi]$.

Proposition 3.2 *Suppose that $v : X \to \mathbb{R}$ lies in $L^2(X)$ and v is not constant (almost everywhere). If the autocorrelation function $\rho(k)$ decays exponentially*[3]*, then $K_c = 1$ except for at most finitely many choices of $c \in [0, 2\pi]$.*

Proof Since $\rho(k)$ decays exponentially, $g(c) = \sum_{k \in -\infty}^{\infty} e^{ikc} \rho(|k|)$ is analytic on $[0, 2\pi]$. Since v is not constant, $\rho(0) = \int_X v^2 \, d\mu - (\int_X v \, d\mu)^2 = \int_X (v - \int_X v)^2 \, d\mu > 0$, and hence g is not the zero function. By analyticity, $S(c) = g(c) > 0$ except for at most finitely many values of c and hence $K_c = 1$ except for these values of c. ∎

Decay of autocorrelations up to a finite cycle Recall that $f : X \to X$ is *mixing* if $\lim_{k \to \infty} \rho(k) \to 0$ for every L^2 observable $v : X \to \mathbb{R}$. The system is *mixing up to a finite cycle* (of length $q \geq 1$) if $X = X_1 \cup \cdots \cup X_q$ where $f(X_j) \subset X_{j+1}$ (computing indices $\bmod q$) and $f^q : X_j \to X_j$ is mixing (with respect to $\mu_q = q\mu | X_j$) for each $j = 1, \ldots, q$.

If $q \geq 2$, then exponential decay of autocorrelations holds only for degenerate observables. The natural property to require is exponential decay for f^q. Given an L^2 observable $v : X \to \mathbb{R}$, define for $j = 1, \ldots, q$ and $m = 0, \ldots, q-1$,

$$\rho_{v \circ f^m, v, X_j}(kq) = \int_{X_j} v \circ f^m \circ f^{kq} \, v \, d\mu_j - \int_{X_j} v \circ f^m \, d\mu_j \int_{X_j} v \, d\mu_j .$$

Definition 3.3 The autocorrelations of v are *summable up to a q cycle* if for each $j = 1, \ldots, q$ and $m = 0, \ldots, q-1$, the series $\sum_{k=0}^{\infty} |\rho_{v \circ f^m, v, X_j}(kq)|$ is convergent.

The autocorrelations of v *decay exponentially up to a q cycle* if $\rho_{v \circ f^m, v, X_j}(kq)$ decays exponentially as $k \to \infty$ for each $j = 1, \ldots, q$ and $m = 0, \ldots, q-1$.

[3]There exist constants $C \geq 1$, $\tau \in (0, 1)$ such that $|\rho(k)| \leq C\tau^k$

Theorem 3.4 *If the autocorrelations of v are summable up to a q cycle, then*

$$S(c) = \sum_{r=-\infty}^{\infty} e^{irc} g_r, \quad \textit{for all } c \neq 2\pi j/q,$$

where writing $r = kq + m$ with $k \in \mathbb{Z}$ and $m \in \{0, 1, \ldots, q-1\}$,

$$g_r = g_{kq+m} = \sum_{j=1}^{q} \rho_{v\circ f^m, v, X_j}(kq).$$

Proof Define

$$v_{q,c} = \sum_{\ell=0}^{q-1} e^{i\ell c} v \circ f^\ell, \quad \rho_{q,c}(k) = \frac{1}{q} \sum_{j=1}^{q} \Big(\int_{X_j} v_{q,c} \circ f^{kq} \bar{v}_{q,c} \, d\mu_j - \Big| \int_{X_j} v_{q,c} \, d\mu_j \Big|^2 \Big).$$

By [14, Theorem A.2],

$$S(c) = \sum_{k=-\infty}^{\infty} e^{ikqc} \rho_{q,c}(k),$$

for $c \neq 2\pi j/q$. Compute that

$$\begin{aligned}
\rho_{q,c}(k) &= \frac{1}{q} \sum_{j=1}^{q} \Big(\int_{X_j} v_{q,c} \circ f^{kq} \, \bar{v}_{q,c} \, d\mu_j - \Big| \int_{X_j} v_{q,c} \, d\mu_j \Big|^2 \Big) \\
&= \frac{1}{q} \sum_{j=1}^{q} \sum_{\ell,\ell'=0}^{q-1} e^{i(\ell-\ell')c} \Big(\int_{X_j} v \circ f^\ell \circ f^{kq} \, v \circ f^{\ell'} \, d\mu_j - \int_{X_j} v \circ f^\ell \, d\mu_j \int_{X_j} v \circ f^{\ell'} \, d\mu_j \Big) \\
&= \frac{1}{q} \sum_{\ell,\ell'=0}^{q-1} e^{i(\ell-\ell')c} \sum_{j=1}^{q} \Big(\int_{X_{j+\ell'}} v \circ f^{\ell-\ell'} \circ f^{kq} \, v \, d\mu_{j+\ell'} - \int_{X_{j+\ell'}} v \circ f^{\ell-\ell'} \, d\mu_{j+\ell'} \int_{X_{j+\ell'}} v \, d\mu_{j+\ell'} \Big) \\
&= \frac{1}{q} \sum_{\ell,\ell'=0}^{q-1} e^{i(\ell-\ell')c} \sum_{j=1}^{q} \Big(\int_{X_j} v \circ f^{\ell-\ell'} \circ f^{kq} \, v \, d\mu_j - \int_{X_j} v \circ f^{\ell-\ell'} \, d\mu_j \int_{X_j} v \, d\mu_j \Big) \\
&= \frac{1}{q} \sum_{j=1}^{q} \sum_{s=-(q-1)}^{q-1} (q - |s|) e^{isc} \Big(\int_{X_j} v \circ f^s \circ f^{kq} \, v \, d\mu_j - \int_{X_j} v \circ f^s \, d\mu_j \int_{X_j} v \, d\mu_j \Big) \\
&= \frac{1}{q} \sum_{j=1}^{q} \sum_{s=-(q-1)}^{q-1} (q - |s|) e^{isc} \rho_{v\circ f^s, v, X_j}(kq).
\end{aligned}$$

For $r = kq+m$ we obtain at most two nonzero contributions to g_r, namely $s = m$ in $\rho_{q,c}(k)$ and $s = -(q-m)$ in $\rho_{q,c}(k+1)$. Hence

$$g_r = \frac{1}{q}\sum_{j=1}^{q}\Big((q-m)\rho_{v\circ f^m,v,X_j}(kq) + m\rho_{v\circ f^{-(q-m)},v,X_j}(kq+q)\Big) = \sum_{j=1}^{q}\rho_{v\circ f^m,v,X_j}(kq)$$

as required. ∎

Corollary 3.5 *If the autocorrelations of v are summable up to a q cycle, then $S(c)$ exists and is continuous except for removable singularities at $c = 2\pi j/q$.*

If the autocorrelations of v decay exponentially up to a q cycle, then $S(c)$ is analytic except for removable singularities at $c = 2\pi j/q$. If moreover $v|X_j$ is not constant (almost everywhere) for at least one j, then $K_c = 1$ except for at most finitely many values of c.

Proof The statements about continuity and analyticity are immediate from Theorem 3.4. In particular, if there is exponential decay up to a q cycle, then the function $g(c) = \sum_{r=-\infty}^{\infty} e^{irc}g_r$ is analytic and hence nonzero except at finitely many points provided $g_0 \neq 0$. If on the other hand, $g_0 = 0$, then

$$0 = g_0 = \sum_{j=1}^{q}\rho_{v,v,X_j}(0) = \sum_{j=1}^{q}\int_{X_j} v^2\,d\mu_j - \Big(\int_{X_j} v\,d\mu_j\Big)^2 = \sum_{j=1}^{q}\mathrm{var}(v|X_j),$$

so $\mathrm{var}(v|X_j) = 0$, and hence $v|X_j$ is constant, for each j. ∎

Remark 3.6 Definition 3.3 and Theorem 3.4 are significant improvements on the corresponding material in [14, Appendix].

Remark 3.7 We have seen that exponential decay of autocorrelations (up to a q cycle) guarantees that $K_c = 1$ with probability one. Surprisingly, it seems nontrivial to weaken the exponential decay hypothesis. The proof of Proposition 3.2 relies crucially on analyticity of the power spectrum. Even if we assume sufficiently rapid decay that $g(c) = \sum_{k\in-\infty}^{\infty} e^{ikc}\rho(|k|)$ is C^∞, then we face the difficulty that the only restriction on the zero set of a C^∞ function is that it is a closed set.

Suppose that summable decay of correlations holds for a large class of observables $\mathcal{B}$ with the property that there is an interval $I \subset (0, 2\pi)$ such that the Fourier series $g(c)$ is identically zero on I for all $v \in \mathcal{B}$. The proof of Proposition 3.2 shows that for every nonconstant observable $v \in \mathcal{B}$, there is an interval $J \subset (0, 2\pi)$ on which $g(c) > 0$ on J. For such examples, where the power spectrum vanishes on an interval I and is typically positive on an interval J, the 0–1 test is inconclusive: we obtain

$K_c = 1$ with positive probability for nonconstant observables in $\mathcal{B}$, but for all $v \in \mathcal{B}$ there is a positive probability that $K_c \in [0, 1)$ or that K_c does not even exist. This situation seems highly pathological, but we do not see how to rule this out.

Hyperbolicity We can overcome the unsatisfactory aspects of Remark 3.7 by assuming some hyperbolicity. In the Collet-Eckmann case ($a \in \mathcal{C}$) for the logistic map, it is known that Hölder observables enjoy exponential decay of correlations up to a finite cycle, so we can apply Theorem 3.4. Alternatively, [14] shows that the power spectrum is bounded away from zero for all $c \neq 2\pi j/q$, so we can apply Proposition 3.1. These comments apply to all maps in the following classes:

- Uniformly expanding maps; Uniformly hyperbolic (Axiom A) diffeomorphisms.
- Nonuniformly expanding/hyperbolic systems in the sense of Young [17], modelled by a Young tower with exponential tails. These enjoy exponential decay of correlations (up to a finite cycle) for Hölder observables. This covers large classes of dynamical systems, including Hénon-like maps, logistic maps and more generally multimodal maps satisfying Collet-Eckmann conditions, and one-dimensional maps with Lorenz-like singularities [2].

Young [18] weakens the decay rates assumed for tower models for nonuniformly expanding/hyperbolic systems. Hölder observables now have subexponential decay of correlations (up to a finite cycle). Provided the decay rate is summable, the argument of [14] still applies, and $S(c)$ is bounded away from zero (except for infinitely degenerate observables).

Example 3.8 A prototypical family of examples is the Pomeau-Manneville intermittency maps $f : [0, 1] \to [0, 1]$ given by $f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha); & 0 \le x \le \frac{1}{2} \\ 2x - 1; & \frac{1}{2} \le x \le 1 \end{cases}$ where $\alpha \in [0, 1)$ is a parameter [16, 11]. When $\alpha = 0$ this is the doubling map with exponential decay of correlations for Hölder observables, so Proposition 3.2 applies. For $\alpha > 0$, let $\beta = \frac{1}{\alpha} - 1$. Then decay of correlations for Hölder observables is at the rate $n^{-\beta}$ [8]. By [14], the power spectrum is bounded below in the summable case ($\beta > 1$, equivalently $\alpha < \frac{1}{2}$) and so $K_c = 1$ for all $c \in (0, 2\pi)$.

Remark 3.9 Numerical experiments for intermittency maps indicate that (i) $K_c = 1$ and (ii) the power spectrum exists and is bounded below, even in the nonsummable case $\beta \le 1$, equivalently $\alpha \in [\frac{1}{2}, 1)$. It remains an interesting problem to prove these statements. By Corollary 5.3, we are at least assured that the power spectrum exists (and hence $K_c = 1$ with positive probability) for $\beta > \frac{1}{2}$ ($\alpha < \frac{2}{3}$).

4 Summable decay without hyperbolicity

If the autocorrelation function is absolutely summable (i.e. $\sum_{k=1}^{\infty} |\rho(k)| < \infty$), then the power spectrum $S(c)$ exists and is continuous for all $c \in (0, 2\pi)$. Indeed, $S(c) = \sum_{k=-\infty}^{\infty} e^{ikc}\rho(|k|)$ on $(0, 2\pi)$ by the Wiener-Khintchine Theorem [9]. It follows that $M_c(n) = S(c)n + o(n)$, and hence that $K_c = 1$ with positive probability.

In this section, we discuss the error term $o(n)$ in more detail. As mentioned in the introduction, this leads to the improved diagnostic $D_c(n)$ for chaos used in [7].

We begin with a formal calculation to express the mean square displacement as follows:

Proposition 4.1 $M_c(n) = \sum_{k=-n}^{n}(n - |k|)e^{ikc}\rho(|k|) + (Ev)^2 \frac{1-\cos nc}{1-\cos c}$.

Proof First, note that

$$\begin{aligned}
M_c(n) = \int_X |p_c(n)|^2\, d\mu &= \sum_{p,q=0}^{n-1} e^{i(p-q)c} \int_X v \circ f^p\, v \circ f^q\, d\mu \\
&= \sum_{p,q=0}^{n-1} e^{i(p-q)c} \int_X v \circ f^{|p-q|}\, v\, d\mu \\
&= \sum_{p,q=0}^{n-1} e^{i(p-q)c} (\rho(|p-q|) + (Ev)^2) \\
&= \sum_{k=-n}^{n} (n - |k|)e^{ikc}\rho(|k|) + (Ev)^2 \sum_{p=0}^{n-1} e^{ipc} \sum_{q=0}^{n-1} e^{-iqc}
\end{aligned}$$

Finally

$$\sum_{p=0}^{n-1} e^{ipc} \sum_{q=0}^{n-1} e^{-iqc} = \frac{1 - e^{inc}}{1 - e^{ic}}\, \frac{1 - e^{-inc}}{1 - e^{-ic}} = \frac{1 - \cos nc}{1 - \cos c}. \qquad \blacksquare$$

The second term in the expression for $M_c(n)$ is bounded in n for fixed c, but is nonuniform in c. Since the term is explicit, it is convenient to remove it. (As demonstrated in [7], this is also greatly advantageous for the numerical implementation of the test.) Hence we define

$$D_c(n) = M_c(n) - (Ev)^2(1 - \cos nc)/(1 - \cos c)\ .$$

Then $M_c(n) = D_c(n) + O(1)$, so it suffices to work with $D_c(n)$ from now on. By Proposition 4.1,

$$D_c(n) = \sum_{k=-n}^{n} (n - |k|)e^{ikc}\rho(|k|). \tag{4.1}$$

Theorem 4.2 *Suppose that $\rho(k)$ is absolutely summable (i.e. $\sum_{k=1}^{\infty} |\rho(k)| < \infty$). Then for all $c \in (0, 2\pi)$,*

$$D_c(n) = S(c)n + e(c,n),$$

where

$$|e(c,n)| \leq 2n \sum_{k=n+1}^{\infty} |\rho(k)| + 2\sum_{k=1}^{n} k|\rho(k)| = o(n).$$

In particular $D_c(n) = S(c)n + o(n)$ uniformly in c.

Proof Write

$$D_c(n) = \sum_{k=-n}^{n} (n - |k|)e^{ikc}\rho(|k|) = S(c)n + e(c,n),$$

where

$$e(c,n) = -n \sum_{k=n+1}^{\infty} (e^{ikc} + e^{-ikc})\rho(k) - \sum_{k=1}^{n} k(e^{ikc} + e^{-ikc})\rho(k).$$

It remains to show that $\sum_{k=1}^{n} k|\rho(k)| = o(n)$. Let $s_n = \sum_{k=1}^{n} |\rho(k)|$ and let $L = \lim s_n$. Define the Cesàro average $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k$, so $L = \lim \sigma_n$. Then $\frac{1}{n}\sum_{k=1}^{n} k|\rho(k)| = \frac{n+1}{n}s_n - \sigma_n \to L - L = 0$. ∎

Corollary 4.3 *If $|\rho(k)| \leq Ck^{-d}$ for $k \geq 1$, where $1 < d < 2$, then in Theorem 4.2,*

$$|e(c,n)| \leq 2C\frac{1}{(d-1)(2-d)}n^{2-d}.$$

Proof The first term of $e(c,n)$ in Theorem 4.2 is dominated by

$$2Cn \sum_{k=n+1}^{\infty} k^{-d} \leq 2Cn \int_{n}^{\infty} x^{-d}\,dx \leq 2Cn^{-(d-2)}/(d-1).$$

The second term is dominated by

$$\begin{aligned} 2C\sum_{k=1}^{n} k^{1-d} &\leq 2C\Big(1 + \int_{1}^{n} x^{1-d}\,dx\Big) = 2Cn^{-(d-2)}/(2-d) - 2C(d-1)/(2-d) \\ &\leq 2Cn^{-(d-2)}/(2-d). \end{aligned}$$

Combining these terms gives the result. ∎

Under stronger assumptions on the decay rate of the autocorrelation function $\rho(k)$, improved estimates for the $o(n)$ term are available.

Theorem 4.4 *Suppose that $\sum_{k=1}^{\infty} k|\rho(k)| < \infty$. Then*

$$D_c(n) = S(c)n + S_0(c) + e(c,n),$$

where $S_0(c) = -\sum_{k=-\infty}^{\infty} e^{ick}|k|\rho(|k|)$ is continuous on $[0,2\pi]$, $S(c)$ is C^1 on $[0,2\pi]$, and

$$|e(c,n)| \le 2\sum_{k=n+1}^{\infty}(k-n)|\rho(k)| = o(1).$$

In particular, $D_c(n) = S(c)n + O(1)$ uniformly in c. (Hence $K_c \in \{0,1\}$ for all $c \in (0,2\pi)$.)

Proof Compute that

$$D_c(n) = \sum_{k=-n}^{n}(n-|k|)e^{ikc}\rho(|k|) = S(c)\,n + S_0(c) + e(c,n),$$

where $e(c,n) = \sum_{k=n+1}^{\infty}(k-n)(e^{ikc}+e^{-ikc})\rho(k)$. ∎

Corollary 4.5 *If $|\rho(k)| \le Ck^{-d}$ for $k \ge 1$, where $d > 2$, then in Theorem 4.4*

$$|e(c,n)| \le 2C\Big\{\frac{1}{(d-1)(d-2)} + \frac{1}{n}\Big\}\frac{1}{n^{d-2}}.$$

Proof By Theorem 4.4,

$$\begin{aligned}
|e(c,n)| &\le 2C\sum_{k=n+1}^{\infty}(k-n)k^{-d} = 2C\sum_{k=n+1}^{\infty}k^{1-d} - 2Cn\sum_{k=n+1}^{\infty}k^{-d} \\
&\le 2C\int_n^{\infty}x^{1-d}\,dx - 2Cn\Big(\int_n^{\infty}x^{-d}\,dx - n^{-d}\Big) \\
&= 2C\Big\{\Big(\frac{1}{d-2} - \frac{1}{d-1}\Big)\frac{1}{n^{d-2}} + \frac{1}{n^{d-1}}\Big\}.
\end{aligned}$$

∎

5 Nonsummable decay of correlations

In this section, we reformulate the 0–1 test in terms of Cesàro averages, and give surprisingly weak sufficient conditions under which $S(c) = \lim_{n\to\infty}\frac{1}{n}\int_X |p_c(n)|^2 d\mu$ exists (for typical values of c).

Let $a_k \in \mathbb{C}$ be a sequence with partial sums $s_n = \sum_{k=-n}^{n} a_k$ and set $\sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$. Recall that the sequence a_k is *Cesàro summable* if $\lim_{n\to\infty} \sigma_n$ exists. If $\lim_{n\to\infty} s_n \to L$ then $\lim_{n\to\infty} \sigma_n = L$ (the converse is not true).

Defining s_n and σ_n as above with $a_k = e^{ikc}\rho(|k|)$, we obtain

$$\sigma_n = \frac{1}{n}\sum_{k=-n}^{n}(n-|k|)\rho(|k|)e^{ikc} = \frac{1}{n}D_c(n),$$

the last equality following from (4.1). Since $S(c) = \lim_{n\to\infty}\frac{1}{n}D_c(n)$, we have proved the following result.

Lemma 5.1 *Let $c \in (0, 2\pi)$. Suppose that the sequence $a_k = e^{ikc}\rho(k)$ is Cesàro summable with limit $L(c)$. Then $S(c) = L(c)$. In particular, if $L(c) > 0$, then $K_c = 1$.* ∎

By Fejér's theorem [10], a special case is provided when $\rho(|k|)$ are Fourier coefficients of an integrable function.

Theorem 5.2 *Suppose that $\rho(|k|)$ are the Fourier coefficients of an L^1 function $g : [0, 2\pi] \to \mathbb{R}^{\geq 0}$. Then the sequence $e^{ikc}\rho(k)$ is Cesàro summable to $g(c)$ almost everywhere. In particular, Lemma 5.1 holds for almost every $c \in (0, 2\pi)$, with $L(c) = g(c)$.*

If g is continuous, then the convergence is uniform in c (and holds for every $c \in (0, 2\pi)$). In particular, $D_c(n) = S(c)n + o(n)$ uniformly in c.

Proof We have written $\sigma_n = \frac{1}{n}D_c(n) = \sum_{k=-n}^{n}(1 - \frac{|k|}{n})e^{ikc}\rho(|k|)$. This is $\sigma_{n-1}(g, c)$ in [10, p.12 (2.9)]).

If g is continuous, then by Fejér's Theorem ([10, Theorem 2.12]), $\sigma_n \to g$ uniformly, and so $D_c(n) = n\sigma_n = ng(c) + o(n)$ uniformly in c.

For general $g \in L^1$, it follows from the discussion in [10, pp. 19-20] that $\sigma_n \to g$ almost everywhere, so that $D_c(n) = n\sigma_n(g, c) = ng(c) + o(n)$ for almost every c. ∎

Corollary 5.3 *Suppose that $\sum_{k\geq 1}\rho(k)^2 < \infty$. Then the first statement of Theorem 5.2 applies with $g(c) = \sum_{k=-\infty}^{\infty} e^{ikc}\rho(|k|)$.*

In particular, $S(c)$ exists almost everywhere, and $D_c(n) = S(c)n + o(n)$.

Proof Since $\sum_{k\geq 1}\rho(k)^2 < \infty$, the function $g(c) = \sum_{k=-\infty}^{\infty} e^{ikc}\rho(|k|)$ lies in L^2 and the Fourier coefficients of g are precisely $\rho(|k|)$. Hence, we can apply the first statement of Theorem 5.2. ∎

6 Correlation method

In [7], we proposed computing K_c as the correlation of the mean-square displacement $M_c(n)$ (or $D_c(n)$) with n, rather than computing the limit of $\log M(n)/\log n$ as in (1.3). In this section we verify that the theoretical value of K_c remains 0 for regular dynamics and 1 for chaotic dynamics.

Given vectors x, y of length n, we define

$$\begin{aligned}
&\operatorname{cov}(x,y) = \frac{1}{n}\sum_{j=1}^{n}(x(j)-\bar{x})(y(j)-\bar{y}), \quad \text{where} \quad \bar{x} = \frac{1}{n}\sum_{j=1}^{n} x(j) \ , \\
&\operatorname{var}(x) = \operatorname{cov}(x,x) \ .
\end{aligned}$$

Form the vectors $\xi = (1, 2, \ldots, n)$ and $\Delta = (D_c(1), D_c(2), \ldots, D_c(n))$. (In particular, $\operatorname{var}(\xi) = \frac{1}{12}(n^2-1)$.) Define the correlation coefficient

$$K_c = \lim_{n\to\infty} \operatorname{corr}(\xi,\Delta) = \frac{\operatorname{cov}(\xi,\Delta)}{\sqrt{\operatorname{var}(\xi)\operatorname{var}(\Delta)}} \in [-1,1] \ .$$

6.1 Quasiperiodic case

For quasiperiodic dynamics, we have the following analogue of Theorem 2.1. However, we require stronger regularity for the observable v, and a Diophantine condition on the frequency ω (in addition to the condition on c).

Theorem 6.1 *Suppose that $X = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ and that $f : X \to X$ is given by $f(x) = x + \omega \bmod 1$. Let $v : X \to \mathbb{R}$ be a nonvanishing C^r observable with $r > 3m/2$. If K_c is computed using the correlation method, then for almost every $\omega \in [0, 2\pi]$ we obtain $K_c = 0$ for almost every $c \in [0, 2\pi]$.*

Proof The proof of Theorem 2.1 shows that $M_c(n)$ (equivalently $D_c(n)$) is bounded for almost every c provided $r > m$. We show for $r > 3m/2$ that $\operatorname{var}(\Delta) = O(1)$ and $\operatorname{cov}(\xi, \Delta) = O(1)$ for almost every ω and c. Moreover, we use the fact that v is nonvanishing to show that $\operatorname{var}(\Delta) = a + O(1/n)$, where $a > 0$. It then follows that

$$\operatorname{corr}(\xi,\Delta) = \frac{O(1)}{\sqrt{\frac{1}{12}(n^2-1)}\sqrt{a+O(1/n)}} = O(1/n),$$

as required.

The starting point is the calculations (2.1) and (2.3) which gives

$$M_c(j) = \sum_{\ell} |v_\ell|^2 \frac{1-\cos j(c+\ell\cdot\omega)}{1-\cos(c+\ell\cdot\omega)} = \sum_{\ell} w_\ell(\cos j\theta_\ell - 1),$$

where $\theta_\ell = c + \ell \cdot \omega$, $w_\ell = -|v_\ell|^2(1-\cos\theta_\ell)^{-1}$. Hence

$$D_c(j) = \sum_\ell w_\ell(\cos j\theta_\ell - 1) - C(1-\cos jc),$$

where $C = (Ev)^2/(1-\cos c)$. Adding a constant (independent of j) to $D_c(j)$ does not alter the value of $\mathrm{corr}(\xi,\Delta)$ so we may replace $D_c(j)$ by

$$\widehat{D}_c(j) = \sum_\ell w_\ell \cos j\theta_\ell + C\cos jc,$$

when proving that $\mathrm{cov}(\xi,\Delta) = O(1)$ and $\mathrm{var}(\Delta) = an + O(1)$ with $a > 0$. Hence it suffices to show that for almost every ω and c there exists $a > 0$ such that

$$\sum_{j=1}^n \widehat{D}_c(j) = O(1), \quad \sum_{j=1}^n j\widehat{D}_c(j) = O(n), \quad \sum_{j=1}^n \widehat{D}_c(j)^2 = an + O(1).$$

Formally,

$$\sum_{j=1}^n \widehat{D}_c(j) = \sum_\ell w_\ell \sum_{j=1}^n \cos j\theta_\ell + C\sum_{j=1}^n \cos jc, \tag{6.1}$$

$$\sum_{j=1}^n j\widehat{D}_c(j) = \sum_\ell w_\ell \sum_{j=1}^n j\cos j\theta_\ell + C\sum_{j=1}^n j\cos jc, \tag{6.2}$$

$$\begin{aligned}\sum_{j=1}^n \widehat{D}_c(j)^2 &= \sum_{\ell,\ell'} w_\ell w_{\ell'} \sum_{j=1}^n \cos j\theta_\ell \cos j\theta_{\ell'} \\ &\quad + 2C\sum_\ell w_\ell \sum_{j=1}^n \cos jc\cos j\theta_\ell + C^2\sum_{j=1}^n \cos^2 jc. \end{aligned} \tag{6.3}$$

For $\varphi \in (0,2\pi)$, we have

$$\sum_{j=1}^n e^{ij\varphi} = e^{i\varphi}\frac{e^{in\varphi}-1}{e^{i\varphi}-1}, \qquad \sum_{j=1}^n je^{ij\varphi} = \frac{ne^{i(n+1)\varphi}}{e^{i\varphi}-1}.$$

In particular, $\sum_{j=1}^n \cos j\varphi = O(1)$ and $\sum_{j=1}^n j\cos j\varphi = O(n)$. By (6.1) and (6.2), formally we have $\sum_{j=1}^n \widehat{D}_c(j) = O(1)$ and $\sum_{j=1}^n j\widehat{D}_c(j) = O(n)$. Turning to (6.3),

$$\sum_{j=1}^n \cos^2 jc = \frac{1}{2}\sum_{j=1}^n (1+\cos 2jc) = \frac{1}{2}n + O(1),$$

for $c \neq \pi$ and formally

$$\sum_\ell w_\ell \sum_{j=1}^n \cos jc \cos j\theta_\ell = \frac{1}{2} \sum_\ell w_\ell \sum_{j=1}^n (\cos j(c+\theta_\ell) + \cos j(c-\theta_\ell)) = O(1),$$

while

$$\sum_{\ell,\ell'} w_\ell w_{\ell'} \sum_{j=1}^n \cos j\theta_\ell \cos j\theta_{\ell'} = \frac{1}{2} \sum_{\ell,\ell'} w_\ell w_{\ell'} \sum_{j=1}^n \{\cos j(\theta_\ell + \theta_{\ell'}) + \cos j(\theta_\ell - \theta_{\ell'})\}$$

$$= \frac{1}{2} \sum_\ell w_\ell^2 n + \frac{1}{2} \sum_\ell w_\ell^2 \sum_{j=1}^n \cos 2j\theta_\ell + \frac{1}{2} \sum_{\ell \neq \ell'} w_\ell w_{\ell'} \sum_{j=1}^n \{\cos j(\theta_\ell + \theta_{\ell'}) + \cos j(\theta_\ell - \theta_{\ell'})\}$$

$$= \frac{1}{2} \sum_\ell w_\ell^2 n + O(1).$$

Hence $\sum_{j=1}^n \widehat{D}_c(j)^2 = an + O(1)$, with $a = \frac{1}{2}(\sum_\ell w_\ell^2 + C^2)$. If v is nonvanishing, then v_ℓ, and hence w_ℓ, is nonzero for at least one ℓ so that $a > 0$.

It remains to justify the formal calculations. We give the details for the first term in (6.3) focusing on the most difficult expression

$$I = \sum_{\ell \neq \ell'} w_\ell w_{\ell'} \sum_{j=1}^n \{\cos j(\theta_\ell + \theta_{\ell'}) + \cos j(\theta_\ell - \theta_{\ell'})\}.$$

Let $\epsilon > 0$. We assume the Diophantine conditions (2.4) and

$$\operatorname{dist} |2c + \ell \cdot \omega, 2\pi\mathbb{Z}| \geq d_2 |\ell|^{-(m+\epsilon)}, \quad \operatorname{dist} |\ell \cdot \omega, 2\pi\mathbb{Z}| \geq d_2 |\ell|^{-(m+\epsilon)}, \tag{6.4}$$

which are satisfied by almost all c and ω for all nonzero ℓ. Proceeding as in the proof of Theorem 2.1,

$$|I| \leq C \sum_{\ell \neq \ell'} |v_\ell|^2 |v_{\ell'}|^2 (1-\cos\theta_\ell)^{-1} (1-\cos\theta_{\ell'})^{-1} (|1 - e^{i(2c+(\ell+\ell')\omega)})|^{-1} + |1 - e^{i(\ell-\ell')\omega}|^{-1})$$

$$\leq C' \sum_{\ell \neq \ell'} |\ell|^{-2r} |\ell'|^{-2r} |\ell|^{m+\epsilon} |\ell'|^{m+\epsilon} (|\ell + \ell'|^{m+\epsilon} + |\ell - \ell'|^{m+\epsilon})$$

$$\leq C'' \sum_{k_1,k_2=1}^\infty k_1^{m-1} k_2^{m-1} k_1^{-2r} k_2^{-2r} k_1^{m+\epsilon} k_2^{m+\epsilon} (k_1 + k_2)^{m+\epsilon}$$

$$\leq C''' \sum_{k_1,k_2=1}^\infty k_1^{m-1} k_2^{m-1} k_1^{-2r} k_2^{-2r} k_1^{m+\epsilon} k_2^{m+\epsilon} (k_1^{m+\epsilon} + k_2^{m+\epsilon})$$

$$= 2C''' \sum_{k_1=1}^\infty k_1^{-(1+2(r-3m/2-\epsilon))} \sum_{k_2=1}^\infty k_2^{-(1+2(r-m-\epsilon/2))} < \infty$$

provided we choose $\epsilon > 0$ so small that $r > 3m/2 + \epsilon$. ∎

6.2 Chaotic case

Recall that $K_c = 1$ in definition (1.3) if and only if $M_c(n) = an + o(n)$ where $a > 0$. Equivalently $D_c(n) = an + o(n)$ with $a > 0$. We show that this is a sufficient condition for $K_c = 1$ via the correlation method.

Theorem 6.2 *Let $c \in (0, 2\pi)$. Suppose that $D_c(n) = an + o(n)$ for some $a > 0$ and that K_c is computed using the correlation method. Then $K_c = 1$.*

Proof We claim that $\mathrm{cov}(\xi, \Delta) = \frac{1}{12}an^2 + o(n^2)$ and $\mathrm{var}(\Delta) = \frac{1}{12}a^2n^2 + o(n^2)$. The result is then immediate.

We verify the claim for $\mathrm{cov}(\xi, \Delta)$. The verification for $\mathrm{var}(\Delta)$ is similar. Write $D_c(n) = an + e(n)$ where $e(n) = o(n)$. Then

$$\begin{aligned}
\mathrm{cov}(\xi, \Delta) &= \frac{1}{n}\sum_{j=1}^{n} j(aj + e(j)) - \Big(\frac{1}{n}\sum_{j=1}^{n} j\Big)\Big(\frac{1}{n}\sum_{j=1}^{n}(aj + e(j))\Big) \\
&= \frac{1}{6}a(n+1)(2n+1) + \frac{1}{n}\sum_{j=1}^{n} je(j) - \frac{1}{2n}(n+1)\Big\{\frac{1}{2}an(n+1) + \sum_{j=1}^{n} e(j)\Big\} \\
&= \frac{1}{12}an^2 + O(n) + \frac{1}{n}\sum_{j=1}^{n} je(j) - \frac{1}{2n}(n+1)\sum_{j=1}^{n} e(j).
\end{aligned}$$

Hence it remains to show that $\sum_{j=1}^{n} je(j) = o(n^3)$ and $\sum_{j=1}^{n} e(j) = o(n^2)$.

Since $e(n) = o(n)$, there is a constant $C > 0$ such that $|e(n)| \le Cn$ for $n \ge 1$. Also, given $\epsilon > 0$, there exists $n_0 \ge 1$ such that $|e(n)| \le \epsilon n$ for all $n \ge n_0$. Choose n_1 such that $Cn_0^2/n_1^2 \le \epsilon$. Then for all $n \ge n_1$,

$$\begin{aligned}
|\frac{1}{n^2}\sum_{j=1}^{n} e(j)| \le \frac{1}{n^2}\Big(\sum_{j=1}^{n_0} |e(j)| + \sum_{j=n_0+1}^{n} |e(j)|\Big) &\le \frac{1}{n_1^2}Cn_0^2 + \frac{1}{n^2}\epsilon\sum_{j=1}^{n} j \\
&\le \epsilon + \epsilon = 2\epsilon,
\end{aligned}$$

so that $\sum_{j=1}^{n} e(j) = o(n^2)$. Similarly $\sum_{j=1}^{n} je(j) = o(n^3)$. ∎

7 Discussion

We have addressed the issue of validity of the 0–1 test as presented in [6, 7]. The original 0–1 test [5] included an equation for a phase variable $\theta(n+1) = \theta(n) + c +$

$v \circ f^n$ rather than a constant "frequency" c. The extra equation for the phase driven by the observable made available theorems from ergodic theory on skew product systems [4, 13, 15]. These theorems rely on the fact that for typical observables v the augmented system with the phase variable is mixing.

In the modified version proposed in [6], the phase variable is not mixing, and the results from ergodic theory are not applicable anymore. Nevertheless, the modified test is more effective, particularly for systems with noise [6]. In this paper, we have verified that the modified test can be rigorously justified. Moreover, our theoretical results are stronger than the corresponding results mentioned in [5] for the original test.

Our main results in this paper are that $K_c = 0$ with probability one in the case of periodic or quasiperiodic dynamics, and that $K_c = 1$ with probability one for "sufficiently chaotic" dynamics. The latter includes dynamical systems with hyperbolicity, (including weakly mixing systems such as Pomeau-Manneville intermittency maps). In particular, nonuniform hyperbolicity assumptions combined with summable autocorrelations for the observable suffice to obtain $K_c = 1$. In the absence of hyperbolicity, we still obtain $K_c = 1$ (with probability one) for observables with exponentially decaying autocorrelations. These results extend to systems that are mixing up to a cycle of finite length.

We also made explicit the connection with power spectra: the test yields $K_c = 1$ with probability one if and only if the power spectrum is well-defined and positive for almost all frequencies. The criteria above – exponential decay of autocorrelations or summable correlations plus hyperbolicity (up to a finite cycle) – are sufficient conditions for existence and positivity of the power spectrum.

There remains the question of whether typical smooth dynamical systems are either quasiperiodic or have power spectra that are defined and positive almost everywhere. This is required for a complete justification of the test for chaos. Unfortunately the current understanding of dynamical systems is inadequate to answer this question, but all numerical studies so far indicate this to be the case. We leave it as a challenge to the skeptical reader to concoct a robust smooth example where the test fails! On the positive side, we showed in this paper that under a mild assumption on autocorrelations, slightly stronger than summable but much weaker than exponential, we obtain either $K_c = 0$ or $K_c = 1$ for each choice of c, though without invoking hyperbolicity we cannot rule out the possibility that both $K_c = 0$ and $K_c = 1$ occur with positive probability.

Our investigations of the validity of the test for chaos enabled us to construct an improved version of our test. The modification which amounts to using $D_c(n)$ rather than the mean square displacement $M_c(n)$ was shown to significantly improve the test in [7]. In addition, we showed in [7] that K_c is better computed by correlating the mean square displacement with linear growth rather than computing the log-

log slope. In this paper, we have shown that our rigorous results apply also to the improved implementation of the test in [7].

Acknowledgements The research of GG was supported in part by the Australian Research Council. The research of IM was supported in part by EPSRC Grant EP/F031807/1.

IM acknowledges the hospitality of the University of Sydney where parts of this work was done, and is greatly indebted to the University of Houston for the use of e-mail.

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