Outline

- Elementary properties of linkage spaces.
- Homology calculations of linkage spaces using Morse theory.
- Cohomology calculations and rigidity results for linkage spaces.
A linkage is a pair \((L, \ell)\) where \(L\) is a finite graph and \(\ell\) is a function assigning each edge of \(L\) a positive number which we think of as the length of the edge.

A configuration of \((L, \ell)\) in a metric space \(M\) is given by a map \(i : L \rightarrow M\) such that \(i\) restricted to each edge is an isometry.

One may also add certain restrictions, e.g. that \(i\) is predescribed on a subgraph \(L_0\).

Denote \(C((L, \ell), M)\) the space of such configurations.
In the 19th century, linkages were of interest in constructing mechanical devices that could trace out a given algebraic curve. For example, the Peaucellier-Lipkin linkage that traces a straight line. Kempe proved that any algebraic curve could be traced by a linkage.

In the 1970’s W. Thurston outlined a proof that any affine variety can be realized as the configuration space of a planar linkage, see also 1978 PhD thesis of Niemann.

**Theorem**

Let $M$ be a compact smooth manifold. Then there is a linkage $L$ whose moduli space is diffeomorphic to a disjoint union of a number of copies of $M$.

A proof can be found in Kapovich-Millson (Topology 41, 2002).
We will now focus on the case where $L$ represents a circle. The linkage is determined by an element $\ell \in \mathbb{R}^n$ which gives the lengths of the edges.
Denote
\[
E_d(\ell) = \left\{ (x_1, \ldots, x_n) \in (S^{d-1})^n \bigg| \sum_{i=1}^n \ell_i x_i = 0 \right\}
\]
the space of polygons based at 0. The configuration space in \(\mathbb{R}^d\) is
\[
E_d(\ell) \times \mathbb{R}^d.
\]

Also, denote
\[
C_d(\ell) = \left\{ (x_1, \ldots, x_{n-1}) \in (S^{d-1})^{n-1} \bigg| \sum_{i=1}^{n-1} \ell_i x_i = -\ell_n e_1 \right\}
\]
the chain space of \(\ell\), obtained by requiring that \(x_n = e_1\).
We have a diagonal action of $O(d)$ on $E_d(\ell)$, and $O(d - 1)$ on $C_d(\ell)$ by fixing the first coordinate in $\mathbb{R}^d$. The quotient spaces

$$
\mathcal{M}_d(\ell) = E_d(\ell)/SO(d)
$$

$$
\mathcal{N}_d(\ell) = E_d(\ell)/O(d)
$$

are the moduli spaces of $\ell$.

Notice that

$$
\mathcal{M}_d(\ell) = C_d(\ell)/SO(d - 1)
$$

and in particular

$$
\mathcal{M}_2(\ell) = C_2(\ell).
$$
Basic topology of linkage spaces

From the definition, all these spaces are compact, but we would like them to be manifolds. Define

\[ F : (S^{d-1})^n \longrightarrow \mathbb{R}^d \]

\[ (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \ell_i x_i \]

The induced map on tangent spaces can only be non-surjective if all tangent spaces \( T_{x_i} S^{d-1} \subset \mathbb{R}^d \) are the same, that is, if \( x_i = \pm x_j \) for all \( i, j \).

So \( E_d(\ell) = F^{-1}(\{0\}) \) is a manifold of dimension \( (n - 1)(d - 1) - 1 \) unless there is a subset \( J \subset \{1, \ldots, n\} \) with

\[ \sum_{i \in J} \ell_i = \sum_{i \notin J} \ell_i \]
This condition can be summarized as $E_1(\ell) = \emptyset$, in which case we call $\ell$ generic.

Similarly, $C_d(\ell)$ is a closed manifold of dimension $(n - 2)(d - 1) - 1$ for generic $\ell$. In particular, $M_2(\ell)$ and $N_2(\ell)$ are closed manifolds of dimension $n - 3$.

Also, $M_3(\ell) = C_3(\ell)/S^1$ is a manifold of dimension $2(n - 3)$, because the $S^1$-action is free. Note that $N_3(\ell) = C_3(\ell)/O(2)$ is not a manifold in general, as the action is not free.

Similarly, $M_d(\ell)$ and $N_d(\ell)$ for $d \geq 4$ are not manifolds in general.
Define $G : \mathbb{R}^n \times (S^{d-1})^n \to \mathbb{R}^d \times \mathbb{R}^n$ by

$$G(\ell_1, \ldots, \ell_n, x_1, \ldots, x_n) = (\sum \ell_ix_i, \ell_1, \ldots, \ell_n)$$

so that $E_d(\ell) = G^{-1}(0, \ell)$.

If $V \subset \mathbb{R}^n$ is the image of a smooth curve, e.g. a line, and $G \pitchfork \{0\} \times V$, then $C = G^{-1}(\{0\} \times V)$ is a manifold. In particular, if $V$ is a closed interval, it is a cobordism between $E_d(\ell)$ and $E_d(\ell')$ and there is an obvious smooth map $C \to [0, 1]$ which is projection to $V$.

If this map has no critical points, then $E_d(\ell)$ and $E_d(\ell')$ are diffeomorphic. This is a Morse-theoretic argument which involves a gradient that flows from one boundary of the cobordism to the other. There is also an $O(d)$-action and the projection is invariant with respect to this action. So the diffeomorphism is $O(d)$-equivariant.
If $J \subset \{1, \ldots, n\}$ define

$$H_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j \in J} x_j = \sum_{i \notin J} x_i \right\}$$

Then

$$\mathbb{R}^n - \bigcup H_J$$

consists of finitely many components, called chambers.

Combinatorially, we can distinguish chambers as follows.

**Definition**

Call $J \subset \{1, \ldots, n\}$

- $\ell$-short, if $\sum_{j \in J} \ell_j < \sum_{i \notin J} \ell_i$
- $\ell$-long, if $\sum_{j \in J} \ell_j > \sum_{i \notin J} \ell_i$

For generic $\ell$, this determines the chamber that $\ell$ is in. In fact, we only need to know this for subsets with $n \in J$. 
Proposition

Let $\ell, \ell'$ be in the same chamber, then

1. $E_d(\ell)$ and $E_d(\ell')$ are $O(d)$-diffeomorphic.
2. $C_d(\ell)$ and $C_d(\ell')$ are $O(d - 1)$-diffeomorphic.
3. $M_d(\ell)$ and $M_d(\ell')$ are homeomorphic.

Also, if we define for $\sigma \in \Sigma_n$ the length vector $\ell^\sigma$ by

$$\ell^\sigma = (\ell_{\sigma_1}, \ldots, \ell_{\sigma_n})$$

it is clear that $M_d(\ell^\sigma) \cong M_d(\ell)$. 

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Configuration Spaces of Linkages
Elementary properties of chambers

There is no general formula for the number of chambers (up to permutations of coordinates). Known cases are

<table>
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<tr>
<th>$n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>7</td>
<td>21</td>
<td>135</td>
<td>2,470</td>
<td>175,428</td>
<td>52,980,624</td>
</tr>
</tbody>
</table>

This coincides with the number of self-dual equivalence classes of threshold functions of $n$ or fewer variables, number of majority (i.e., decisive and weighted) games with $n$ players.

For $n = 4$, the three cases are represented by

- $\ell = (1, 1, 1, 4)$ with $\mathcal{M}_2(\ell) = \emptyset$.
- $\ell = (1, 1, 1, 2)$ with $\mathcal{M}_2(\ell) = S^1$.
- $\ell = (1, 2, 2, 2)$ with $\mathcal{M}_2(\ell) = S^1 \sqcup S^1$. 
For $n = 5$, we have

- $\ell = (1, 1, 1, 1, 5)$ with $M_2(\ell) = \emptyset$.
- $\ell = (1, 1, 3, 3, 3)$ with $M_2(\ell) = T^2 \sqcup T^2$.
- $\ell = (1, 1, 1, 1, 3)$ with $M_2(\ell) = S^2$.
- $\ell = (1, 2, 2, 2, 4)$ with $M_2(\ell) = T^2$.
- $\ell = (1, 1, 2, 2, 3)$ with $M_2(\ell) = M_2$.
- $\ell = (1, 1, 1, 2, 2)$ with $M_2(\ell) = M_3$.
- $\ell = (1, 1, 1, 1, 1)$ with $M_2(\ell) = M_4$.

The moduli spaces can be derived from the Betti numbers in these cases, but one can also see this directly.

For $n = 6$ one can still give a list of the diffeomorphism types for the 21 cases, however, they are no longer distinguished by their Betti numbers.
Want to use Morse Theory to get a better understanding of the topology.

A classical smooth function on $\mathcal{M}_2(\ell)$ is the signed area function, with critical points the configurations that fit on a circle. Recently, Panina-Zhukova have done the index calculations for the critical points.

This does not give rise to a perfect Morse function in general, for example, for $n = 5$ and $\ell = (1, 1, 1, 1, 1)$ one gets two local maxima and two local minima.

Another approach is to consider cobordism as before, and then look at “wall-crossing” when going from one chamber to another. The critical points are easy to control, but one needs to know how the homology changes during the surgery.
The distance function of a robot arm

Let

$$W = (S^{d-1})^n$$

and given a length vector $\ell \in \mathbb{R}^n$, define $f_\ell : W \to \mathbb{R}$ by

$$f_\ell(z_1, \ldots, z_n) = - \left| \sum_{i=1}^n \ell_i z_i \right|^2$$

The maximum is $E_d(\ell)$ which is a Morse-Bott critical manifold for generic $\ell$, and other critical manifolds are given by collinear configurations.

This Morse-Bott function is also very useful for $C_d(\ell)$, when restricted to

$$W' = \{(z_1, \ldots, z_n) \in W \mid z_n = e_1\}$$
When we restrict $f_\ell$ to $W'$ the critical points corresponding to collinear configurations are isolated Morse critical points. In fact, these critical points correspond to $J \subset \{1, \ldots, n\}$ long, and the Morse index is equal to $(d - 1)(n - |J|)$. (See M. Farber, *Invitation to topological robotics*, for detailed index calculation)