FREE AND PROPERLY DISCONTINUOUS ACTIONS OF DISCRETE GROUPS ON HOMOTOPY CIRCLES

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Abstract. Let $G \times \Sigma(1) \to \Sigma(1)$ be a free, properly discontinuous and cellular action of a group G on a finite dimensional CW-complex $\Sigma(1)$ that has the homotopy type of the circle. We determine all virtually cyclic groups G that act on $\Sigma(1)$ together with the induced action $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$, and we classify the orbit spaces $\Sigma(1)/G$.

Then, we study the same questions for certain families of groups. First, we consider the family of groups with vcd ≤ 1 which includes semi-direct products $\mathbb{Z}_n \rtimes F$ and $F \rtimes \mathbb{Z}_n$ and amalgamated products of finite groups with bounded orders since these groups have vcd $= 1$. Next, we study locally cyclic groups consisting of subgroups of the rationals $\mathbb{Q}$ with vcd ≤ 2 and subgroups of the quotient $\mathbb{Q}/\mathbb{Z}$ with vcd $= \infty$. The results obtained depend upon the subfamily in question. In particular, for an action of any subgroup of $\mathbb{Q}/\mathbb{Z}$ there is only one orbit space up to homotopy and the induced action on $H^1(\Sigma(1), \mathbb{Z})$ is trivial.

Introduction. Finite groups with free and cellular actions on n-homotopy spheres $\Sigma(n)$ (a finitely dimensional CW-complex with the homotopy type of the n-sphere $\mathbb{S}^n$) have been fully classified, and one can find a classification in a table by Suzuki-Zassenhaus, see e.g., [1, Chapter IV, Theorem 6.15]. Further, the complete calculation of the number of homotopy types was obtained in a series of papers [11]-[16]. Finally, following [4, Proposition 10.2], for any action of a finite group

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G on an odd dimensional homotopy sphere $\Sigma(2n+1)$ the induced action $G \to \mathrm{Aut}(H^{2n+1}(\Sigma(2n+1), \mathbb{Z}))$ is trivial.

Actions of infinite discrete groups on $\Sigma(2n+1)$ and $H^{2n+1}(\Sigma(2n+1), \mathbb{Z})$ have been studied as well. For more about this subject, we refer the reader to the papers [2], [7], [25], [27] and [35]. For example, [2, Corollary 1.3] states that a discrete group G acts freely and properly discontinuously on $\mathbb{R}^m \times \mathbb{S}^n$ for some $m, n > 0$ if and only if G is a countable group with periodic cohomology (with any coefficients) after some n-step with $n \geq 0$.

Most of the results which appear in the above-mentioned papers apply to odd-dimensional homotopy spheres different from $\Sigma(1)$, and either the result is not true for a $\Sigma(1)$ or the techniques used do not apply, see e.g., [27]. The purpose of this paper is to study groups with free, properly discontinuous and cellular actions on a homotopy circle $\Sigma(1)$, noting that we need somewhat different methods than those used for $\Sigma(n)$ with $n > 1$. We address the following problems:

- *when a group G acts;*
- *what is the minimal dimension of $\Sigma(1)$ on which G acts;*
- *the description of the action $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$.*

We begin by exhibiting a one-to-one correspondence (Proposition 1.8) between equivalence classes of actions of a group G on $\Sigma(1)$ and equivalence classes of extensions $e \to \mathbb{Z} \to \pi \to G \to e$, where π is a group with finite cohomological dimension, $\mathrm{cd}\,\pi < \infty$, where the induced action $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is given by the above extension. Using this result, we then study our problems for several subfamilies of groups. Here are the main results of this work. Recall that an infinite virtually cyclic group is the middle term of a short exact sequence of the form

$$e \to \mathbb{Z} \to G \to F \to e,$$

where F is a finite group. If the extension is central we say that the group G is of type I, otherwise it is of type II. For virtually cyclic groups we have:

Corollary 2.6. *Let $G \times \Sigma(1) \to \Sigma(1)$ be an action, where G is a virtually cyclic group.*

(1) *If G is finite, then $G \cong \mathbb{Z}_n$ for some $n \geq 1$;*

(2) *if G is of type I, then $G \cong \mathbb{Z}_n \rtimes \mathbb{Z}$ for some $n \geq 1$;*

(3) *if G is of type II, then $G \cong \mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n}$ for some $n \geq 1$, and $\mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n} \cong \mathbb{Z} \rtimes_\theta \mathbb{Z}_{2n}$ with $\theta(1_{2n}) = -1$.*

Theorem 2.10. *Let G be a virtually cyclic group. Then the orbit space $\Sigma(1)/G$ of an action $G \times \Sigma(1) \to \Sigma(1)$ of a virtually cyclic group G on $\Sigma(1)$ has the following homotopy type:*

(1) *of a circle if the group G is finite;*

(2) *either of the torus or the Klein bottle if the group G is infinite.*

Then, we study possible actions of semi-direct products $\mathbb{Z}_n \rtimes F$ (Theorem 3.2) and $F \rtimes \mathbb{Z}_n$ (Theorem 3.9), and free products with an amalgamated subgroup (Theorem 3.11) on $\Sigma(1)$. We point out that Theorem 3.11 shows that the class of all groups acting on $\Sigma(1)$ is closed with respect to free products.

The family of locally-cyclic groups is divided into two subfamilies, where the first one consists of subgroups of the rationals $\mathbb{Q}$ and the second one subgroups of $\mathbb{Q}/\mathbb{Z}$.

Finally, such actions of those groups and corresponding orbit spaces are investigated. Given a set P of primes, write $\mathbb{Z}_P$ for the localization of $\mathbb{Z}$ with respect to the muliplicative system generated by P. Then, the following results are presented:

Theorem 3.17. *There are $2^{\aleph_0}$ distinct homotopy types of orbit spaces $\Sigma(1)/\mathbb{Z}_P$ with respect to actions of $\mathbb{Z}_P$ on $\Sigma(1)$ and any such action induces the trivial action on $H^1(\Sigma(1), \mathbb{Z})$.*

Theorem 3.19. *For any subgroup $\mathbb{A} < \mathbb{Q}/\mathbb{Z}$ there is an action of $\mathbb{A}$ on some $\Sigma(1)$ and exactly one homotopy type of the orbit spaces $\Sigma(1)/\mathbb{A}$ for all its possible actions on homotopy circles $\Sigma(1)$. Further, any such action determines the trivial action on $H^1(\Sigma(1), \mathbb{Z})$.*

The paper is divided into three sections. In Section 1, general facts on free, properly discontinuous and cellular actions of groups on $\Sigma(1)$ are presented. Also the questions mentioned there are shown in Proposition 1.8 to be equivalent to an algebraic problem in terms of extension of the group of integers $\mathbb{Z}$. Other results are stated in Propositions 1.3 and 1.7. The latter one shows that there is a one-to-one correspondence between equivalence classes of actions $G \times \Sigma(1) \to \Sigma(1)$ of a group G on homotopy circles $\Sigma(1)$ and equivalence classes of extensions $e \to \mathbb{Z} \to \pi \to G \to e$ with $\operatorname{cd} \pi < \infty$, the cohomological dimension of π.

Section 2 is devoted to classifications of all virtually cyclic groups G acting on $\Sigma(1)$ and homotopy types of corresponding orbit spaces $\Sigma(1)/G$. These are Corollary 2.6 and Theorem 2.10.

Section 3 takes up the study of other families of groups. For certain groups with vcd ≤ 1, the main results are stated in Theorems 3.2 and 3.9. Next, we present a family of groups which contains two subfamilies. One is the family of subgroups of

$\mathbb{Q}$, the additive groups of rationals which have cd ≤ 2. For this subfamily the main result is Theorem 3.17. The second subfamily is formed by subgroups of $\mathbb{Q}/\mathbb{Z}$. They have infinite virtual cohomological dimension and the main result on actions of those groups is Theorem 3.19.

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1. Actions on $\Sigma(1)$ and extensions.

A CW-complex $\Sigma(n)$ is said to be a *homotopy n-sphere*, if $\dim \Sigma(n) < \infty$ and $\Sigma(n)$ is homotopy equivalent to the n-sphere $\mathbb{S}^n$. For $n = 1$, a space $\Sigma(1)$ is simply called a *homotopy circle*.

From now on, we assume that any action $G \times \Sigma(n) \to \Sigma(n)$ is free, properly discontinuous and cellular.

Notice that $n \leq \dim \Sigma(n)$ and for an action $G \times \Sigma(n) \to \Sigma(n)$ there is a fibration

$$\Sigma(n) \longrightarrow \Sigma(n) \times_G \widetilde{K(G,1)} \longrightarrow K(G,1),$$

where $\widetilde{K(G,1)}$ is the universal covering of the Eilenberg-MacLane space $K(G,1)$ and $\Sigma(n) \times_G \widetilde{K(G,1)}$ is the Borel construction which is homotopy equivalent to the orbit space $\Sigma(n)/G$. Consequently, in view of the fibration above there are isomorphisms $\pi_k(\Sigma(n)) \cong \pi_k(\Sigma(n)/G)$ for $k > 1$ and $n \geq 1$, $\pi_1(\Sigma(n)/G) \cong G$ for $n > 1$, and there is an extension

$$e \to \mathbb{Z} \to \pi_1(\Sigma(1)/G) \to G \to e \tag{1.1}$$

of groups. Notice that the action of G on the automorphism group $\mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ induced by $G \times \Sigma(1) \to \Sigma(1)$ corresponds to the action $G \to \mathrm{Aut}(\mathbb{Z})$ given by the extension (1.1).

Lemma 1.1. *If $G \times \Sigma(1) \to \Sigma(1)$ is an action, then:*

(1) *the cohomology $H^m(\Sigma(1)/G, A) = 0$ for $m > \dim \Sigma(1)$ and any $\pi_1(\Sigma(1)/G)$-local system A;*

(2) *the group $\pi_1(\Sigma(1)/G)$ is torsion-free.*

Proof. The inequality $\dim(\Sigma(1)/G) \leq \dim \Sigma(1) < \infty$ implies (1).

Item (2) is a consequence of (1) and [4, Chapter VIII, Corollary 2.5].

□

Write $\operatorname{cd} G$ (resp. $\operatorname{gd} G$, $\operatorname{vcd} G$) for cohomological (resp. geometric, virtual cohomological) dimension of a group G [4, Chapter VIII]. Given a subgroup $H < G$, we have the orbit space $\widetilde{K(G,1)}/H = K(H,1)$. Consequently, we deduce $\operatorname{gd} H \leq \operatorname{gd} G$.

Certainly, $\operatorname{cd} G \leq \operatorname{gd} G$ for any group G and, in view of [10], it follows that: $\operatorname{cd} G = \operatorname{gd} G$ provided $\operatorname{cd} G \neq 2$ and $\operatorname{cd} G = 2$ implies $\operatorname{gd} G \leq 3$.

Remark 1.2. The Eilenberg-Ganea Conjecture [10] states that $\operatorname{cd} G = 2$ implies $\operatorname{gd} G = 2$.

Notice that for an action $G \times \Sigma(1) \to \Sigma(1)$, there is an extension (1.1) with $\operatorname{gd} \pi_1(\Sigma(1)/G) \leq \dim \Sigma(1)$. The converse also holds and it is a particular case of the following:

Proposition 1.3. *Given an extension*

$$e \to G' \to G \to G'' \to e$$

there is a CW-complex $\tilde{X}$ of the homotopy type of $K(G',1)$, with $\dim \tilde{X} \leq \operatorname{gd} G$, and an action

$$G'' \times \tilde{X} \longrightarrow \tilde{X}$$

such that the orbit space $\tilde{X}/G'' \simeq K(G,1)$.

In particular, for an extension $e \to \mathbb{Z} \to \pi \to G \to e$ with $\operatorname{gd} \pi < \infty$, there is such $\Sigma(1)$ that $\operatorname{gd} \pi \leq \dim \Sigma(1)$ and an action $G \times \Sigma(1) \to \Sigma(1)$ with $\Sigma(1)/G \simeq K(\pi,1)$.

Proof. Let $\widetilde{X} = \widetilde{K(G,1)}/G'$. Then the extension

$$e \to G' \to G \to G'' \to e$$

leads to an action

$$G'' \times \widetilde{X} \longrightarrow \widetilde{X}.$$

From the definition of $\widetilde{X}$ and the action of G'' on $\widetilde{X}$ we have $\widetilde{X}/G'' = \widetilde{K(G,1)}/G$, which in turn is homeomorphic to $K(G,1)$.

Now, consider an extension $e \to \mathbb{Z} \to \pi \to G \to e$ and an Eilenberg-MacLane space $K(\pi,1)$ with $\dim K(\pi,1) = \operatorname{gd} \pi$. Then as above, we derive an action

$$G \times \Sigma(1) \longrightarrow \Sigma(1)$$

with $\Sigma(1) = \widetilde{K(\pi,1)}/\mathbb{Z}$ and the proof is complete.

□

Remark 1.4. Any extension $e \to \mathbb{Z} \to \pi \to G \to e$ with $\operatorname{gd}\pi < \infty$ and the multiplications $\mathbb{Z} \to \mathbb{Z}$ by $n \geq 1$ produce a sequence of extensions $e \to \mathbb{Z} \to \pi \to G_n \to e$. Groups $G_n \cong \pi/n\mathbb{Z}$ with $n \geq 1$ are possibly distinct and act freely, and properly discontinuously on some $\Sigma(1)$ and all of them yield the same orbit space.

Now write $\mathbb{Z}_n$ for the cyclic group of order n.

Example 1.5. (1) Given a group G with $\operatorname{gd} G < \infty$, the obvious extension

$$e \to \mathbb{Z} \longrightarrow \mathbb{Z} \times G \longrightarrow \mathbb{Z}_n \times G \to e$$

yields, in view of Proposition 1.3, an action

$$(\mathbb{Z}_n \times G) \times \Sigma(1) \longrightarrow \Sigma(1)$$

with $\dim \Sigma(1) = \operatorname{gd} G + 1$.

(2) The Artin braid group B_n on n strands determines an epimorphism $B_n \to S_n \to e$, where S_n is the symmetric group on n letters. Its kernel gives the pure Artin braid group P_n on n strands. Hence, we have the extension

$$e \to P_n \longrightarrow B_n \longrightarrow S_n \to e.$$

Given a subgroup $G < S_n$, we follow [6], to denote by B_n^G the G-Artin braid group defined as the preimage of G in B_n by the projection map of the extension above. In view of [6, Proposition 2.4(1)], the center $\mathcal{Z}(B_n^G) \cong \mathbb{Z}$. Then, we get the extension

$$e \to \mathcal{Z}(B_n^G) \cong \mathbb{Z} \longrightarrow B_n^G \longrightarrow B_n^G/\mathcal{Z}(B_n^G) \to e.$$

Because $P_n < B_n^G < B_n$ and, $\operatorname{cd} B_n = \operatorname{cd} P_n = n-1$ (by [19]), the result [4, Proposition 2.4, Chapter VIII] leads to $\operatorname{cd} B_n^G = n-1$. Therefore $\operatorname{gd} B_n^G = n-1$ for $n \neq 3$ and $\operatorname{gd} B_3^G \leq 3$. Then Proposition 1.3 yields an action

$$(B_n^G/\mathcal{Z}(B_n^G)) \times \Sigma(1) \to \Sigma(1)$$

with $\dim \Sigma(1) = n-1$ for $n \neq 3$ and $\dim \Sigma(1) \leq 3$ for $n = 3$. We point out that by [6, Proposition 2.4(1)] we have $B_n^G/\mathcal{Z}(B_n^G) \cong \Gamma_0^{n+1, G\times\{e\}}$, where $\Gamma_0^{n+1, G\times\{e\}}$ is the $G \times \{e\}$-mapping class group of S_0^{n+1}, the orientable surface of genus 0 with $n+1$ punctures.

We have been informed by Fred Cohen on the extension

$$e \to \mathbb{Z} \to B_3 \to SL_2(\mathbb{Z}) \to e \ ([24, \text{Theorem } 10.5])$$

which, in view of Remark 1.4, leads to $e \to \mathbb{Z} \to B_3 \to B_3/n\mathbb{Z} \to e$ for $n \geq 1$, where $SL_2(\mathbb{Z})$ denotes the special 2×2-linear group over $\mathbb{Z}$. In particular, $n = 2$ yields

$$e \to \mathbb{Z} \to B_3 \to PSL_2(\mathbb{Z}) \to e$$

with $\mathbb{Z} \cong \mathcal{Z}(B_3)$ and $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$. Hence, Proposition 1.3 yields actions

$$(B_3/n\mathbb{Z}) \times \Sigma(1) \to \Sigma(1) \text{ with } \dim \Sigma(1) \leq 3 \text{ and } n \geq 1.$$

To show the next result, we notice that Proposition 1.3 and [4, Proposition 2.4, Chapter VIII] say:

Remark 1.6. For any extension $e \to G' \to G \to G'' \to e$ it follows that:

$$\operatorname{gd} G \leq \operatorname{gd} G' + \operatorname{gd} G'' \text{ and } \operatorname{cd} G \leq \operatorname{cd} G' + \operatorname{cd} G''.$$

Proposition 1.7. *If* $\operatorname{vcd} G < \infty$ *and there is an action* $G \times \Sigma(n) \to \Sigma(n)$ *then* $\operatorname{vcd} G \leq \dim \Sigma(n) - n$ *for* $n \geq 1$. *In particular,* G *is finite provided* $\dim \Sigma(n) = n$.

Further, if $n = 1$ *then* $\Sigma(1)$ *could be chosen such that* $\dim \Sigma(1) = \operatorname{vcd} G + 1$ *for* $\operatorname{vcd} G \neq 1$ *and* $\dim \Sigma(1) \leq 3$ *for* $\operatorname{vcd} G = 1$.

Proof. Given a subgroup $G' \leq G$ with $[G : G'] < \infty$ and $\operatorname{cd} G' = m < \infty$, consider the induced action $G' \to \operatorname{Aut}(H^n(\Sigma(n), \mathbb{Z})) \cong \mathbb{Z}_2$ and write $G'' = \operatorname{Ker}(G' \to \operatorname{Aut}(H^n(\Sigma(n), \mathbb{Z})) \cong \mathbb{Z}_2$. Then $[G : G''] = [G : G'][G' : G''] \leq 2[G : G'] < \infty$.

Next, for any G''-local coefficient system A, the Leray-Serre spectral sequence $E_2^{p,q} = H^p(G'', H^q(\Sigma(n), A))$ determined by the fibration $\Sigma(n) \to \Sigma(n)/G'' \to K(G'', 1)$ converges to $H^{p+q}(\Sigma(n)/G'', A)$. Because $E_2^{m+n,0} = 0$, it follows that

$$H^{m+n}(\Sigma(n)/G'', A) \cong H^m(G'', H^n(\Sigma(n), A)).$$

In view of the Universal Coefficient Theorem, we have an isomorphism $H^n(\Sigma(n), A) \cong \operatorname{Hom}(H^n(\Sigma(n), \mathbb{Z}), A) \cong A$ of $\mathbb{Z}G''$-modules.

Therefore, $H^{m+n}(\Sigma(n)/G'', A) \cong H^m(G'', A)$ and consequently, $\operatorname{vcd} G \leq \dim \Sigma(n) - n$.

To show the last statement first, consider an action $G \times \Sigma(1) \to \Sigma(1)$ and the associated extension $e \to \mathbb{Z} \to \pi = \pi_1(\Sigma(1)/G) \xrightarrow{p} G \to e$. Because $\operatorname{vcd} G < \infty$, we can assume that there is such a normal subgroup $G' < G$ that $\operatorname{cd} G' < \infty$ and the index $[G : G'] < \infty$. Then, the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(G') & \longrightarrow & G' & \longrightarrow & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \xrightarrow{\;p\;} & G & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & e & \longrightarrow & \pi/p^{-1}(G') & \longrightarrow & G/G' & \longrightarrow & e
\end{array}$$

determined by that extension above and Serre's Theorem [4, Theorem 3.1, Chapter VIII] yield $\operatorname{cd}\pi = \operatorname{cd} p^{-1}(G')$. Next the extension $e \to \mathbb{Z} \to p^{-1}(G') \to G' \to e$ and Remark 1.6 imply

$$\operatorname{cd}\pi = \operatorname{cd} p^{-1}(G') \leq \operatorname{cd} G' + 1 = \operatorname{vcd} G + 1.$$

First, for $\operatorname{vcd} G = 0$ we get $\operatorname{cd}\pi \leq 1$ and π is a free group. Thus there is an action of $G \times \Sigma(1) \to \Sigma(1)$ with $\dim\Sigma(1) = 1$.

Next, $\operatorname{vcd} G = 1$ implies $\operatorname{cd}\pi = \operatorname{cd} p^{-1}(G') \leq 2$. Because the group $\mathbb{Z}$ is normal in $p^{-1}(G')$, we deduce that $\operatorname{cd}\pi = 2$ and $\operatorname{gd}\pi \leq 3$. Hence, there is an action $G \times \Sigma(1) \to \Sigma(1)$ with $\dim\Sigma(1) \leq 3$.

Finally let $\operatorname{vcd} G \geq 2$. Because $\operatorname{gd}\pi = \operatorname{cd}\pi$ for $\operatorname{cd}\pi \neq 2$ and $\operatorname{gd}\pi \leq 3$ for $\operatorname{cd}\pi = 2$, we deduce that $\operatorname{gd}\pi \leq \operatorname{vcd} G + 1$. Then the first part of this proof leads to $\operatorname{gd}\pi = \operatorname{vcd} G + 1$ and the proof is complete.

□

We say that actions $\varphi : G\times\Sigma(1) \to \Sigma(1)$ and $\varphi' : G\times\Sigma'(1) \to \Sigma'(1)$ are *equivalent* if there is a homotopy equivalence $f : \Sigma(1) \to \Sigma'(1)$ such that the diagram

$$\begin{array}{ccc}
G \times \Sigma(1) & \xrightarrow{\;\varphi\;} & \Sigma(1) \\
{\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\
G \times \Sigma'(1) & \xrightarrow[\;\varphi'\;]{} & \Sigma'(1)
\end{array}$$

commutes. Then, we are in a position to state:

Proposition 1.8. *There is a one-to-one correspondence between equivalence classes of actions $G \times \Sigma(1) \to \Sigma(1)$ of a group G on homotopy circles $\Sigma(1)$ and equivalence classes of extensions $e \to \mathbb{Z} \to \pi \to G \to e$ with $\operatorname{cd}\pi < \infty$.*

In particular for $\operatorname{cd} G < \infty$, then equivalence classes of actions $G \times \Sigma(1) \to \Sigma(1)$ on homotopy circles $\Sigma(1)$ with a fixed action τ of G on $H^1(\Sigma(1), \mathbb{Z}) \cong \mathbb{Z}$ are in a *one-to-one correspondence with the cohomology group $H^2_\tau(G, \tilde{\mathbb{Z}})$, where the action of G on $\mathbb{Z}$ is determined by τ.*

Proof. Certainly, equivalent actions

$$\varphi : G \times \Sigma(1) \to \Sigma(1) \text{ and } \varphi' : G \times \Sigma'(1) \to \Sigma'(1)$$

yield equivalent extensions

$$e \to \mathbb{Z} \to \pi_1(\Sigma(1)/G) \to G \to e \text{ and } e \to \mathbb{Z} \to \pi_1(\Sigma'(1)/G) \to G \to e.$$

Now, equivalent extensions

$$e \to \mathbb{Z} \to \pi \to G \to e \text{ and } e \to \mathbb{Z} \to \pi' \to G \to e$$

determine an isomorphism $\alpha : \pi \to \pi'$ such that the diagram

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & e \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \longrightarrow & G & \longrightarrow & e \end{array}$$

commutes. Then for corresponding actions

$$\varphi : G \times \Sigma(1) \to \Sigma(1) \text{ and } \varphi' : G \times \Sigma'(1) \to \Sigma'(1)$$

given by Proposition 1.3 there is a homotopy equivalence $f : \Sigma(1) \to \Sigma'(1)$ (determined by $\alpha : \pi \to \pi'$) which leads to an equivalence of that actions.

Certainly, the second statement follows from the first one and the proof is complete.

□

2. Virtually cyclic groups acting on $\Sigma(1)$. A *virtually cyclic* group is a group that has a cyclic subgroup of finite index. Every virtually cyclic group in fact has a normal cyclic subgroup of finite index (namely, the core of any cyclic subgroup of finite index), and virtually cyclic groups are also known as *cyclic-by-finite groups*. Consequently, an infinite virtually cyclic group is the middle term of a short exact sequence of the form $e \to \mathbb{Z} \to G \to F \to e$, where F is a finite group. If the extension is central we say that the group G is of *type I*, otherwise it is of *type II*.

A finite-by-cyclic group (that is, a group G with a finite normal subgroup H such that G/H is cyclic) is always virtually cyclic. A finite-by-dihedral group is always virtually cyclic as well. In fact, these two families constitute all virtually cyclic groups. The statements below follow from the result of C.T.C. Wall [29, Theorem 5.12] which is basically:

Theorem 2.1. *Let G be an infinite finitely generated group. Then the following are equivalent:*

(1) *G is a group with two ends;*

(2) *G has an infinite cyclic subgroup of finite index;*

(3) *G has a finite normal subgroup $F \trianglelefteq G$ with the quotient $G/F \cong \mathbb{Z}$ or $\mathbb{Z}_2 \star \mathbb{Z}_2 \cong D_\infty$, the infinite dihedral group;*

(4) *G is of the form:*

(i) *$F \rtimes \mathbb{Z}$, a semi-direct product with F finite*

or

(ii) *$G_1 \star_F G_2$, a free product with an amalgamated finite subgroup F, where $[G_i : F] = 2$ for $i = 1, 2$.*

Notice that Theorem 2.1 implies:

Corollary 2.2. *Groups of the following three types are virtually cyclic. Moreover, every virtually cyclic group is exactly one of these three types:*

(1) *finite;*

(2) *finite-by-(infinite cyclic);*

(3) *finite-by-(infinite dihedral).*

In particular, every torsion-free virtually cyclic group is either trivial or infinite cyclic.

In view of [17, Chapter 2, Proposition 19], we may state:

Remark 2.3. A virtually cyclic group G is of type I, if it satisfies Theorem 2.1(4)-(i) and of type II, if it satisfies Theorem 2.1(4)-(ii).

To study actions of virtually cyclic groups, we need:

Proposition 2.4. *Given an action $G \times \Sigma(1) \to \Sigma(1)$ and any finite subgroup $F \leq G$, there is an isomorphism $F \cong \mathbb{Z}_n$ for some $n \geq 1$.*

Proof. Certainly, we may assume that G is a finite group. Because the group $\pi_1(\Sigma(1)/G)$ is torsion-free (Lemma 1.1(2)), the extension

$$e \to \mathbb{Z} \to \pi_1(\Sigma(1)/G) \to G \to e$$

and Corollary 2.2 lead to an isomorphism $\pi_1(\Sigma(1)/G) \cong \mathbb{Z}$ and the proof is complete. $\square$

Remark 2.5. The groups $GL_n(\mathbb{Z})$ (general linear over $\mathbb{Z}$) for $n \geq 2$ and $SL_n(\mathbb{Z})$ (special linear over $\mathbb{Z}$) for $n \geq 3$ do not act on $\Sigma(1)$ independently of the $\dim \Sigma(1)$.

Certainly, in view of [26, Chapter IX.14], the groups $GL_2(\mathbb{Z})$ and $SL_3(\mathbb{Z})$ contain non-cyclic finite subgroups. But $GL_2(\mathbb{Z}) < GL_n(\mathbb{Z})$ for $n \geq 2$ and $SL_3(\mathbb{Z}) < SL_n(\mathbb{Z})$ for $n > 2$. Hence, by means of Proposition 2.4, do not exist actions of $GL_n(\mathbb{Z})$ for $n \geq 2$ and $SL_n(\mathbb{Z})$ for $n > 2$ on any $\Sigma(1)$ independently of the $\dim \Sigma(1)$.

Then basing on Theorem 2.1, we derive:

Corollary 2.6. *Let $G \times \Sigma(1) \to \Sigma(1)$ be an action, where G is a virtually cyclic group.*

(1) *If G is finite, then $G \cong \mathbb{Z}_n$ for some $n \geq 1$;*
(2) *if G is of type I, then $G \cong \mathbb{Z}_n \rtimes \mathbb{Z}$ for some $n \geq 1$;*
(3) *if G is of type II, then $G \cong \mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n}$ for some $n \geq 1$, and $\mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n} \cong \mathbb{Z} \rtimes_\theta \mathbb{Z}_{2n}$ with $\theta(1_{2n}) = -1$.*

Proof. Item (1) follows immediately from Proposition 2.4.

Now, we prove item (2). For G being of type I, again by Proposition 2.4, we get an isomorphism $G \cong \mathbb{Z}_n \rtimes_\theta \mathbb{Z}$ for some $n \geq 1$ and an action $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$.

Next we prove item (3). Let G be a group of type II. Then by Theorem 2.1, it holds $G \cong G_1 \star_F G_2$ with $[G_i : F] = 2$ for $i = 1, 2$. Because, in view of [29, Theorem 1.6], the natural maps $G_i \to G_1 \star_F G_2$ are injective for $i = 1, 2$, it follows again from Proposition 2.4 that $G \cong \mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n}$ for some $n \geq 1$.

Further, let x (resp. y) denote the generator of the first (resp. second) copy of $\mathbb{Z}_{2n}$ in $\mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n}$. Then $\langle xy^{-1} \rangle \cong \mathbb{Z}$ is a normal subgroup of $\mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n}$ and $\mathbb{Z}_{2n} \star_{\mathbb{Z}_n} \mathbb{Z}_{2n} \cong \mathbb{Z} \rtimes \mathbb{Z}_{2n}$, where a generator of $\mathbb{Z}$ is sent to xy^{-1} and a generator of $\mathbb{Z}_{2n}$ to x.

$\square$

Next, consider an action $G \times \Sigma(1) \to \Sigma(1)$ for any of the groups G listed in Corollary 2.6.

Remark 2.7. In view of [4, Proposition 10.2], the induced action $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is trivial when restricted to any finite subgroup of G.

The following result [18, Corollary 1.3] will be crucial to show Proposition 2.9 and Theorem 2.10:

Lemma 2.8. *Let $e \to A \to G \to B \to e$ be a central extension with G torsion-free and B finite. Then G and hence B is abelian.*

Now, we can state:

Proposition 2.9. *Let G be any of the groups listed in Corollary 2.6. Then there is an action:*

(1) $\mathbb{Z}_n \times \mathbb{S}^1 \to \mathbb{S}^1$ for any $n \geq 1$;

(2) $(\mathbb{Z}_n \rtimes_\theta \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$ if and only if $\theta(1) = \pm 1$ for an action $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$. Further:

(i) if $\theta(1) = -1$, then an action $(\mathbb{Z}_n \rtimes_\theta \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$ on any $\Sigma(1)$ for $n \geq 3$ induces the non-trivial action $\mathbb{Z}_n \rtimes_\theta \mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ given by the composition $Z_n \rtimes_\theta \mathbb{Z} \to \mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$;

(ii) if $\theta(1) = 1$, then an action $(\mathbb{Z}_n \rtimes_\theta \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$ on any $\Sigma(1)$ for $n \geq 3$ induces the trivial action $\mathbb{Z}_n \rtimes_\theta \mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$. For G isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}_2 \oplus \mathbb{Z}$ there are actions $G \times \Sigma(1) \to \Sigma(1)$ with trivial and non-trivial induced actions $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$;

(3) $(\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n}) \times \Sigma(1) \to \Sigma(1)$. Further, any action $(\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n}) \times \Sigma(1) \to \Sigma(1)$ induces the trivial action $\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$.

Proof. Item (1) is obvious because of the inclusion $\mathbb{Z}_n \leq \mathbb{S}^1$ for any $n \geq 1$.

Now, we prove item (2). Suppose there is an action $(\mathbb{Z}_n \rtimes_\theta \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$ with a non-trivial action $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$. The extension

$$e \to \mathbb{Z} \to \pi = \pi_1(\Sigma(1)/(\mathbb{Z}_n \rtimes_\theta \mathbb{Z})) \xrightarrow{p} \mathbb{Z}_n \rtimes_\theta \mathbb{Z} \to e$$

associated with that action leads to the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(\mathbb{Z}_n) & \longrightarrow & \mathbb{Z}_n & \longrightarrow & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \xrightarrow{p} & \mathbb{Z}_n \rtimes_\theta \mathbb{Z} & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & e & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & e
\end{array}$$

with exact rows. But, the group $p^{-1}(\mathbb{Z}_n)$ is torsion-free and Remark 2.7 yields the trivial action of $\mathbb{Z}_n$ on $H^1(\Sigma(1), \mathbb{Z}) \cong \mathbb{Z}$. Thus, in view of Lemma 2.8, the extension $e \to \mathbb{Z} \to p^{-1}(\mathbb{Z}_n) \to \mathbb{Z}_n \to e$ is central. Hence, the group $p^{-1}(\mathbb{Z}_n)$ is abelian and $p^{-1}(\mathbb{Z}_n) \cong \mathbb{Z}$. Consequently, we get either $\pi \cong \mathbb{Z} \oplus \mathbb{Z}$ or $\pi \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z} = \langle a, b;\ bab^{-1} =$

$a^{-1}\rangle$. If $n > 2$, then $\mathbb{Z}_n \rtimes_\theta \mathbb{Z}$ is not abelian and it follows that $\pi \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}$. If $n = 2$, then θ is trivial and will be treated later.

But, any element of the group $\mathbb{Z} \rtimes_{-1} \mathbb{Z}$ has a unique form $a^r b^s$ for some $r, s \geq 0$. A routine calculation shows that any infinite cyclic subgroup $\langle a^r b^s \rangle \leq \mathbb{Z} \rtimes_{-1} \mathbb{Z}$ is normal if and only if $s = 0$ or $r = 0$ and s is even. First consider the case $s = 0$. Then, there is an extension

$$e \to \mathbb{Z} \cong \langle a^r \rangle \to \mathbb{Z} \rtimes_{-1} \mathbb{Z} \to \mathbb{Z}_n \rtimes_\theta \mathbb{Z} \to e \tag{2.2}$$

with $r \geq 1$ which yields $r = n$. Next, the relation $ba^r b^{-1} = a^{-r}$ implies that $\theta(1) = -1$.

Let now consider the case $r = 0$ and $s = 2s_1$. Then, we have the extension

$$e \to \mathbb{Z} \cong \langle b^{2s_1} \rangle \to \mathbb{Z} \rtimes_{-1} \mathbb{Z} \to \mathbb{Z}_n \rtimes_\theta \mathbb{Z} \to e.$$

But such an extension cannot exist because the group $\mathbb{Z}_n \rtimes_\theta \mathbb{Z}$ is virtually cyclic of type I and the quotient $\mathbb{Z} \rtimes_{-1} \mathbb{Z}/\langle b^{2s_1} \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2s_1}$ is virtually cyclic of type II. Then, for a non-trivial θ, we have shown that $\theta(1) = -1$.

Now, we construct actions which correspond to the cases above.

Given $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$ with $\theta(1) = \pm 1$, we define:

$$\circ : (\mathbb{Z}_n \rtimes_\theta \mathbb{Z}) \times (\mathbb{S}^1 \times \mathbb{R}^1) \to \mathbb{S}^1 \times \mathbb{R}^1$$

by $(1_n, 0) \circ (x, t) = (e^{\frac{2\pi i}{n}} x, t)$, $(0, 1) \circ (x, t) = (\beta(x), t + 1)$ for $(x, t) \in \mathbb{S}^1 \times \mathbb{R}^1$ and $n \geq 1$, where $\beta(x) = x$ if $\theta(1) = 1$ and $\beta(x) = \bar{x}$ (the complex conjugation) if $\theta(1) = -1$.

For $n = 1, 2$ we have only one group, $\mathbb{Z}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}$, respectively. The extensions:

$$e \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to e, \quad e \to \mathbb{Z} \to \mathbb{Z} \rtimes_{-1} \mathbb{Z} \to \mathbb{Z} \to e$$

and

$$e \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z} \to e, \quad e \to \mathbb{Z} \to \mathbb{Z} \rtimes_{-1} \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z} \to e$$

lead to actions

$$\mathbb{Z} \times \Sigma(1) \to \Sigma(1) \text{ and } (\mathbb{Z}_2 \oplus \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$$

with trivial and non-trivial induced actions $\mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ and $(\mathbb{Z}_2 \oplus \mathbb{Z}) \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$.

Now, we show the further parts of item (2).

(i): If $\theta(1) = -1$, then from the extension (2.2) it follows that the action $\mathbb{Z}_n \rtimes_{-1} \mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is non-trivial.

(ii): Now, suppose that $\theta(1) = 1$. Then, following the same steps as above, we can show that an action $(\mathbb{Z}_n \oplus \mathbb{Z}) \times \Sigma(1) \to \Sigma(1)$ yields the extension

$$e \to \mathbb{Z} \to \pi \to \mathbb{Z}_n \oplus \mathbb{Z} \to e$$

with $\pi \cong \mathbb{Z} \rtimes \mathbb{Z}$. We claim that $\pi \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n \geq 3$. Suppose by contradiction that $\pi \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}$. Then the surjective map $\mathbb{Z} \rtimes_{-1} \mathbb{Z} \to \mathbb{Z}_n \oplus \mathbb{Z}$ factors through the abelianization $(\mathbb{Z} \rtimes_{-1} \mathbb{Z})^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$, so we obtain a surjective map $\mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_n \oplus \mathbb{Z}$. Since $n \geq 3$ this is a contradiction and the result follows. It immediately follows that the homomorphism $\mathbb{Z}_n \oplus \mathbb{Z} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is trivial for $n \geq 3$. For $n = 1, 2$, as we have shown above, the action of $\mathbb{Z}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}$ can be either trivial or non-trivial.

Finally, we show item (3). We define an action of $\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n} \cong \mathbb{Z} \rtimes \mathbb{Z}_{2n}$ on $\Sigma(1) = \mathbb{S}^1 \times \mathbb{R}^1$ as follows:

$$\circ : (\mathbb{Z} \rtimes \mathbb{Z}_{2n}) \times (\mathbb{S}^1 \times \mathbb{R}^1) \to \mathbb{S}^1 \times \mathbb{R}^1$$

given by $(0, 1_{2n}) \circ (x, t) = (e^{2\pi i/n}x, -t)$, $(1, 0) \circ (x, t) = (x, t + 1)$ for $(x, t) \in \mathbb{S}^1 \times \mathbb{R}^1$.

Next, given an action $(\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n}) \times \Sigma(1) \to \Sigma(1)$, in view of [4, Proposition 10.2], the induced action $\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is trivial because $\mathbb{Z}_{2n} *_{\mathbb{Z}_n} \mathbb{Z}_{2n}$ is generated by elements of finite order and the proof is complete.

□

Now, we aim to classify homotopy types of orbits spaces $\Sigma(1)/G$ for all actions $G \times \Sigma(1) \to \Sigma(1)$ of virtually cyclic groups G on a homotopy circle $\Sigma(1)$. By Lemma 1.1, the orbit space $\Sigma(1)/G$ is an Eilenberg-MacLane space of type $K(\pi, 1)$ with a torsion-free group $\pi = \pi_1(\Sigma(1)/G)$ and there is an extension

$$e \to \mathbb{Z} \to \pi \to G \to e.$$

Theorem 2.10. *Let G be a virtually cyclic group. Then the orbit space $\Sigma(1)/G$ of an action $G \times \Sigma(1) \to \Sigma(1)$ of a virtually cyclic group G on $\Sigma(1)$ has the homotopy type:*

(1) *of a circle if the group G is finite;*

(2) *either of the 2-torus or the Klein bottle, if the group G is infinite.*

Proof. Item (1): If G is a finite group, then by Proposition 2.4 we get $G \cong \mathbb{Z}_n$ for some $n \geq 1$. Hence,

$$e \to \mathbb{Z} \to \pi \to \mathbb{Z}_n \to e$$

is the associated extension to the action $\mathbb{Z}_n \times \Sigma(1) \to \Sigma(1)$. In view of Remark 2.7 this extension is central and Lemma 2.8 yields that $\pi \cong \mathbb{Z}$ and so $\Sigma(1)/G \simeq \mathbb{S}^1$.

Item (2): Consider the extension $e \to \mathbb{Z} \to \pi \to G \to e$ associated to the action $G \times \Sigma(1) \to \Sigma(1)$. Since G is virtually cyclic, the group π is virtually polycyclic and it follows that π has Hirsch length 2. Because π is torsion-free, it is the fundamental group of a two dimensional flat manifold which is either either the 2-torus or the Klein bottle and the proof is complete.

$\square$

3. Arbitrary groups acting on $\Sigma(1)$.

We begin from the study of groups G with $\operatorname{vcd} G \leq 1$ acting on a $\Sigma(1)$ such that $\dim \Sigma(1) \leq 2$. Observe that G is finite provided $\operatorname{vcd} G = 0$ or $\dim \Sigma(1) = 1$ (Proposition 1.7), and by Proposition 2.4, we get $G \cong \mathbb{Z}_n$ for some $n \geq 1$. Hence, we consider groups G with $\operatorname{vcd} G = 1$ which act on $\Sigma(1)$ with $\dim \Sigma(1) > 1$.

Notice that $\operatorname{vcd}(F \rtimes_{\theta_1} \mathbb{Z}_n) = \operatorname{vcd}(\mathbb{Z}_n \rtimes_{\theta_2} F) = 1$ for any free group F and actions $\theta_1 : \mathbb{Z}_n \to \operatorname{Aut}(F)$ and $\theta_2 : F \to \operatorname{Aut}(\mathbb{Z}_n)$. Further, $F \rtimes_{\theta_1} \mathbb{Z}_n$ and $\mathbb{Z}_n \rtimes_{\theta_2} F$ are fundamental groups of finite graphs of finite groups. In order to study actions on homotopy circles of such groups, consider a slightly more general situation. Given a group G and its subgroup $G' < G$ write $\operatorname{Aut}_{G'}(G)$ for the subgroup of $\operatorname{Aut}(G)$ given by automorphisms which send G' to itself. Notice that any epimorphism $p : G \to H$ induces an action

$$\bar{p} : \operatorname{Aut}_{\operatorname{Ker} p}(G) \to \operatorname{Aut}(H).$$

Lemma 3.1. *Let G, H be groups that $\operatorname{cd} G < \infty$ and H acts on some $\Sigma(1)$. Then:*

(1) *for any action $\theta : H \to \operatorname{Aut}(G)$, there is an action of the group $G \rtimes_\theta H$ on some $\Sigma(1)$;*

(2) *for an action $\theta : G \to \operatorname{Aut}(H)$ which admits a factorization*

$$\begin{array}{ccc} & & G \\ & \swarrow^{\theta'} & \downarrow{\scriptstyle\theta} \\ \operatorname{Aut}_{\mathbb{Z}}(\pi) & \xrightarrow{\bar{p}} & \operatorname{Aut}(H), \end{array}$$

where $\bar{p}$ is induced by an extension $e \to \mathbb{Z} \to \pi \xrightarrow{p} H \to e$ determined by an action of H, there is an action of the group $H \rtimes_\theta G$ on some $\Sigma(1)$.

Further, if there is $g_0 \in G$ such that $\theta(g_0) \notin \operatorname{Im} \bar{p}$ for any $\bar{p} : \operatorname{Aut}_{\mathbb{Z}}(\pi) \to \operatorname{Aut}(H)$ as above, then the group $H \rtimes_\theta G$ cannot act on any $\Sigma(1)$.

Proof. Item (1): The action of H on some $\Sigma(1)$ yields an extension

$$e \to \mathbb{Z} \to \pi \xrightarrow{p} H \to e.$$

Then, given $\theta : H \to \mathrm{Aut}(G)$, consider $\theta' = \theta p : \pi \to \mathrm{Aut}(G)$. Thus, we get the extension

$$e \to \mathbb{Z} \to G \rtimes_{\theta'} \pi \to G \rtimes_{\theta} H \to e$$

and, in view of Remark 1.6, it holds $\mathrm{gd}(G \rtimes_{\theta'} \pi) < \infty$. Hence Proposition 1.3 leads to an action

$$(G \rtimes_{\theta} H) \times \Sigma(1) \to \Sigma(1)$$

for some $\Sigma(1)$.

Item (2): The factorization $\theta = \bar{p}\theta'$ and the extension $e \to \mathbb{Z} \to \pi \xrightarrow{p} H \to e$ determined by an action of H on some $\Sigma(1)$ lead to an extension

$$e \to \mathbb{Z} \to \pi \rtimes_{\theta'} G \to H \rtimes_{\theta} G \to e$$

and, in view of Remark 1.6, it holds $\mathrm{gd}(\pi \rtimes_{\theta'} G) < \infty$. Hence, Proposition 1.3 yields an action

$$(H \rtimes_{\theta} G) \times \Sigma(1) \to \Sigma(1)$$

for some $\Sigma(1)$.

Suppose that there is $g_0 \in G$ such that $\theta(g_0) \notin \mathrm{Im}\,\bar{p}$ for any $\bar{p} : \mathrm{Aut}_{\mathbb{Z}}(\pi) \to \mathrm{Aut}(H)$ as above and the group $H \rtimes_{\theta} G$ acts on some $\Sigma(1)$. Then the subgroup $H \rtimes \mathbb{Z} \cong H \rtimes \langle g_0 \rangle < H \rtimes_{\theta} G$ acts on that $\Sigma(1)$ and we get the associated extension

$$e \to \mathbb{Z} \to \pi' \xrightarrow{p'} H \times_{\theta} \mathbb{Z} \to e.$$

Next, the extension above leads to the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(H) & \longrightarrow & H & \longrightarrow & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \xrightarrow{p'} & H \rtimes \mathbb{Z} & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & e & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & e.
\end{array}$$

Hence, $\pi' \cong p'^{-1}(H) \rtimes \mathbb{Z}$ and so $\theta(g_0) \in \mathrm{Im}\,\bar{p}'$. This yields a contradiction and the proof follows.

□

Setting $\mathrm{rk}(F)$ for the rank of a free group F, we are in a position to show:

Theorem 3.2. *For an action $\theta : F \to \mathrm{Aut}(\mathbb{Z}_n)$ there is an action of the group $\mathbb{Z}_n \rtimes_\theta F$ on $\Sigma(1)$ with $\dim \Sigma(1) \leq 2$ if and only if $\theta(x) = \pm 1$ for any element $x \in F$ and exactly one homotopy type of the orbit spaces $\Sigma(1)/(\mathbb{Z}_n \rtimes_\theta F)$ provided $n \geq 3$, and $2^{rk(F)}$ distinct such homotopy types with $n = 1, 2$ for all possible actions of such $\mathbb{Z}_n \rtimes_\theta F$ on homotopy circles $\Sigma(1)$. Further:*

(1) *any action $(\mathbb{Z}_n \rtimes_\theta F) \times \Sigma(1) \to \Sigma(1)$ on a $\Sigma(1)$ induces the action $\varphi : \mathbb{Z}_n \rtimes_\theta F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ given by $\varphi(x) = \theta(x)$ for $x \in F$;*

(2) *if θ is trivial, then any action $(\mathbb{Z}_n \rtimes_\theta F) \times \Sigma(1) \to \Sigma(1)$ on a $\Sigma(1)$ for $n \geq 3$ induces the trivial action $\mathbb{Z}_n \rtimes_\theta F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$. For G isomorphic either to F or $\mathbb{Z}_2 \oplus F$ there are actions $G \times \Sigma(1) \to \Sigma(1)$ with trivial and non-trivial induced actions $G \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$.*

Proof. Suppose $\theta : F \to \mathrm{Aut}(\mathbb{Z}_n)$ satisfies $\theta(x) = \pm 1$ for any element $x \in F$. Then θ lifts to $\theta' : F \to \mathrm{Aut}(\mathbb{Z})$, and in view of Proposition 1.3, the extension $e \to \mathbb{Z} \to \mathbb{Z} \rtimes_{\theta'} F \to \mathbb{Z}_n \rtimes_\theta F \to e$, leads to an action

$$(\mathbb{Z}_n \rtimes_\theta F) \times \Sigma(1) \to \Sigma(1)$$

with $\dim \Sigma(1) \leq 2$. Notice that the induced action $\varphi : \mathbb{Z}_n \rtimes_\theta F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is trivial on $\mathbb{Z}_n$, and $\varphi(x) = \theta'(x)$ for any element $x \in F$.

Conversely, suppose there is an action $(\mathbb{Z}_n \rtimes_\theta F) \times \Sigma(1) \to \Sigma(1)$, by contradiction, and assume that $\theta(x_0) \neq \pm 1$ for an element $x_0 \in F$. Consider the map $\theta_0 : \langle x_0 \rangle \to \mathrm{Aut}(\mathbb{Z}_n)$ with $\theta_0(x_0) = \theta(x_0)$. Then, in view of Proposition 2.9(2), the subgroup $\mathbb{Z}_n \rtimes_{\theta_0} \langle x_0 \rangle < \mathbb{Z}_n \rtimes_\theta F$ does not act on any $\Sigma(1)$, which is a contradiction. Consequently, it follows that $\theta(x_0) = \pm 1$.

Now, we study the homotopy type of the orbit spaces. Given an action $(\mathbb{Z}_n \rtimes_\theta F) \times \Sigma(1) \to \Sigma(1)$, we have an associated extension $e \to \mathbb{Z} \to \pi \xrightarrow{p} \mathbb{Z}_n \rtimes_\theta F \to e$ with a torsion-free group π and the action $\varphi : \mathbb{Z}_n \rtimes_\theta F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ which restricts to the trivial one on $\mathbb{Z}_n$. Hence, the extension

$$e \to \mathbb{Z} \to p^{-1}(\mathbb{Z}_n) \to \mathbb{Z}_n \to e$$

yields $p^{-1}(\mathbb{Z}_n) \cong \mathbb{Z}$. Further, the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(\mathbb{Z}_n) & \longrightarrow & \mathbb{Z}_n & \longrightarrow & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \xrightarrow{p} & \mathbb{Z}_n \rtimes_\theta F & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & e & \longrightarrow & F & \longrightarrow & F & \longrightarrow & e
\end{array}$$

leads to an isomorphism

$$\pi \cong \mathbb{Z} \rtimes_{\theta'} F,$$

where $\theta' : F \to \mathrm{Aut}(\mathbb{Z})$ is the lifting of $\theta : F \to \mathrm{Aut}(\mathbb{Z}_n)$. Because θ' is uniquely determined by θ for $n \geq 3$, there is exactly one homotopy type of the orbit spaces $\Sigma(1)/(\mathbb{Z}_n \rtimes_\theta F)$ for all possible actions of such $\mathbb{Z}_n \rtimes_\theta F$ on homotopy circles $\Sigma(1)$ for $n \geq 3$. Further, Proposition 1.3 leads to $\dim \Sigma(1) \leq 2$.

Now, the group $\mathrm{Aut}(\mathbb{Z}_n)$ is trivial for $n = 1, 2$. Hence, there are $2^{\mathrm{rk}(F)}$ homotopy type of the orbit spaces $\Sigma(1)/(\mathbb{Z}_n \rtimes_\theta F)$ for $n = 1, 2$.

Item (1): Given an extension $e \to \mathbb{Z} \to F \rtimes_{\theta'} \mathbb{Z} \to F \rtimes_\theta \mathbb{Z}_n \to e$ which arises from an action $(F \rtimes_\theta \mathbb{Z}_n) \times \Sigma(1) \to \Sigma(1)$, the induced action $\varphi : F \rtimes_\theta \mathbb{Z}_n \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is completely determined by $\varphi(x) = \theta'(x)$ for any element $x \in F$, and the result follows.

Item (2): Now, suppose that θ is trivial. Then an action $(\mathbb{Z}_n \times F) \times \Sigma(1) \to \Sigma(1)$ yields the extension

$$e \to \mathbb{Z} \to \pi \to \mathbb{Z}_n \times F \to e$$

with $\pi \cong \mathbb{Z} \rtimes_{\theta'} F$. We claim that $\pi \cong \mathbb{Z} \times F$ for $n \geq 3$. Suppose by contradiction that θ' is non-trivial. Then the surjective map $\mathbb{Z} \rtimes_{\theta'} F \to \mathbb{Z}_n \times F$ implies a surjective map $\mathbb{Z}_2 \oplus F^{ab} \to \mathbb{Z}_n \oplus F^{ab}$, where F^{ab} denotes the abelianization of F. Since $n \geq 3$ this is a contradiction and the result follows. It immediately follows that the induced action $\mathbb{Z}_n \times F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ is trivial for $n \geq 3$.

For $n = 1, 2$, the extensions:

$$e \to \mathbb{Z} \to \mathbb{Z} \rtimes_{-1} F \to F \to e, \quad e \to \mathbb{Z} \to \mathbb{Z} \times F \to F \to e$$

and

$$e \to \mathbb{Z} \to \mathbb{Z} \rtimes_{-1} F \to \mathbb{Z}_2 \times F \to e, \quad e \to \mathbb{Z} \to \mathbb{Z} \times F \to \mathbb{Z}_2 \times F \to e$$

lead to actions

$$F \times \Sigma(1) \to \Sigma(1) \text{ and } (\mathbb{Z}_2 \times F) \times \Sigma(1) \to \Sigma(1)$$

with trivial and non-trivial induced actions $F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ and $\mathbb{Z}_2 \times F \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$.

□

Now, let $F = \langle X \rangle$ be a free group with a basis X and let $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ be an action. In view of Proposition 1.3, any extension $e \to \mathbb{Z} \to \pi \xrightarrow{p} F \rtimes_\theta \mathbb{Z}_n \to e$ with $\mathrm{gd}\, \pi < \infty$, leads to an action $(F \rtimes_\theta \mathbb{Z}_n) \times \Sigma(1) \to \Sigma(1)$ and the induced action

$\varphi : F \rtimes_\theta \mathbb{Z}_n \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z})) \cong \{\pm 1\}$. Notice that [21, Proposition 1, Chapter 10] yields the presentation

$$\pi = \big\langle X, s, t |\; xsx^{-1} = \varphi'(x)(s),\; tst^{-1} = s^m,\; txt^{-1} = \theta(1_n)(x)s^{m_x} \text{ for some } m, m_x \in \mathbb{Z}\big\rangle,$$

where $\varphi' = \varphi|_F$ and $1_n \in \mathbb{Z}_n$. But, by means of Remark 2.7, the induced action $\varphi : F \rtimes_\theta \mathbb{Z}_n \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z})) \cong \{\pm 1\}$ restricts to the trivial action on $\mathbb{Z}_n$. Hence, the extension

$$e \to \mathbb{Z} \to p^{-1}(\mathbb{Z}_n) \to \mathbb{Z}_n \to e$$

is central and yields $p^{-1}(\mathbb{Z}_n) \cong \mathbb{Z}$, so it follows that $s = t^{\pm n}$. Consequently, we get

$$\pi = \big\langle X, t |\; xt^n x^{-1} = \varphi'(x)(t^n),\; txt^{-1} = \theta'(1)(x)t^{nm_x} \text{ for some } m_x \in \mathbb{Z}\big\rangle,$$

where $\theta' : \mathbb{Z} \to \mathrm{Aut}(F)$ is the composition of $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ with the quotient map $\mathbb{Z} \to \mathbb{Z}_n$.

Further, the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(F) & \longrightarrow & F & \longrightarrow & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \xrightarrow{p} & F \rtimes_\theta \mathbb{Z}_n & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & e & \longrightarrow & \mathbb{Z}_n & \longrightarrow & \mathbb{Z}_n & \longrightarrow & e
\end{array}$$

leads to an isomorphism $p^{-1}(F) \cong \mathbb{Z} \rtimes_{\varphi'} F$. Consequently, in view of Serre's Theorem [4, Theorem 3.1], we get

$$\mathrm{cd}(\pi) = \mathrm{cd}(\mathbb{Z} \rtimes_{\varphi'} F) = 2$$

provided π is torsion-free.

Let I, Λ, L be arbitrary sets and write $(*_G G_i)_{i\in I}$ for the free product with an amalgamated subgroup $G < G_i$ for $i \in I$. In particular, if $G = e$, the trivial group, then $(*_G G_i)_{i\in I} = (*G_i)_{i\in I}$, the free product of groups G_i for $i \in I$. Let $F^{\theta(1_n)} < F$ be the fixed point subgroup of the automorphism $\theta(1_n)$. If n is a prime, then [9, Theorem 3] determines a decomposition into the free product

$$F = F^{\theta(1_n)} * (*F_i)_{i\in I} * (*F_\lambda)_{\lambda\in\Lambda},$$

where each factor is $\theta(1_n)$-invariant and:

(i) for each $i \in I$, $F_i = \langle y_{i,1}, \ldots, y_{i,n}\rangle$ such that

$$\theta(1_n)(y_{i,r}) = y_{i,r+1 (\bmod\ n)};$$

(ii) for each $\lambda \in \Lambda$, there is a set J_λ with $F_\lambda = \langle y_{\lambda,1}, \ldots, y_{\lambda,n-1}, z_j \mid j \in J_\lambda \rangle$ such that

$$\theta(1_n)(y_{\lambda,s}) = y_{\lambda,s+1} \text{ for } s = 1, \ldots, n-2,$$

$$\theta(1_n)(y_{\lambda,n-1}) = (y_{\lambda,1} \cdots y_{\lambda,n-1})^{-1},$$

$$\theta(1_n)(z_j) = y_{\lambda,1}^{-1} z_j y_{\lambda,1} \text{ for } j \in J_\lambda \text{ and } \lambda \in \Lambda.$$

Further, set $F^{\theta(1_n)} = \langle x_l \mid l \in L \rangle$.

Motivated by the result above, from now on, we study actions of groups of the form $F \rtimes_\theta \mathbb{Z}_n$ for any integer $n \geq 1$ and homomorphisms $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ which admit a decomposition

$$F = F^{\theta(1_n)} * (*F_i)_{i \in I} * (*F_\lambda)_{\lambda \in \Lambda}$$

satisfying (i) and (ii) above. At least if n is a prime, as result of [9, Theorem 3] stated above, we are considering all possible homomorphisms θ.

Because there is $t \in F \rtimes_\theta \mathbb{Z}_n$ with $\theta(1_n)(x) = txt^{-1}$ for all $x \in F$, we derive:

Lemma 3.3. *Given an extension $e \to \mathbb{Z} \to \pi \to F \rtimes_\theta \mathbb{Z}_n \to e$, write $\varphi : F \to \mathrm{Aut}(\mathbb{Z})$ for the induced action. If $F = F^{\theta(1_n)} * (*F_i)_{i \in I} * (*F_\lambda)_{\lambda \in \Lambda}$ satisfies* (i) *and* (ii) *above then:*

(1) *no restriction on $\varphi(x_l)$ for $l \in L$;*

(2) *$\varphi(y_{i,1}) = \cdots = \varphi(y_{i,n})$ for $i \in I$;*

(3) *$\varphi(y_{\lambda,1}) = \cdots = \varphi(y_{\lambda,n-1}) = 1$ for all $\lambda \in \Lambda$;*

(4) *no restriction on $\varphi(z_j)$ for $j \in J_\lambda$ and $\lambda \in \Lambda$.*

Proof. Certainly, (1), (2), (3) (for an odd n) and (4) are obvious.

(3) (for an even n): suppose that $\varphi(y_{\lambda,1}) = \cdots = \varphi(y_{\lambda,n-1}) = -1$. Then, the relations $y_{\lambda,1} t^n y_{\lambda,1}^{-1} = t^{-n}$, $t y_{\lambda,r} t^{-1} = y_{\lambda,r+1} t^{nm_r}$ for $r = 1, \ldots, n-2$ and $t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm_{n-1}}$ imply $\sum_{k=1}^{n'} m_{2k-1} = -1$. Hence, we derive $(y_{\lambda,n-1} t)^n = e$. Since the group π is torsion-free, we must have $y_{\lambda,n-1} t = e$ which is a contradiction since $y_{\lambda,n-1} \neq t^{-1}$. So, we must have $\varphi(y_\lambda, 1) = \cdots = \varphi(y_{\lambda,n-1}) = 1$ for any $n \geq 1$. □

Generalized Baumslag-Solitar groups are well known to be torsion-free as they are the fundamental groups of finite graphs of groups with $\mathbb{Z}'s$ as vertex and edge groups. In particular, the one-relator *Baumsalg-Solitar* group $BS(m,n) = \langle x, t; xt^m x^{-1} = t^n \rangle$ with $m, n \in \mathbb{Z} \backslash \{0\}$ is torsion-free as the fundamental groups of graphs of infinite cyclic groups where the graph is a 1-loop. We will use in the sequel that:

Remark 3.4. The one-relator group $BS(n,-n) = \langle x, t |\, xt^n x^{-1} = t^{-n} \rangle$ is torsion-free for any $n \in \mathbb{Z}\backslash\{0\}$.

Lemma 3.5. *Let J be an arbitrary set. If $F = \langle y_1, \ldots, y_{n-1}, z_j\,, j \in J \rangle$, $\theta(1_n)(y_r) = y_{r+1}$ for $r = 1, \ldots, n-2$, $\theta(1_n)(y_{n-1}) = (y_1 \cdots y_{n-1})^{-1}$ and $\theta(1_n)(z_j) = y_1^{-1} z_j y_1$ for $j \in J$, then for any extension $e \to \mathbb{Z} \to \pi \to F \rtimes_\theta \mathbb{Z}_n \to e$ with a torsion-free group π it follows that:*

(1) $\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^n,\ j \in J,\ t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm}, t z_j t^{-1} = y_1^{-1} z_j y_1, j \in J \rangle$ for any n and $m = 0, \ldots, n-2$

or

*(2) $\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^n,$
$j \in J, j \neq j_0,\ z_{j_0} t^n z_{j_0}^{-1} = t^{-n}, t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} =$
$(y_1 \cdots y_{n-1})^{-1} t^{-n(1+n')}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{n(1+2m')}, j \in J \rangle$ for $n = 2n'$.*

Proof. Let $F = \langle y_1, \ldots, y_{n-1}, z_j\,, j \in J \rangle$, $\theta(1_n)(y_r) = y_{r+1}$ for $r = 1, \ldots, n-2$, $\theta(1_n)(y_{n-1}) = (y_1 \cdots y_{n-1})^{-1}$ and $\theta(1_n)(z_j) = y_1^{-1} z_j y_1$ for $j \in J$.

Then, in view of Lemma 3.3, we get

$\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^{\pm n},\ j \in J,$
$t y_r t^{-1} = y_{r+1} t^{nm_r},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm_{n-1}}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{nm_j}, j \in J \rangle$.

The substitution $t' = t$, $y_1' = y_1$, $y_r' = y_r t^{\sum_{k=1}^{r-1} nm_k}$ for $r = 2, \ldots, n-1$ and $z_j' = z_j$ for $j \in J$ implies

$\pi \cong \langle y_1', \ldots, y_{n-1}', z_j', j \in J, t' |\, {y'}_r {t'}^n {y'}_r^{-1} = {t'}^n,\ r = 1, \ldots, n-1,\ {z'}_j {t'}^n {z'}_j^{-1} = {t'}^{\pm n},\ j \in J, t' {y'}_r {t'}^{-1} = {y'}_{r+1},\ r = 1, \ldots, n-2,\ t' y_{n-1}' {t'}^{-1} = (y_1' \cdots y_{n-1}')^{-1} {t'}^{nm},\ t' {z'}_j {t'}^{-1} = {y'}_1^{-1} {z'}_j {y'}_1,\ j \in J \rangle$, where $m = \sum_{r=1}^{n-1} (n-r) m_r$.

The infinity of the order of t implies $-2n = 2nm + n^2 m_j$ for $z_j t^n z_j^{-1} = t^{-n}$. Thus, n or m_j is even.

Thus, n or m_j is even. Hence, $m \equiv -1 \,(\mathrm{mod}\, n)$ for any n and an even m_j and $m \equiv -1 - n' \,(\mathrm{mod}\, n)$ for $n = 2n'$ and m_j odd. Further, the infinity of the order of t implies $m_j = 0$ for $z_j t^n z_j^{-1} = t^n$ with $j \in J$.

Because $m \equiv -1 \,(\mathrm{mod}\, n)$ implies $(y_1 t^{-m})^n = e$, we get

$\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^{\pm n},\ j \in J,$
$t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{nm_j}, j \in J \rangle$,

where $m_j = 0$ with $z_j t^n z_j^{-1} = t^n$ for any n or $m_j = 2m' + 1$ with $z_j t^n z_j^{-1} = t^{-n}$ for some $m' \geq 0$, an even n and $j \in J$.

Consequently,

item (1): $\pi \cong \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^n,\ j \in J,\ t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{nm_j}, j \in J \rangle$ for any n and $m = 0, \ldots, m-2$

or $J = J_+ \cup J_-$ and

$\pi \cong \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^n,\ j \in J_+,\ z_j t^n z_j^{-1} = t^{-n},\ j \in J_-, t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2,\ t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{-n(1+n')},\ t z_j t^{-1} = y_1^{-1} z_j y_1 t^{n(1+m')}, j \in J \rangle$ for $n = 2n'$.

Further, if $j_0 \in J_-$ then the substitution $t' = t$, $y'_r = y_r$ for $r = 1, \ldots, n-1$, $z'_j = z_j$ for $j \in J_+$ and $z'_j = z_{j_0} z_j$ for $j \in J_-$ implies

item (2): $\pi \cong \langle y'_1, \ldots, y'_{n-1}, z'_j, j \in J, t' |\, {y'}_r t'^n {y'}_r^{-1} = t'^n,\ r = 1, \ldots, n-1,\ {z'}_j t'^n {z'}_j^{-1} = t'^n,\ j \in J,\ j \neq j_0,\ {z'}_{j_0} t'^n {z'}_{j_0}^{-1} = t'^{-n},\ t' {y'}_r t'^{-1} = {y'}_{r+1},\ r = 1, \ldots, n-2,\ t' y'_{n-1} t'^{-1} = (y'_1 \cdots y'_{n-1})^{-1} t'^{-n(1+n')},\ t' {z'}_j t'^{-1} = {y'}_1^{-1} {z'}_j {y'}_1 t'^{n(1+2m')}, j \in J \rangle$ for $n = 2n'$.

To examine this group, consider the extension

$$e \to \mathbb{Z} \rtimes F \to \pi \to \mathbb{Z}_n \to e,$$

where $\mathbb{Z}$ is sent onto the subgroup $\langle t'^n \rangle < \pi$.

Suppose that π is non-torsion-free. Because $\mathbb{Z} \rtimes F$ is torsion-free, there are integers $k \geq 0$, $l > 0$ and $w \in F$ such that $(t'^l t'^{nk} w)^n = e$. The even length of the word w and t being of infinite order lead to a contradiction $kn^2 + n = 0$.

So, the length of the word w is odd. Now, the projection of $(t'^l t'^{nk} w)^n$ on the abelianization of the free group $\langle x_j;\, j \in J \rangle$ yields a non-trivial element which leads again to a contradiction.

Consequently, π is torsion-free and the proof is complete.

$\square$

Now, we are in a position to state:

Proposition 3.6. *Let F be a free group and $n \geq 1$. Then, for any extension $e \to \mathbb{Z} \to \pi \to F \rtimes_\theta \mathbb{Z}_n \to e$ with a torsion-free group π and the trivial induced action of $\mathbb{Z}_n$ on $\mathbb{Z}$, it follows that:*

(1) $\pi \cong \mathbb{Z} \rtimes_\tau F$, where the action $\tau : F \to \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$ is trivial for $n > 2$ and is arbitrary for $n = 1, 2$ provided $F^{\theta(1_n)} = F$ is any free group. Further, for any two non-trivial $\tau_1, \tau_2 : F \to \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$ the groups $\mathbb{Z} \rtimes_{\tau_i} F$ are isomorphic $i = 1, 2$;

(2) $\pi \cong F \rtimes_{\theta'} \mathbb{Z}$ for some action $\theta' : \mathbb{Z} \to \mathrm{Aut}(F)$ or $\pi \cong \langle x, t |\, x t^n x^{-1} = t^{-n} \rangle$ provided $F = \langle y_1, \ldots, y_n \rangle$ and $\theta(1_n)(y_r) = y_{r+1(\bmod n)}$ for $r = 1, \ldots, n$.

Proof. Item (1): Let $F = \langle X \rangle$ and $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ be the trivial action, $xt^nx^{-1} = t^{\pm n}$ and $txt^{-1} = xt^{nm_x}$ for $x \in X$. Because the order of t is infinite, $t^nxt^{-n} = xt^{n^2m_x}$ and $xt^nx^{-1} = t^{\pm n}$ imply $\pm n - n = n^2m_x$. For $n > 2$ this equation has a solution only in the case $n - n = n^2m_x$ which implies $m_x = 0$. If $n = 2$ we get either $m_x = 0$ for $2 - 2 = 4m_x$ or $m_x = -1$ for $-2 - 2 = 4m_x$. Finally, if $n = 1$ we get either $m_x = 0$ or $m_x = -2$.

Hence, $\pi \cong \langle X, t |\, xtx^{-1} = t^{\tau(x)}\ for\ x \in X \rangle \cong \mathbb{Z} \rtimes_\tau F$ where $\tau : F \to \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$ is trivial of $n > 2$. In case $n = 1, 2$ the action τ is arbitrary.

Notice that for any two non-trivial actions $\tau_1, \tau_2 : F \to \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$ the groups $\mathbb{Z} \rtimes_{\tau_1} F$ and $\mathbb{Z} \rtimes_{\tau_2} F$ are isomorphic because there is an automorphism $\alpha : F \to F$ such that $\tau_1 \circ \alpha = \tau_2$.

Item (2): The presentation

$$\pi = \langle y_1, \ldots, y_n, t |\, y_rt^ny_r^{-1} = t^{\pm n},\ ty_rt^{-1} = y_{r+1\,(\bmod\, n)}t^{nm_r}\ \text{for}\ r = 1, \ldots, n \rangle$$

and the order of t imply $\pm n - n = \sum_{r=1}^n m_r$. The substitutions: $t' = t$, $y'_1 = y_1$, $y'_r = y_rt^{\sum_{k=1}^{r-1} nm_k}$ for $r = 2, \ldots, n$ lead to an isomorphism

$\pi \cong \langle y'_1, \ldots, y'_n, t' |\, y'_rt'^ny'^{-1}_r = t'^{\pm n},\ t'y'_rt'^{-1} = y'_{r+1}\ \text{for}\ r = 1, \ldots, n-1\ \text{and}\ t'y'_nt^{-1} = y'_1t'^{(\pm 1 - 1)n} \rangle$.

Hence, $\pi \cong \langle y'_1, \ldots, y'_n, t' |\, y'_rt'^ny'^{-1}_r = t'^n,\ t'y'_rt'^{-1} = y'_{r+1(\bmod\, n)} \rangle \cong F \rtimes_{\theta'} \mathbb{Z}$ for an appropriate action action $\theta' : \mathbb{Z} \to \mathrm{Aut}(F)$ or

$$\pi \cong \langle y'_1, \ldots, y'_n, t' |\, y'_rt'^ny'^{-1}_r = t'^{-n},\ t'y'_rt'^{-1} = y'_{r+1}\ \text{for}\ r = 1, \ldots, n-1\ \text{and}$$

$t'y'_nt'^{-1} = y'_1t'^{-2n} \rangle \cong \langle x, t'' |\, xt''^nx^{-1} = t''^{-n} \rangle$ for $x = y'_1$, $t'' = t'$ and, in view of Remark 3.4, the result follows and the proof is complete.

□

Proposition 3.7. *For $F = \langle y_1, \ldots, y_{n-1}, z_j\,, j \in J \rangle$, $\theta(1_n)(y_r) = y_{r+1}$ for $r = 1, \ldots, n-2$, $\theta(1_n)(y_{n-1}) = (y_1 \cdots y_{n-1})^{-1}$ and $\theta(1_n)(z_j) = y_1^{-1}z_jy_1$ for $j \in J$ and $n \geq 1$ and an extension $e \to \mathbb{Z} \to \pi \to F \rtimes_\theta \mathbb{Z}_n \to e$ with $n \geq 1$, a torsion-free group π and the trivial induced action of $\mathbb{Z}_n$ on $\mathbb{Z}$, it follows that:*

(1) $\pi \cong F \rtimes_\tau \mathbb{Z}$ for some $\tau : \mathbb{Z} \to \mathrm{Aut}(F)$ with $\tau^n(1) = \mathrm{id}_F$ or

(2) $\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_rt^ny_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_jt^nz_j^{-1} = t^n$, $j \in J, j \neq j_0$, $z_{j_0}t^nz_{j_0}^{-1} = t^{-n}, ty_rt^{-1} = y_{r+1},\ r = 1, \ldots, n-2, ty_{n-1}t^{-1} = (y_1 \cdots y_{n-1})^{-1}t^{-n(1+n')}, tz_jt^{-1} = y_1^{-1}z_jy_1t^{n(1+2m')}, j \in J \rangle$ for $n = 2n'$ and some $m' \in \mathbb{Z}$. Further, none of groups in (1) is isomorphic to a group in (2).

Proof. We make use of presentations of π stated in Lemma 3.5.

Item (1): if $\pi = \langle y_1, \ldots, y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1, \ldots, n-1,\ z_j t^n z_j^{-1} = t^n,\ j \in J,\ t y_r t^{-1} = y_{r+1},\ r = 1, \ldots, n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm}, t z_j t^{-1} = y_1^{-1} z_j y_1, j \in J \rangle$ for any n and $m = 0, \ldots, n-2$ then the map

$$\alpha : \pi \longrightarrow \mathbb{Z}$$

given by: $\alpha(t) = 1$, $\alpha(y_1) = \cdots = \alpha(y_{n-1}) = m$ and $\alpha(z_j) = 0$ for $j \in J$ is a well-defined homomorphism which yields the splitting extension

$$e \to K \to \pi \overset{\alpha}{\to} \mathbb{Z} \to e$$

with $K = \operatorname{Ker} \alpha$. Thus,

$$\pi \cong K \rtimes_\tau \mathbb{Z}$$

for $\tau : \mathbb{Z} \to \operatorname{Aut}(K)$ with $\tau(1)(x) = txt^{-1}$ for $x \in K$ $\tau^n(1) = \mathrm{id}_K$. It follows that π is an HNN-extension and, in view of [5, Thorem 17.1], the group K might be chosen to be an amalgamated product.

We aim to show an isomorphism $K \cong F$. The Reidemeister-Schreier rewriting procedure [22, Theorem 2.8, p. 91] yields that K is generated by:

$\gamma_{r,k} = t^k y_r t^{-k-m}$ for $r = 1, \ldots, n-1$ and $k \in \mathbb{Z}$,

$\delta_{j,k} = t^k z_j t^{-k}$ for $j \in J$ and $k \in \mathbb{Z}$.

The relation: $y_r t^n y_r^{-1} = t^n$ leads to

$$\gamma_{r,k+n} = \gamma_{r,k}$$

for $r = 1, \ldots, n-1$;

$z_j t^n z_j^{-1} = t^n$ to

$$\delta_{j,k+n} = \delta_{j,k}$$

for $j \in J$ and $k \in \mathbb{Z}$;

$t y_r t^{-1} = y_{r+1}$ for $r = 1, \ldots, n-2$ to $\gamma_{r,k+1} = \gamma_{r+1,k}$ for $r = 1, \ldots, n-2$ and $k \in \mathbb{Z}$. Consequently,

$$\gamma_{n-1,k} = \gamma_{n-2,k+1} = \cdots = \gamma_{1,n+k-2}$$

for any $k \in \mathbb{Z}$.

Next, $t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm}$ yields

$$\gamma_{n-1,k+1} \gamma_{1,k+m} \cdots \gamma_{n-1,k+m(n-1)} = e$$

for any $k \in \mathbb{Z}$.

Finally, $t z_j t^{-1} = y_1^{-1} z_j y_1$ leads to

$$\delta_{j,k+1} \gamma_{j,k-m}^{-1} \delta_{j,k-m}^{-1} \gamma_{1,k-m} = e$$

for $j \in J$ and $k \in \mathbb{Z}$. Thus, we get the presentation:

$$K = \big\langle \gamma_{r,k}, \delta_{j,k} |\, \gamma_{r,k+n} = \gamma_{r,k},\ \delta_{j,k+n} = \delta_{j,k},\ \gamma_{n-1,k} = \gamma_{n-2,k+1} = \cdots = \gamma_{1,n+k-2},$$

$\delta_{j,k+1}\gamma_{j,k-m}^{-1}\delta_{j,k-m}^{-1}\gamma_{1,k-m} = e$ for $r = 1,\dots,n-1$, $j \in J$ and $k \in \mathbb{Z}\big\rangle$.

Routine computations show that

$$F \cong K = \big\langle \gamma_{1,k}, \delta_{0,j} |\ k = 0,\dots,n-2,\ j \in J \big\rangle$$

and consequently

$$\pi \cong F \rtimes_\tau \mathbb{Z}$$

for some $\tau : \mathbb{Z} \to \mathrm{Aut}(F)$.

Item (2): in view of Lemma 3.5, remaining groups have the presentations:
$\pi = \big\langle y_1,\dots,y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1,\dots,n-1,\ z_j t^n z_j^{-1} = t^n,$
$j \in J, j \neq j_0,\ z_{j_0} t^n z_{j_0}^{-1} = t^{-n}, t y_r t^{-1} = y_{r+1},\ r = 1,\dots,n-2, t y_{n-1} t^{-1} =$
$(y_1 \cdots y_{n-1})^{-1} t^{-n(1+n')}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{n(1+2m')}, j \in J\big\rangle$ for $n = 2n'$.

The groups $\pi = \big\langle y_1,\dots,y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1,\dots,n-1,\ z_j t^n z_j^{-1} =$
$t^n,\ j \in J,\ t y_r t^{-1} = y_{r+1},\ r = 1,\dots,n-2, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{nm},$
$t z_j t^{-1} = y_1^{-1} z_j y_1, j \in J\big\rangle$ for any n and $m = 0,\dots,n-2$

and

$\pi = \big\langle y_1,\dots,y_{n-1}, z_j, j \in J, t |\, y_r t^n y_r^{-1} = t^n,\ r = 1,\dots,n-1,\ z_j t^n z_j^{-1} = t^n,$
$j \in J, j \neq j_0,\ z_{j_0} t^n z_{j_0}^{-1} = t^{-n}, t y_r t^{-1} = y_{r+1},\ r = 1,\dots,n-2, t y_{n-1} t^{-1} =$
$(y_1 \cdots y_{n-1})^{-1} t^{-n(1+n')}, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{n(1+2m')}, j \in J\big\rangle$ for $n = 2n'$

are not isomorphic because their abelianizations are isomorphic to

$$\bigoplus_{r=1}^{n-1} \mathbb{Z} \oplus \bigoplus_{j \in J} \mathbb{Z} \oplus \mathbb{Z}_n \quad\text{and}\quad \bigoplus_{r=1}^{n-2} \mathbb{Z} \oplus \mathbb{Z}_{2n} \oplus \bigoplus_{j \in J} \mathbb{Z} \oplus \mathbb{Z}_n,$$

respectively and the proof follows.

□

To state the main result, we need the lemma below a proof of which is straightforward.

Lemma 3.8. *Let H and G_i be groups with actions $\tau_i : H \to \mathrm{Aut}(G_i)$ for $i \in I$.*

(1) *If $\tau : H \to \mathrm{Aut}(*G_i)_{i\in I}$ with $\tau|_{G_i} = \tau_i$ for $i \in I$, then*

$$(*G_i)_{i\in I} \rtimes_\tau H \cong (*_H(G_i \rtimes_{\tau_i} H))_{i\in I};$$

(2) *if* $e \to \pi' \to \pi \xrightarrow{p} (*G_i \rtimes_\tau H)_{i\in I} \to e$ *are extensions and* $\pi_i = p^{-1}(G_i \rtimes_{\tau_i} H)$, *then* $e \to \pi' \to \pi_i \to G_i \rtimes_{\tau_i} H \to e$ *are extensions for* $i \in I$ *and*

$$\pi \cong *_{p^{-1}(H)}\pi_i.$$

Now, given an action $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ with $n \geq 1$ and the decomposition $F = F^{\theta(1_n)} * (*_{i\in I}F_i) * (*_{\lambda\in\Lambda}F_\lambda)$ write: $\theta_0 : \mathbb{Z}_n \to \mathrm{Aut}(F^{\theta(1_n)})$, $\theta_i : \mathbb{Z}_n \to \mathrm{Aut}(F_i)$ and $\theta_\lambda : \mathbb{Z}_n \to \mathrm{Aut}(F_\lambda)$ for the induced actions with $i \in I$ and $\lambda \in \Lambda$. Then, in view of Lemma 3.8(2), any extension $e \to \mathbb{Z} \to \pi \to F \rtimes_\theta \mathbb{Z}_n \to e$ leads to a family of extensions:

$$e \to \mathbb{Z} \to \pi_0 \to F^{\theta(1_n)} \rtimes_{\theta_0} \mathbb{Z}_n \to e,$$

$$e \to \mathbb{Z} \to \pi_i \to F_i \rtimes_{\theta_i} \mathbb{Z}_n \to e \text{ for } i \in I,$$

$$e \to \mathbb{Z} \to \pi_\lambda \to F_\lambda \rtimes_{\theta_\lambda} \mathbb{Z}_n \to e \text{ for } \lambda \in \Lambda$$

and an isomorphism

$$\pi \cong *_{\mathbb{Z}}(\pi_0 * (*\pi_i)_{i\in I} * (*\pi_\lambda)_{\lambda\in\Lambda}).$$

Consequently, Propositions 3.6 and 3.7 yield:

Theorem 3.9. *Let* F *be a free group,* $n \geq 1$, $\theta : \mathbb{Z}_n \to \mathrm{Aut}(F)$ *an action and* $F = F^{\theta(1_n)} * (*_{i\in I}F_i) * (*_{\lambda\in\Lambda}F_\lambda)$.

There is an action of the group $F \rtimes_\theta \mathbb{Z}_n$ *on some homotopy circle* $\Sigma(1)$ *with* $\dim \Sigma(1) \leq 2$ *and the homotopy type of the orbit space*

$$\Sigma(1)/(F \rtimes_\theta \mathbb{Z}_n) \cong K(\pi, 1),$$

where $\pi \cong *_{\mathbb{Z}}(\pi_0 * (*_{i\in I}\pi_i * (*_{\lambda\in\Lambda}\pi_\lambda)))$.
Further:

(1) $\pi_0 \cong \mathbb{Z} \rtimes_\tau F^{\theta(1_n)}$ *for all possible actions* $\tau : F^{\theta(1_n)} \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z})) \cong \{\pm 1\}$. *In particular,* $\pi \cong \mathbb{Z} \times F^{\theta(1_n)}$ *for* $n > 2$;

(2) $\pi_i \cong F_i \rtimes_{\theta'_i} \mathbb{Z}$ *with* $F_i = \langle y_{i,1}, \ldots, y_{i,n}\rangle$ *and* $\theta'_i : \mathbb{Z} \to \mathbb{Z}_n \xrightarrow{\theta_i} \mathrm{Aut}(F_i)$, *if the induced action* $F_i \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ *is trivial and* $\pi_i \cong \langle x, t |\, xt^n x^{-1} = t^{-n}\rangle$ *otherwise;*

(3) $\pi_\lambda \cong F_\lambda \rtimes_{\tau_\lambda} \mathbb{Z}$ *with* $F_\lambda = \langle y_{\lambda,1}, \ldots, y_{\lambda,n-1}, z_j \mid j \in J_\lambda\rangle$ *and some* $\tau_\lambda : \mathbb{Z} \to \mathrm{Aut}(F_\lambda)$ *for any* n, *if the induced action* $F_\lambda \to \mathrm{Aut}(H^1(\Sigma(1), \mathbb{Z}))$ *is trivial and* $\pi_\lambda \cong \langle y_{\lambda,1}, \ldots, y_{\lambda,n-1}, z_j, j \in J_\lambda, t |\, y_{\lambda,r}t^n y_{\lambda,r}^{-1} = t^n,\, r = 1, \ldots, n-1,\, z_j t^n z_j^{-1} = t^n,\, j \in J_\lambda, j \neq j_0,\, , z_{j_0} t^n z_{j_0}^{-1} = t^{-n}, t y_r t^{-1} = y_{r+1},\, r = 1, \ldots, n-2,\, t y_{n-1} t^{-1} = (y_1 \cdots y_{n-1})^{-1} t^{-n(1+n')},\, t z_j t^{-1} = y_1^{-1} z_j y_1 t^{n(1+2m')}, j \in J_\lambda\rangle$ *for* $n = 2n'$, *otherwise.*

Notice that Theorems 3.2 and 3.9 show that the condition $\operatorname{vcd} G \leq \dim \Sigma(1) - 1$ stated in Proposition 1.7 does not suffice for the existence of an action of G on $\Sigma(1)$.

There are many more groups G with $\operatorname{vcd} G \leq 1$. In view of [31, Chapter II, Proposition 11], $\operatorname{vcd} G \leq 1$ provided G is the fundamental group of a finite graph of finite groups. Further, according to [4, Chapter VIII, Example 2 page 228]: "if we drop the requirement that the graph of finite groups be finite and require instead that there be a bound on the orders of the finite groups, then it is still true that $\operatorname{vcd} G \leq 1$". Further, if $\operatorname{vcd} G \leq 1$, then G is the fundamental group of a graph of finite groups. A proof of that result can be found e.g., in [29].

In order to study a new family of groups G with $\operatorname{vcd} G \leq 1$, we begin by a slightly more general situation. Namely, consider groups of the form $(*_G G_i)_{i\in I}$. Because $K((*G_i)_{i\in I}, 1) = \bigvee_{i\in I} K(G_i, 1)$ and, in view of [3, Proposition 1.1] (see also [4, Theorem 7.3, Chapter II]), it holds $\operatorname{gd}(G_1 *_G G_2) \leq \max\{\operatorname{gd} G_i, \operatorname{gd} G + 1;\ i = 1, 2\}$, we derive

$$\operatorname{gd}(*_G G_i)_{i\in I} < \infty \tag{3.3}$$

provided there is an upper bound of the set $\{\operatorname{gd} G_i |$ for $i \in I\}$. Then, $\operatorname{vcd}(*_G G_i)_{i\in I} = 1$ provided G_i for $i \in I$ and G are finite groups with bounded orders. If $(*_G G_i)_{i\in I}$ acts on a $\Sigma(1)$, then by means of Proposition 2.4, we get $(*_G G_i)_{i\in I} \cong (*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i\in I}$ for some $k, n_i \geq 1$. Notice that in view of monomorphisms $\mathbb{Z}_k \hookrightarrow \mathbb{Z}_{n_i}$, it holds $k \mid n_i$ for $i \in I$.

Further, we may state:

Proposition 3.10. *Let G_i be groups and $H < G < G_i$ such subgroups that H is normal in G_i for $i \in I$. Then there is an extension*

$$e \to H \longrightarrow (*_G G_i)_{i\in I} \longrightarrow (*_{(G/H)}(G_i/H))_{i\in I} \to e.$$

In particular, for $H = G$, we have the extension

$$e \to G \longrightarrow (*_G G_i)_{i\in I} \longrightarrow (*(G_i/G))_{i\in I} \to e.$$

Proof. By the functoriality of the amalgamated free product, we derive an epimorphism

$$(*_G G_i)_{i\in I} \xrightarrow{\varphi} (*_{(G/H)}(G_i/H))_{i\in I} \to e$$

with $H \subseteq \operatorname{Ker} \varphi$.

Now, let S_i be a set of representatives of the set G_i/G of right cosets for $i \in I$. In view of [31, Theorem 1], every element $x \in *_G G_i$ may be presented by a unique reduced word (i.e., there is a sequence $(g, s_{i_1}, \ldots, s_{i_n})$ with $g \in G$ and $s_{i_k} \in S_{i_k}$,

$s_{i_k} \neq e$ for $k = 1, \ldots, n$ such that $x = gs_{i_1} \cdots s_{i_n}$ in a unique form). Because $\bar{S}_i = \{\bar{s}_i = Hs_i;\ s_i \in S_i\}$ is a set of representatives of the set $(G_i/H)/(G/H) \cong G_i/G$ of right cosets for $i \in I$, every element $\bar{x} \in *_{(G/H)}(G_i/H)$ may be also presented by a unique reduced word.

Now, for $x = gs_{i_1} \cdots s_{i_n} \in \operatorname{Ker} \varphi$, we get $\bar{e} = \bar{g}\bar{s}_{i_1} \cdots \bar{s}_{i_n}$. Consequently $s_{i_1} = \cdots = s_{i_n} = e$, $x = g \in H$ and the result follows.

$\square$

Notice that Proposition 1.8, Proposition 3.10 and (3.3) lead to:

Theorem 3.11. *Let $e \to \mathbb{Z} \to \pi_i \to G_i \to e$ be extensions such that $\operatorname{gd} \pi_i < \infty$ and $\mathbb{Z} < \pi < \pi_i$ for $i \in I$. Then there is an action*

$$(*_{(\pi/\mathbb{Z})} G_i)_{i \in I} \times \Sigma(1) \longrightarrow \Sigma(1)$$

with $\dim \Sigma(1) \leq \max\{\operatorname{gd} \pi_i, \operatorname{gd} \pi + 1;\ i \in I\}$ provided there is an upper bound of the set $\{\operatorname{gd} \pi_i |$ for $i \in I\}$.

In particular, for $\pi = \mathbb{Z}$, there is an action

$$(*G_i)_{i \in I} \times \Sigma(1) \longrightarrow \Sigma(1)$$

with $\dim \Sigma(1) \leq \max\{\operatorname{gd} \pi_i, 2 \mid\ i \in I\}$.

In view of the extensions $0 \to \mathbb{Z} \to \mathbb{Z}_i \to \mathbb{Z}_{n_i} \to 0$ for with $\mathbb{Z}_i = \mathbb{Z}$ for $i \in I$, Theorem 3.11 and (3.3) yield:

Corollary 3.12. *Given a set I and positive integers k, n_i with $k \mid n_i$ for $i \in I$, there is an action*

$$(*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i \in I} \times \Sigma(1) \longrightarrow \Sigma(1)$$

with $\dim \Sigma(1) = 2$. Further, the induced action of $*_{\mathbb{Z}_k} \mathbb{Z}_{n_i}$ on $H^1(\Sigma(1), \mathbb{Z})$ is trivial and there is exactly one homotopy type of the orbit spaces $\Sigma(1)/(*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})$ for all possible actions of $*_{\mathbb{Z}_k} \mathbb{Z}_{n_i}$ on homotopy circles $\Sigma(1)$.

Proof. Certainly Theorem 3.11 leads to an action $(*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i \in I} \times \Sigma(1) \longrightarrow \Sigma(1)$ with $\dim \Sigma(1) = 2$. In view of Remark 2.7, any such an action determines the trivial action of $(*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i \in I}$ on $H^1(\Sigma(1), \mathbb{Z})$.

Further, given an extension $e \to \mathbb{Z} \to \pi \to (*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i \in I} \to e$ with a torsion-free group π, we can easily see that there is a system of commutative diagrams

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_i & \longrightarrow & \mathbb{Z}_{n_i} & \longrightarrow & e \\ & & \| & & \downarrow & & \downarrow & & \\ e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & (*_{\mathbb{Z}_k} \mathbb{Z}_{n_i})_{i \in I} & \longrightarrow & e \end{array}$$

with $\mathbb{Z}_i = \mathbb{Z}$ for $i \in I$ which determine the commutative diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \longrightarrow & (*_{\mathbb{Z}}\mathbb{Z}_i)_{i\in I} & \longrightarrow & (*_{\mathbb{Z}_k}\mathbb{Z}_{n_i})_{i\in I} & \longrightarrow & e \\
 & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & (*_{\mathbb{Z}_k}\mathbb{Z}_{n_i})_{i\in I} & \longrightarrow & e,
\end{array}$$

where $\beta : (*_{\mathbb{Z}_k}\mathbb{Z}_{n_i})_{i\in I} \to (*_{\mathbb{Z}_k}\mathbb{Z}_{n_i})_{i\in I}$ is an automorphism. Hence, $\alpha : (*_{\mathbb{Z}}\mathbb{Z}_i)_{i\in I} \to \pi$ is an isomorphism and we derive that there is exactly one homotopy type of the orbit spaces $\Sigma(1)/(*_{\mathbb{Z}_k}\mathbb{Z}_{n_i})_{i\in I}$. This completes the proof.

□

It is well-known that $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ for the infinite dihedral group D_∞. Further, by [31], there are isomorphisms $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ and $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Hence, Proposition 3.12 leads to actions of those groups on $\Sigma(1)$ with $\dim \Sigma(1) = 2$ (cf. Example 1.5(2)).

On the other hand, there are groups G with $\operatorname{vcd} G = 1$ which cannot act on any $\Sigma(1)$ independently of $\dim \Sigma(1)$. In fact, by means of [30], it holds $\operatorname{vcd} SL_2(\mathbb{Z}) = 1$. Hence, $\operatorname{vcd} GL_2(\mathbb{Z}) = \operatorname{vcd}(SL_2(\mathbb{Z}) \rtimes \mathbb{Z}_2) = 1$ as well. But by Remark 2.5 the group $GL_2(\mathbb{Z})$ cannot act on any $\Sigma(1)$. Further, given a group G with $\operatorname{vcd} G = 1$ and a finite group G_0, we get $\operatorname{vcd}(G \rtimes G_0) = \operatorname{vcd}(G_0 \rtimes G) = 1$. Nevertheless by means of Proposition 2.4 groups $G \rtimes G_0$ and $G_0 \rtimes G$ cannot act on any $\Sigma(1)$ provided G_0 is non-cyclic.

To conclude our study of groups G with $\operatorname{vcd} G = 1$, we consider the situation where $\Sigma(1)$ is a manifold. This includes the case of $\Sigma(1)$ as a real vector bundle over $\mathbb{S}^1$ which is an n-manifold without boundary with $n \geq 2$.

Proposition 3.13. *If a finitely generated group G with $\operatorname{vcd} G < \infty$ acts on an n-manifold M with $n \geq 2$ and of the homotopy type of the circle, then:*

(1) $\operatorname{vcd} G \leq n - 2$ *if and only if M/G is not compact. In particular, $G \cong \mathbb{Z}_m$ for some $m \geq 1$, if and only if M/G is not compact provided for $n = 2$;*

(2) *G is any of the groups listed in Corollary* 2.6(2)-(3), *if and only if M/G is compact provided $n = 2$.*

Proof. Item (1): Let $G' < G$ such that $G : G' < \infty$ and $\operatorname{cd} G' \leq n - 2$.

$\Rightarrow$: Suppose that M/G is compact. Then, M/G' is also compact and $H^n(M, \mathbb{Z}) \neq 0$. Because $\operatorname{vcd} G' \leq n - 2$, the Leray-Cartan spectral sequence associated with the fibration $M \to M/G' \to K(G', 1)$ leads to a contradiction $H^k(M, \mathbb{Z}) = 0$ for $k \geq m$.

$\Leftarrow$: If $G'' = \operatorname{Ker}(G' \to \operatorname{Aut}(\mathbb{Z}))$, then $G' : G'' \leq 2$ and $\operatorname{vcd} G = \operatorname{vcd} G''$. Further, Proposition 1.7 leads to $\operatorname{vcd} G \leq n - 1$. Now, given a G''-module A, the extension

$e \to \mathbb{Z} \to \pi \to G \to e$ associated with the G-action on M leads to $e \to \mathbb{Z} \to \pi'' \to G'' \to e$. Hence, the corresponding Lyndon-Hochschild spectral sequence yields $H^{n-1}(G'', A) = 0$ and $\operatorname{vcd} G \leq n-2$.

If $n = 2$, then $\operatorname{vcd} G = 0$ and Proposition 2.4 leads to $G \cong \mathbb{Z}_m$ for some $m \geq 1$.

Item (2) $\Rightarrow$: Let G be any of the groups listed in Corollary 2.6(2)-(3). Then Theorem 2.10 shows that M/G is compact.

$\Leftarrow$: If M/G is compact then [4, Proposition 8.1, Chapter VIII] yields $\operatorname{cd} \pi_1(M/G) = 2$ and by [29] we get for ends $e(G) = e(M/G) = 2$. Hence, Theorem 2.1 and Corollary 2.6 imply the result and the proof is complete.

□

At the end, we present a family of groups with an infinite virtual cohomological dimension and acting on a homotopy circle. To conclude that, we study the family of the locally cyclic groups. We begin by presenting a very explicit description of these groups. First, we recall that a group G is called *locally cyclic* (resp. *locally finite*) if each of its finitely generated subgroups is cyclic (resp. finite). Notice that locally cyclic (resp. finite) groups are closed with respect to subgroups and quotient groups. Further, [34, Theorem 2.3] implies that any locally cyclic group has period 2 after 0- or 1-step and, by means of Proposition 2.4, any torsion locally finite group acting on some $\Sigma(1)$ is locally cyclic.

In view of [28, II.2.k. Theorem], locally cyclic groups are characterized as follows.

Theorem 3.14. *A group is locally cyclic if and only if it is isomorphic to a subgroup of homomorphic image of additive rationals* $\mathbb{Q}$.

Further, we show:

Proposition 3.15. *A group G is locally cyclic if and only if G is a subgroup of* $\mathbb{Q}$ *or* $\mathbb{Q}/\mathbb{Z}$.

Proof. Certainly, any subgroup of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ is locally cyclic.

Let now G be a locally cyclic group. Then, by Theorem 3.14, the group G is isomorphic to a subgroup of homomorphic image of $\mathbb{Q}$. Hence, there are such subgroups $\mathbb{A} < \mathbb{B} < \mathbb{Q}$ that $G = \mathbb{B}/\mathbb{A}$. If $\mathbb{A}$ is trivial, then $G = \mathbb{B} < \mathbb{Q}$.

If $\mathbb{A}$ is non-trivial, then $n\mathbb{Z} < \mathbb{A}$ for some $n \geq 1$. Then for any $\frac{x}{y} \in G$, we get $ny\frac{x}{y} = nx \in \mathbb{A}$ and so $\mathbb{Q}/\mathbb{A}$ is a torsion group. Because $\mathbb{Q}/\mathbb{A}$ is also divisible, we obtain that

$$\mathbb{Q}/\mathbb{A} \cong \bigoplus \mathbb{Z}_{q^\infty}$$

for some primes q and

$$\mathbb{Q}/\mathbb{A} < \mathbb{Q}/\mathbb{Z} \cong \bigoplus \mathbb{Z}_{p^\infty}$$

for all primes p. Because, $\mathbb{B}/\mathbb{A} < \mathbb{Q}/\mathbb{A}$, we derive that $\mathbb{B}/\mathbb{A} < \mathbb{Q}/\mathbb{Z}$ and the proof is complete.

□

So, our family of groups are divided into two subfamilies, where the first one consists of subgroups of $\mathbb{Q}$ and the second one subgroups of $\mathbb{Q}/\mathbb{Z}$. We will see that the first subfamily have the cohomological dimension two and the second one has the infinite cohomology dimension.

First, we show that $\operatorname{gd} \mathbb{A} = 2$ for any non-cyclic subgroup $\mathbb{A} < \mathbb{Q}$ of the additive group of the rationals $\mathbb{Q}$. To aim that, consider the direct system

$$\mathbb{Z} \overset{i_2}{\hookrightarrow} \mathbb{Z} \hookrightarrow \cdots \hookrightarrow \mathbb{Z} \overset{i_n}{\hookrightarrow} \mathbb{Z} \hookrightarrow \cdots$$

with $\lim\limits_{\to} \mathbb{Z} \cong \mathbb{Q}$, where $i_n : \mathbb{Z} \hookrightarrow \mathbb{Z}$ is the multiplication by n for $n \geq 2$. Next, fix maps $f_n : X_{n-1} = \mathbb{S}^1 \to X_n = \mathbb{S}^1$ of degree n for $n \geq 2$ and consider the telescope construction

$$\mathcal{T}(\mathbb{S}^1) = (\bigsqcup_{n=1}^{\infty} X_n \times I)/\sim,$$

where $I = [0,1]$ is the unit interval and $(x_n, 1) \sim (f_{n+1}(x_n), 0)$ for $x_n \in X_n$ and $n \geq 1$. Then $\mathcal{T}(\mathbb{S}^1)$ is a CW-complex with $\dim \mathcal{T}(\mathbb{S}^1) = 2$. Notice that the maps $f_n : X_{n-1} = \mathbb{S}^1 \to X_n = \mathbb{S}^1$ above lift, via the exponential map $\exp : \mathbb{R}^1 \to \mathbb{S}^1$, to $g_n : Y_{n-1} = \mathbb{R}^1 \to Y_n = \mathbb{R}^1$ for $n \geq 2$. Then the corresponding telescope construction $\mathcal{T}(\mathbb{R}^1)$ leads to the universal covering of $\mathcal{T}(\mathbb{R}^1) \to \mathcal{T}(\mathbb{S}^1)$, where $\mathcal{T}(\mathbb{R}^1)$ is contractible. Hence, $\pi_1(\mathcal{T}(\mathbb{S}^1)) \cong \lim\limits_{\to} \mathbb{Z} \cong \mathbb{Q}$, $\pi_n(\mathcal{T}(\mathbb{S}^1)) = 0$ for $n \neq 1$ and so $\mathcal{T}(\mathbb{S}^1) = K(\mathbb{Q}, 1)$. Consequently, $\operatorname{gd} \mathbb{A} = 2$ for any non-cyclic subgroup $\mathbb{A} < \mathbb{Q}$ and, in view of Proposition 1.3, there is an action

$$\mathbb{A} \times \Sigma(1) \to \Sigma(1)$$

with $\dim \Sigma(1) \leq 3$.

First, we study actions of non-cyclic subgroups $\mathbb{A} < \mathbb{Q}$.

Proposition 3.16. *Let $\mathbb{A} < \mathbb{Q}$ be a noncyclic subgroup. Then:*

(1) *for any central extension $e \to \mathbb{Z} \to \pi \to \mathbb{A} \to e$ the group π is abelian torsion-free and with rank two;*

(2) $H^2(\mathbb{A}, \mathbb{Z}) \cong \operatorname{Ext}(\mathbb{A}, \mathbb{Z}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_p \mathbb{Z}_{p^\infty}$ *provided p is a prime with $p\mathbb{A} \subsetneqq \mathbb{A}$ and $\mathbb{Z}$ the trivial $\mathbb{A}$-module;*

(3) $H^2_\tau(\mathbb{A},\tilde{\mathbb{Z}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *for any non-trivial* $\mathbb{A}$*-module structure* τ *on* $\mathbb{Z}$ *provided that* $2\mathbb{A} \lneqq \mathbb{A}$.

Proof. Item (1): Suppose $e \to \mathbb{Z} \to \pi \overset{\alpha}{\to} \mathbb{A} \to e$ is a central extension with a noncyclic $\mathbb{A} < \mathbb{Q}$ and the commutator $[x,y] \neq 0$ for some $x,y \in \pi$. Because $\alpha([x,y]) = e$, we may set $[x,y] = m \neq 0$ for some $m \in \mathbb{Z}$. Let $\alpha(x) = \frac{k}{l}$ for some $k,l \in \mathbb{Z}$. But $\mathbb{A} < \mathbb{Q}$ is noncyclic, so there is such $\frac{n}{p} \in \mathbb{A}$ that p is a prime with $p \nmid l,m,n$. If $\alpha(z) = \frac{kn}{p}$, then $\alpha(z^p) = \alpha(x^{ln}) = kn$ and $z^p = ax^{ln}$ for some $a \in \mathbb{Z}$. Because the extension $e \to \mathbb{Z} \to \pi \to \mathbb{A} \to e$ is central, the formula (3.4) leads to a contradiction $lmn = [x^{ln}, y] = [z^p, y] = p[z,y]$. Further, the isomorphisms $\mathbb{Z}\otimes\mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{A} \otimes \mathbb{Q} \cong \mathbb{Q}$ imply $\pi \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}$. But π is torsion-free, as an extension of torsion-free groups, so we deduce the rank of $\mathbb{A}$ is two.

Item (2): In view of (1) we get that $H^2(\mathbb{A},\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{A},\mathbb{Z})$. By [20, Chapter III] the group $\mathrm{Ext}(\mathbb{A},\mathbb{Z})$ is divisible with $\mathrm{card}(H^2(\mathbb{A},\mathbb{Z})) = 2^{\aleph_0}$. If p is a prime such that $p\mathbb{A} \lneqq \mathbb{A}$ then the extension $e \to p\mathbb{A} \to \mathbb{A} \to \mathbb{Z}_p \to e$ implies that the Prüffer group $\mathbb{Z}_{p^\infty}$ is a direct summand of $\mathrm{Ext}(\mathbb{A},\mathbb{Z})$.

Item (3): Given $\mathbb{A} < \mathbb{Q}$ and an extension $e \to \mathbb{Z} \to \pi \to \mathbb{Z} \to e$, any action of $\mathbb{A}$ on $\mathbb{Z}$ is trivial provided $2\mathbb{A} = \mathbb{A}$. But $2\mathbb{A} \lneqq \mathbb{A}$ implies an isomorphism $\mathbb{Z}/2\mathbb{A} \cong \mathbb{Z}_2$ and so an existence of a non-trivial action of $\mathbb{A}$ on $\mathbb{Z}$. Then, for a non-trivial an $\mathbb{A}$-module structure τ on $\mathbb{Z}$, the Lyndon-Hochschild-Serre spectral sequence applied to the extension

$$e \to 2\mathbb{A} \to \mathbb{A} \to \mathbb{Z}_2 \to e$$

leads to $E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(2\mathbb{A},\mathbb{Z})) \Rightarrow H^{p+q}_\tau(\mathbb{A},\tilde{\mathbb{Z}})$ and the extension

$$e \to E_2^{0,2} \to H^2(\mathbb{A},\tilde{\mathbb{Z}}) \to E_2^{2,0} \to e.$$

But $E_2^{2,0} = H^2(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}_2$ and, in view of Proposition 3.16(2), it holds $E_2^{0,2} = (\bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{p\mathbb{A}\lneqq\mathbb{A}} \mathbb{Z}_{p^\infty})^{\mathbb{Z}_2} \cong \mathbb{Z}_2$. Consequently $H^2_\tau(\mathbb{A},\tilde{\mathbb{Z}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the proof is complete.

□

Given a set P of primes, write $\mathbb{Z}_P$ for the localization of $\mathbb{Z}$ with respect to the muliplicative system generated by P. Based on the above we obtain the following result.

Theorem 3.17. *There are* $2^{\aleph_0}$ *distinct homotopy types of orbit spaces* $\Sigma(1)/\mathbb{Z}_P$ *with respect to actions of* $\mathbb{Z}_P$ *on* $\Sigma(1)$ *and any such an action induces the trivial action on* $H^1(\Sigma(1),\mathbb{Z})$.

Proof. First, suppose that P is an infinite set of primes. Then for any subset $P' \subseteq P$ there is an extension

$$e \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}_{P'} \oplus \mathbb{Z}_{P\setminus P'} \xrightarrow{\beta} \mathbb{Z}_P \to e,$$

where $\alpha(1) = (1, -1)$ and the restriction maps $\beta|_{\mathbb{Z}_{P'}}$, $\beta|_{\mathbb{Z}_{P\setminus P'}}$ are the inclusion maps $\mathbb{Z}_{P'} \hookrightarrow \mathbb{Z}_P$, $\mathbb{Z}_{P'} \hookrightarrow \mathbb{Z}_P$, respectively. Notice that for any two distinct subsets $P, P'' \subseteq P$, the groups $\mathbb{Z}_{P'} \oplus \mathbb{Z}_{P\setminus P'}$ and $\mathbb{Z}_{P''} \oplus \mathbb{Z}_{P\setminus P''}$ are not isomorphic.

If $P = \{p_1, \ldots, p_n\}$ for some primes $p_1, \ldots, p_n$, then $\mathbb{Z}_P = \mathbb{Z}[\frac{1}{p_1 \cdots p_n}]$. Next for any two distinct primes $p, p' \notin P$ and any sequence $(m_k)_{k\geq 1}$ of natural numbers consider the subgroup $\mathbb{A}(p, (m_k)) = \big\langle 1, \frac{p^{km_k}}{(p_1\cdots p_n)^{km_k}} \text{ for } k \geq 1 \big\rangle$ and $\mathbb{A}'(p', (m_k)) = \big\langle 1, \frac{p'^{km_k}}{(p_1\cdots p_n)^{km_k}} \text{ for } k \geq 1 \big\rangle$ of $\mathbb{Z}_P$. Then we get an extension

$$e \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{A}(p, (m_k)) \oplus \mathbb{A}'(p', (m_k)) \xrightarrow{\beta} \mathbb{Z}_P \to e,$$

with the maps α and β defined as above. Notice that the groups $\mathbb{A}(p, (m_k)) \oplus \mathbb{A}'(p', (m_k))$ and $\mathbb{A}(p, (\bar{m}_k)) \oplus \mathbb{A}'(p', (\bar{m}_k))$ are not isomorphic for two distinct sequences (m_k) and $(\bar{m}_k)$ and this completes the proof.

□

We close the paper with the study of actions on $\Sigma(1)$ of groups of the second subfamily which are subgroups of $\mathbb{Q}/\mathbb{Z}$. Let $p : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the quotient map. Then, $\operatorname{gd} p^{-1}(\mathbb{A}) \leq 2$ and the extension $e \to \mathbb{Z} \to p^{-1}(\mathbb{A}) \to \mathbb{A} \to e$ yields, in view of Proposition 1.3, an action

$$\mathbb{A} \times \Sigma(1) \to \Sigma(1)$$

with $\dim \Sigma(1) \leq 2$. Further, notice that for any non-cyclic subgroup $\mathbb{A}$, this extension leads also to

$$e \to \mathbb{Z} \to \operatorname{Ext}(\mathbb{A}, \mathbb{Z}) \to \operatorname{Ext}(p^{-1}(\mathbb{A}), \mathbb{Z}) \to e.$$

Because $\operatorname{card}(\operatorname{Ext}(p^{-1}(\mathbb{A}), \mathbb{Z})) = 2^{\aleph_0}$, we conclude that

$$\operatorname{card}(\operatorname{Ext}(\mathbb{A}, \mathbb{Z})) = 2^{\aleph_0}$$

for any non-cyclic $\mathbb{A} < \mathbb{Q}/\mathbb{Z}$.

To present the next result, we need:

Lemma 3.18. *If $0 \to A \to G \xrightarrow{p} B \to 0$ is a central extension with a torsion-free abelian group A and a torsion abelian group B, then G is an abelian group.*

Proof. First, invoke one Hall's commutator formulae

$$[g_1g_2, g_3] = [g_1, [g_2, g_3]][g_2, g_3][g_1, g_3] \tag{3.4}$$

for any $g_1, g_2, g_3 \in G$. Because B is abelian, $[g_2, g_3] \in A$. But A is in the center of G, so we get $[g_1g_2, g_3] = [g_2, g_3][g_1, g_3]$. In particular, $[g_1^2, g_2] = [g_1, g_2]^2$ and the inductive argument leads to $[g_1^n, g_2] = [g_1, g_2]^n$ for any $n > 0$.

Let now $p(g_1)^n = 0$ for some $n > 0$. Then $g_1^n \in A$ and $[g_1^n, g_2] = [g_1, g_2]^n = 0$. Because $[g_1, g_2] \in A$ which is torsion-free, the result follows.

□

Theorem 3.19. *For any subgroup $\mathbb{A} < \mathbb{Q}/\mathbb{Z}$ there is an action of $\mathbb{A}$ on some $\Sigma(1)$ and exactly one homotopy type of the orbit spaces $\Sigma(1)/\mathbb{A}$ for all its possible actions on homotopy circles $\Sigma(1)$. Further, any such an action induces the trivial action on $H^1(\Sigma(1), \mathbb{Z})$.*

Proof. Certainly, we may assume that $\mathbb{A} < \mathbb{Q}/\mathbb{Z}$ is a non-trivial subgroup. Then, in view of the canonical extension

$$e \to \mathbb{Z} \to p^{-1}(\mathbb{A}) \to \mathbb{A} \to e,$$

we get $\operatorname{gd} p^{-1}(\mathbb{A}) \leq 2$ and Proposition 1.3 leads to an action $\mathbb{A} \times \Sigma(1) \to \Sigma(1)$ with $\dim \Sigma(1) \leq 2$.

Now, given an action $\mathbb{A} \times \Sigma(1) \to \Sigma(1)$, consider the associated extension

$$e \to \mathbb{Z} \xrightarrow{\alpha} \pi \xrightarrow{\beta} \mathbb{A} \to 0.$$

Because the group $\mathbb{A}$ is torsion, in view of Remark 2.7, the induced action of $\mathbb{A}$ on $H^1(\Sigma(1), \mathbb{Z})$ is trivial. Then, by Lemma 3.18 we get that π is abelian. Because $\mathbb{A}$ is torsion, for any non-trivial $x \in \pi$ there are uniquely determined relatively prime integers n, m such that $mx = n\alpha(1)$, where m is the order of $\beta(x)$. Then the map

$$\varphi : \pi \to p^{-1}(\mathbb{A})$$

given by $\varphi(0) = 0$ and $\varphi(x) = \frac{n}{m}$ provided $mx = n\alpha(1)$ is a well-defined injection. To show that φ is a homomorphism, take $x_1, x_2 \in \pi$ with $m_1x_1 = n_1\alpha(1)$ and $m_2x_2 = n_2\alpha(1)$. Because the least common multiple $[m_1, m_2]$ is the order of $\beta(x_1 + x_2)$, we derive that $[m_1, m_2](x_1 + x_2) = (\frac{[m_1,m_2]}{m_2}n_1 + \frac{[m_1,m_2]}{m_1}n_2)\alpha(1)$. Consequently, $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$. Further, the relation $\varphi\alpha(1) = 1$ leads to such a monomorphism

$\bar{\varphi} : \mathbb{A} \hookrightarrow \mathbb{A}$ that the diagram

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \pi & \xrightarrow{\beta} & \mathbb{A} & \longrightarrow & e \\
 & & \Big\| & & \varphi \Big\downarrow & & \bar{\varphi} \Big\downarrow & & \\
e & \longrightarrow & \mathbb{Z} & \longrightarrow & p^{-1}(\mathbb{A}) & \xrightarrow{p} & \mathbb{A} & \longrightarrow & e
\end{array}$$

commutes. Next, notice that a torsion abelian group $\mathbb{B}$ is co-Hopfian if and only if each its p-primary component $\mathbb{B}_p$ is co-Hopfian, where p runs over the set of all primes. This follows because any homomorphism preserves the p-primary components. The p-primary component of $\mathbb{A}$ is a subgroup of the p-primary component of $\mathbb{Q}/\mathbb{Z}$, which in turn is the Prüffer group $\mathbb{Z}_{p^\infty}$. The only proper subgroups of $\mathbb{Z}_{p^\infty}$ are the finite groups $\mathbb{Z}_{p^n}$, where $0 \leq n < \infty$ which are certainly co-Hopfian. But the group $\mathbb{Z}_{p^\infty}$ is co-Hopfian, so we deduce that any monomorphism $\mathbb{A} \hookrightarrow \mathbb{A}$ is an automorphism. Consequently, $\varphi : \pi \hookrightarrow p^{-1}(\mathbb{A})$ is an isomorphism and the proof is complete.

□

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