Missing proofs for MA222 2015/2016

There were 4 theorems that were stated without proofs. In the case of Tychonoff’s theorem this was intentional. The remaining 3 proofs were omitted due to lack of time. Here are the omitted proofs (they are not examinable).

**Theorem 1** Let $X_i = \mathbb{R}$ (standard topology) and $X = \prod_{i \in A} X_i$ (product topology). Then $X$ is metrizable if and only if $A$ is countable.

**Proof** The case $A$ countable was covered in class (for general metrizable spaces $X_i$). Here we consider the uncountable case.

Define $A = \{ x \in X : x_i = 1 \text{ for all but finitely many } i \in A \}$. We show that $0 \in \overline{A}$ but that $0$ is not the limit of a sequence $x_n \in A$. This shows that $X$ cannot be metrizable.

First, we show that $0 \in \overline{A}$. Let $U = \prod_{i \in A} U_i$ be a basis element containing $0$. Then $U_i = \mathbb{R}$ except for $i = \alpha_1, \ldots, \alpha_k$ for some $k \geq 0$. Also $0 \in U_i$ for all $i \in A$.

Define $x_i = \begin{cases} 0 & i = \alpha_1, \ldots, \alpha_k \\ 1 & \text{otherwise} \end{cases}$. Then $x \in U \cap A$ so $U$ intersects $A$. Hence $0 \in \overline{A}$.

Next suppose that $x_n$ is a sequence in $A$. We show that $x_n \nrightarrow 0$ completing the proof. Write $x_n = (x_{ni})$. Define

$$A_n = \{ i \in A : x_{ni} \neq 1 \text{ for all but finitely many } i \in A \}, \quad A' = \bigcup_{n=1}^{\infty} A_n.$$  

Then $A_n$ is finite and so $A'$ is a countable subset of $A$. Since $A$ is uncountable, we can choose $\alpha \in A \setminus A'$. Then $x_{n\alpha} = 1$ for each $n$. Let $U = \prod_{i \in A} U_i$ where $U_i = \begin{cases} (-1,1) & i = \alpha \\ \mathbb{R} & \text{otherwise} \end{cases}$. Then $U$ is a neighbourhood of $0$ but $x_n \not\in U$ for all $n$. ■

**Theorem 2** Let $X_i, \ i \in A$ be a collection of connected spaces and let $X = \prod_{i \in A} X_i$ (product topology). Then $X$ is connected.

**Proof** Fix $b = (b_i) \in X$. For $\alpha_1, \ldots, \alpha_n \in A$, $n \geq 1$, define

$$X(\alpha_1, \ldots, \alpha_n) = \{ x \in X : x_i = b_i \text{ for } i = \alpha_1, \ldots, \alpha_n \}.$$  

Define the map $f : X_{\alpha_1} \times \cdots \times X_{\alpha_n} \rightarrow X(\alpha_1, \ldots, x_n)$, $f(x_{\alpha_1}, \ldots x_{\alpha_n}) = y$ where $y_i = \begin{cases} x_i & i = \alpha_1, \ldots, \alpha_n \\ b_i & \text{otherwise} \end{cases}$. It is easy to check that $f$ is a homeomorphism. This means
that $X(\alpha_1, \ldots, \alpha_n)$ is homeomorphic to a finite product of connected spaces and hence is connected. In addition $b \in X(\alpha_1, \ldots, \alpha_n)$ for all choices of $\alpha_1, \ldots, \alpha_n, n \geq 1$. Hence

$$Y = \bigcup_{\alpha_1, \ldots, \alpha_n \in \mathcal{A}, n \geq 1} X(\alpha_1, \ldots, \alpha_n)$$

is connected. It follows that $\bar{Y}$ is connected. We show that $\bar{Y} = X$.

Let $x \in X$ and let $U = \prod_{i \in \mathcal{A}} U_i$ be a neighbourhood of $x$. We show that $U$ intersects $Y$. Now there exists $\alpha_1, \ldots, \alpha_n \in \mathcal{A}, n \geq 1$, such that $U_i = X_i$ for all $i \neq \alpha_1, \ldots, \alpha_n$. Let $y_i = \begin{cases} x_i & i = \alpha_1, \ldots, \alpha_n \\ b_i & \text{otherwise} \end{cases}$. Then $y \in U \cap X(\alpha_1, \ldots, \alpha_n) \subset U \cap Y$.

**Theorem 3 (Arzelà-Ascoli Theorem)** Let $X$ be a compact metric space and let $S \subset C(X)$ be a subspace. Then $S$ is compact if and only if $S$ is closed, bounded and equicontinuous.

**Proof** “$\Rightarrow$” Done in class.

“$\Leftarrow$” Since $C(X)$ is complete and $S$ is closed, it is enough to show that $S$ is totally bounded.

Let $\epsilon > 0$. Since $S$ is equicontinuous, for any $x \in M$, there exists $\delta(x) > 0$ such that

$$y \in B(x, \delta(x)), f \in S \implies |f(x) - f(y)| < \epsilon/3. \quad (1)$$

Since $X$ is compact and $\{B(x, \delta(x)) : x \in X\}$ is an open cover, there exists $n \geq 1$ and $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^n B(x_i, \delta(x_i)). \quad (2)$$

Since $S$ is bounded, there exists $K > 0$ such that $\|f\| \leq K$ for all $f \in S$. Define the bounded set $Q \in \mathbb{Z}$ given by

$$Q = \{q \in \mathbb{Z} : -K - \epsilon < q\epsilon/3 < (q + 1)\epsilon/3 < K + \epsilon\}.$$

Let

$$Q^* = \{q = (q_1, \ldots, q_n) \in Q^n : \text{there exists } h \in S$$

$$\text{with } h(x_i) \in [q_i\epsilon/3, (q_i + 1)\epsilon/3], \text{ for } i = 1, \ldots, n\}. \quad (3)$$

For each $q \in Q^*$, choose $h_q \in S$ such that

$$h_q(x_i) \in [q_i\epsilon/3, (q_i + 1)\epsilon/3] \quad \text{for } i = 1, \ldots, n.$$
We claim that $S \subset \bigcup_{q \in Q^*} B(h_q, \epsilon)$. Since $|Q^*| < \infty$, this completes the proof.

It remains to verify the claim. Let $f \in S$. Since $\|f\| \leq K$, there exists $q \in Q^n$ such that $f(x_i) \in [q_i \epsilon/3, (q_i + 1)\epsilon/3]$ for $i = 1, \ldots, n$. In particular, $q \in Q^*$ (because of $f$) and $h_q(x_i) \in [q_i \epsilon/3, (q_i + 1)\epsilon/3]$ for $i = 1, \ldots, n$. It follows that

$$|f(x_i) - h_q(x_i)| \leq \epsilon/3 \quad \text{for } i = 1, \ldots, n. \quad (3)$$

Let $x \in X$. By (2), $x \in B(x_i, \delta(x_i))$ for some $i = 1, \ldots, n$. By (1) (twice) and (3),

$$|f(x) - h_q(x)| \leq |f(x) - f(x_i)| + |f(x_i) - h_q(x_i)| + |h_q(x_i) - h_q(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$ 

Since $x$ is arbitrary, $\|f - h_q\| < \epsilon$ and so $f \in B(h_q, \epsilon)$ verifying the claim. 

\[\square\]