How we solve Diophantine equations

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A Diophantine problem is the problem of finding integer or rational solutions to a given polynomial equation in one or several variables with rational coefficients.
Examples

Find \((x, y) \in \mathbb{Q}^2\) satisfying \(x^2 - 5y^2 = 3\).

Find \((x, y) \in \mathbb{Z}^2\) satisfying \(x^2 + y^2 = -3\).

Find \((x, y, z) \in \mathbb{Z}^3\) satisfying \(x^2 - 5y^2 = 3z^2\). This is a homogeneous equation of degree 2.

Given an integer \(n \geq 3\), find all \((x, y, z) \in \mathbb{Z}^2\) satisfying \(x^n + y^n = z^n\). This is the famous Fermat equation.
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Non-Examples

Solving Diophantine equations

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What is a Diophantine equation

The Hasse principle

Elliptic curves

Birch and Swinnerton-Dyer conjecture

Unique factorisation

Non-Examples

$n! = m(m+1)$ is not a Diophantine equation in the above sense, because of the factorial.

$x^2 + y^2 = z^2$ is a very interesting equation, but not polynomial in the variables, so not Diophantine.

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History

A 16th century edition of “Arithmetica” by Diophantus of Alexandria, translated into Latin:
We want to find rational solutions to $x^2 - 5y^2 = 3$ or, equivalently, integral solutions to $x^2 - 5y^2 = 3z^2$ with $z \neq 0$. 
Idea: Consider the equation \( x^2 - 5y^2 = 3z^2 \) modulo 3:

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$\Rightarrow z \equiv 0 \pmod{3}$

$\Rightarrow x^2 - 5y^2 \equiv 0 \pmod{27}$

$\Rightarrow \ldots$
Since $x$ and $y$ cannot be divisible by arbitrarily large powers of 3, we obtain a contradiction, so there are no integer solutions to $x^2 - 5y^2 = 3z^2$. 
This is the **method of infinite descent**, due to Pierre de Fermat.
Moral of the story: for an equation to have integer solutions, it must have solutions modulo $p^n$ for any prime number $p$ and any $n \in \mathbb{N}$. It must also have real solutions.
Theorem (H. Minkowski): A homogeneous equation of degree 2 has an integer solution \textit{if and only if} it has a real solution and solutions modulo all prime powers. In other words, the obvious necessary conditions are also sufficient.
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Moreover, a quadratic equation in two variables has either no rational solutions or infinitely many. Once we find one, we find them all:
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Famous example, due to Ernst Selmer:

$$3x^3 + 4y^3 + 5z^3 = 0$$

has a non-zero solution in the reals and non-zero solutions modulo all prime powers, but no integral solutions!
Equations of degree 3 differ from those of degree 2 in many other ways. E.g. an equation of the form $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Q}$, can have 0, or finitely many, or infinitely many solutions.
An equation of the form

\[ E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q} \]

describes an elliptic curve.
Addition law on elliptic curves

Given a point on the curve $E$, we cannot quite repeat the conic trick for finding a new point, but given two points, we can find a third one:
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**Theorem** (Mordell): Given any elliptic curve $E$, the group $E(\mathbb{Q})$ is finitely generated. Thus, it is isomorphic to $\Delta \oplus \mathbb{Z}^{r(E)}$, where $\Delta$ is a finite abelian group, and $r(E) \geq 0$.

The integer $r(E)$ is called the rank of $E$ and is a very mysterious invariant.
One important ingredient in the proof of Mordell’s theorem is Fermat’s technique of infinite descent. This technique has been vastly generalised.
Even though elliptic curves do not satisfy the Hasse principle, we can still try to count solutions modulo primes. Denote the number of solutions modulo $p$ by $N_E(p)$. It turns out that $N_E(p) = p + 1 - a_p$, where

$$|a_p| \leq 2\sqrt{p}.$$ 

So, $N_e(p) \sim p$ as $p \to \infty$. 


In the 1960s, Bryan Birch and Peter Swinnerton-Dyer computed

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Birch and Swinnerton-Dyer conjecture

This led them to conjecture that

\[ f_E(X) \sim c_E(\log X)^{r(E)}. \]

This is the naive form of the famous Birch and Swinnerton-Dyer conjecture. It is a very deep kind of local-global principle, of which the Hasse principle is the simplest example.
Suppose that we want to find integer solutions to

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**Idea:** Work in the slightly bigger ring\[ R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\}. \]
Factorise

\[ x^3 = y^2 + 2 = (y + \sqrt{-2})(y - \sqrt{-2}). \]
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**Step 1.** Show that the two factors \((y + \sqrt{-2})\) and \((y - \sqrt{-2})\) are coprime in the ring \(R = \mathbb{Z}[\sqrt{-2}]\).
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**Step 2.** Deduce that \((y + \sqrt{-2}) = u \cdot \alpha^3\) for a unit \( u \in R^\times\) and some \( \alpha = a + b\sqrt{-2} \in R\). But the only units in \( R \) are \( \pm 1 \) and they are both cubes, so can be incorporated into \( \alpha \).
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**Step 3.** Expand and equate coefficients to find the only solutions are \(b = 1, a = \pm 1\), which correspond to \(x = 3, y = \pm 5\).
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- We implicitly used in Step 2 that in $R$, any element can be factorised uniquely into irreducibles, just like in $\mathbb{Z}$. 
If we tried to do this for the equation

\[ y^2 = x^3 - 1, \]

working in the ring \( \mathbb{Z}[\sqrt{-1}] \), then we would have to be careful with the units, since there are the additional units \( \pm i \) (they are still all cubes, but in other circumstances they might not be). In fact, if \( d > 0 \) is square-free and congruent to 3 modulo 4, then \( \mathbb{Z}[\sqrt{d}] \) has infinitely many units!
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If we tried to do this for the equation

\[ y^2 = x^3 - 6, \]

then things would go completely wrong, since the ring \( \mathbb{Z}[\sqrt{-6}] \) does not have unique factorisation into irreducibles.
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**Open question:** Are there infinitely many real quadratic fields, whose ring of integers has unique factorisation?
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