Topology of Configuration Spaces - Configuration Spaces of Linkages

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Outline

- Elementary properties of linkage spaces.
- Homology calculations of linkage spaces using Morse theory.
- Cohomology calculations and rigidity results for linkage spaces.

Linkages

A *linkage* is a pair (L, ℓ) where L is a finite graph and ℓ is a function assigning each edge of L a positive number which we think of as the length of the edge.

A *configuration* of (L, ℓ) in a metric space M is given by a map $i\colon L \to M$ such that i restricted to each edge is an isometry.

One may also add certain restrictions, e.g. that i is predescribed on a subgraph L_0.

Denote $C((L, \ell), M)$ the space of such configurations.

Manifolds as configuration spaces of planar linkages

In the 19th century, linkages were of interest in constructing mechanical devices that could trace out a given algebraic curve. For example, the Peaucellier-Lipkin linkage that traces a straight line. Kempe proved that any algebraic curve could be traced by a linkage.

In the 1970's W. Thurston outlined a proof that any affine variety can be realized as the configuration space of a planar linkage, see also 1978 PhD thesis of Niemann.

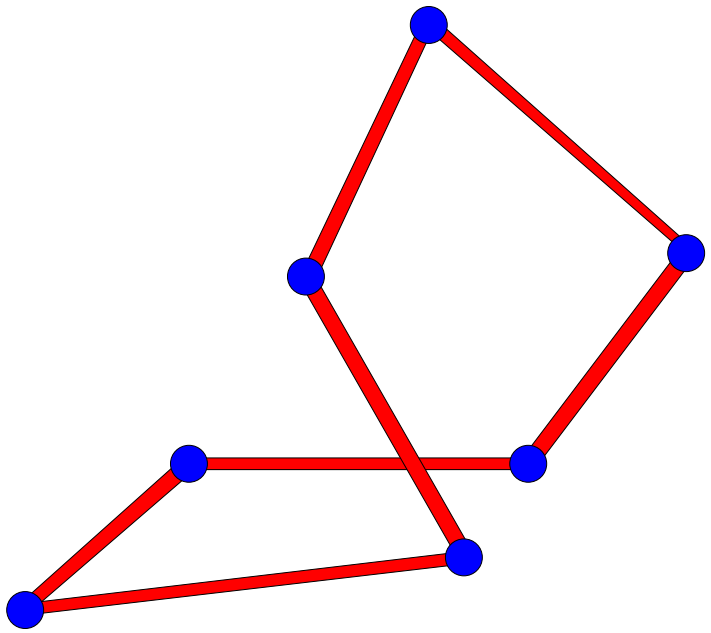
Theorem

Let M be a compact smooth manifold. Then there is a linkage L whose moduli space is diffeomorphic to a disjoint union of a number of copies of M.

A proof can be found in Kapovich-Millson (Topology 41, 2002).

Closed linkages

We will now focus on the case where L represents a circle. The linkage is determined by an element $\ell \in \mathbb{R}^n$ which gives the lengths of the edges.



Denote

$$E_d(\ell) \quad = \quad \left\{ (x_1, \ldots, x_n) \in (S^{d-1})^n \;\middle|\; \sum_{i=1}^{n} \ell_i x_i = 0 \right\}$$

the space of polygons based at 0. The configuration space in $\mathbb{R}^d$ is

$$E_d(\ell) \times \mathbb{R}^d.$$

Also, denote

$$\mathcal{C}_d(\ell) \quad = \quad \left\{ (x_1, \ldots, x_{n-1}) \in (S^{d-1})^{n-1} \;\middle|\; \sum_{i=1}^{n-1} \ell_i x_i = -\ell_n \, \mathbf{e}_1 \right\}$$

the chain space of ℓ, obtained by requiring that $x_n = \mathbf{e}_1$.

We have a diagonal action of $\mathbf{O}(d)$ on $E_d(\ell)$, and $\mathbf{O}(d-1)$ on $\mathcal{C}_d(\ell)$ by fixing the first coordinate in $\mathbb{R}^d$. The quotient spaces

$$\begin{aligned} \mathcal{M}_d(\ell) &= E_d(\ell)/\mathbf{SO}(d) \\ \mathcal{N}_d(\ell) &= E_d(\ell)/\mathbf{O}(d) \end{aligned}$$

are the moduli spaces of ℓ.

Notice that

$$\mathcal{M}_d(\ell) = \mathcal{C}_d(\ell)/\mathbf{SO}(d-1)$$

and in particular

$$\mathcal{M}_2(\ell) = \mathcal{C}_2(\ell).$$

Basic topology of linkage spaces

From the definition, all these spaces are compact, but we would like them to be manifolds. Define

$$\begin{aligned} F\colon (S^{d-1})^n &\longrightarrow \mathbb{R}^d \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^{n} \ell_i x_i \end{aligned}$$

The induced map on tangent spaces can only be non-surjective if all tangent spaces $T_{x_i} S^{d-1} \subset \mathbb{R}^d$ are the same, that is, if $x_i = \pm x_j$ for all i, j.

So $E_d(\ell) = F^{-1}(\{0\})$ is a manifold of dimension $(n-1)(d-1)-1$ unless there is a subset $J \subset \{1, \ldots, n\}$ with

$$\sum_{i \in J} \ell_i = \sum_{i \notin J} \ell_i$$

This condition can be summarized as $E_1(\ell) = \emptyset$, in which case we call ℓ *generic*.

Similarly, $\mathcal{C}_d(\ell)$ is a closed manifold of dimension $(n-2)(d-1)-1$ for generic ℓ. In particular, $\mathcal{M}_2(\ell)$ and $\mathcal{N}_2(\ell)$ are closed manifolds of dimension $n-3$.

Also, $\mathcal{M}_3(\ell) = \mathcal{C}_3(\ell)/S^1$ is a manifold of dimension $2(n-3)$, because the S^1-action is free. Note that $\mathcal{N}_3(\ell) = \mathcal{C}_3(\ell)/\mathbf{O}(2)$ is not a manifold in general, as the action is not free.

Similarly, $\mathcal{M}_d(\ell)$ and $\mathcal{N}_d(\ell)$ for $d \geq 4$ are not manifolds in general.

Define $G\colon \mathbb{R}^n \times (S^{d-1})^n \to \mathbb{R}^d \times \mathbb{R}^n$ by

$$G(\ell_1, \ldots, \ell_n, x_1, \ldots, x_n) \;=\; (\sum \ell_i x_i, \ell_1, \ldots, \ell_n)$$

so that $E_d(\ell) = G^{-1}(0, \ell)$.

If $V \subset \mathbb{R}^n$ is the image of a smooth curve, e.g. a line, and $G \pitchfork \{0\} \times V$, then $C = G^{-1}(\{0\} \times V)$ is a manifold. In particular, if V is a closed interval, it is a cobordism between $E_d(\ell)$ and $E_d(\ell')$ and there is an obvious smooth map $C \to [0, 1]$ which is projection to V.

If this map has no critical points, then $E_d(\ell)$ and $E_d(\ell')$ are diffeomorphic. This is a Morse-theoretic argument which involves a gradient that flows from one boundary of the cobordism to the other. There is also an $\mathbf{O}(d)$-action and the projection is invariant with respect to this action. So the diffeomorphism is $\mathbf{O}(d)$-equivariant.

If $J \subset \{1, \ldots, n\}$ define

$$H_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{j \in J} x_j = \sum_{i \notin J} x_i \right\}$$

Then

$$\mathbb{R}^n - \bigcup H_J$$

consists of finitely many components, called *chambers*.

Combinatorially, we can distinguish chambers as follows.

Definition

Call $J \subset \{1, \ldots, n\}$

- ℓ-short, if $\sum_{j \in J} \ell_j < \sum_{i \notin J} \ell_i$
- ℓ-long, if $\sum_{j \in J} \ell_j > \sum_{i \notin J} \ell_i$

For generic ℓ, this determines the chamber that ℓ is in. In fact, we only need to know this for subsets with $n \in J$.

Proposition

Let ℓ, ℓ' be in the same chamber, then

1. $E_d(\ell)$ *and* $E_d(\ell')$ *are* $\mathbf{O}(d)$*-diffeomorphic.*
2. $\mathcal{C}_d(\ell)$ *and* $\mathcal{C}_d(\ell')$ *are* $\mathbf{O}(d-1)$*-diffeomorphic.*
3. $\mathcal{M}_d(\ell)$ *and* $\mathcal{M}_d(\ell')$ *are homeomorphic.*

Also, if we define for $\sigma \in \Sigma_n$ the length vector ℓ^σ by

$$\ell^\sigma \quad = \quad (\ell_{\sigma 1}, \ldots, \ell_{\sigma n})$$

it is clear that $\mathcal{M}_d(\ell^\sigma) \cong \mathcal{M}_d(\ell)$.

Elementary properties of chambers

There is no general formula for the number of chambers (up to permutations of coordinates). Known cases are

n	4	5	6	7	8	9	10
	3	7	21	135	2,470	175,428	52,980,624

This coincides with the number of self-dual equivalence classes of threshold functions of n or fewer variables, number of majority (i.e., decisive and weighted) games with n players.

For $n = 4$, the three cases are represented by

- $\ell = (1, 1, 1, 4)$ with $\mathcal{M}_2(\ell) = \emptyset$.
- $\ell = (1, 1, 1, 2)$ with $\mathcal{M}_2(\ell) = S^1$.
- $\ell = (1, 2, 2, 2)$ with $\mathcal{M}_2(\ell) = S^1 \sqcup S^1$.

For $n = 5$, we have

- $\ell = (1, 1, 1, 1, 5)$ with $\mathcal{M}_2(\ell) = \emptyset$.
- $\ell = (1, 1, 3, 3, 3)$ with $\mathcal{M}_2(\ell) = T^2 \sqcup T^2$.
- $\ell = (1, 1, 1, 1, 3)$ with $\mathcal{M}_2(\ell) = S^2$.
- $\ell = (1, 2, 2, 2, 4)$ with $\mathcal{M}_2(\ell) = T^2$.
- $\ell = (1, 1, 2, 2, 3)$ with $\mathcal{M}_2(\ell) = M_2$.
- $\ell = (1, 1, 1, 2, 2)$ with $\mathcal{M}_2(\ell) = M_3$.
- $\ell = (1, 1, 1, 1, 1)$ with $\mathcal{M}_2(\ell) = M_4$.

The moduli spaces can be derived from the Betti numbers in these cases, but one can also see this directly.

For $n = 6$ one can still give a list of the diffeomorphism types for the 21 cases, however, they are no longer distinguished by their Betti numbers.

Want to use Morse Theory to get a better understanding of the topology.

A classical smooth function on $\mathcal{M}_2(\ell)$ is the signed area function, with critical points the configurations that fit on a circle. Recently, Panina-Zhukova have done the index calculations for the critical points.

This does not give rise to a perfect Morse function in general, for example, for $n = 5$ and $\ell = (1, 1, 1, 1, 1)$ one gets two local maxima and two local minima.

Another approach is to consider cobordism as before, and then look at "wall-crossing" when going from one chamber to another. The critical points are easy to control, but one needs to know how the homology changes during the surgery.

The distance function of a robot arm

Let

$$W \;=\; (S^{d-1})^n$$

and given a length vector $\ell \in \mathbb{R}^n$, define $f_\ell \colon W \to \mathbb{R}$ by

$$f_\ell(z_1, \ldots, z_n) \;=\; -\left|\sum_{i=1}^{n} \ell_i z_i\right|^2$$

The maximum is $E_d(\ell)$ which is a Morse-Bott critical manifold for generic ℓ, and other critical manifolds are given by collinear configurations.

This Morse-Bott function is also very useful for $\mathcal{C}_d(\ell)$, when restricted to

$$W' \;=\; \{(z_1, \ldots, z_n) \in W \mid z_n = \mathbf{e}_1\}$$

When we restrict f_ℓ to W' the critical points corresponding to collinear configurations are isolated Morse critical points. In fact, these critical points correspond to $J \subset \{1, \ldots, n\}$ long, and the Morse index is equal to $(d-1)(n-|J|)$. (See M. Farber, *Invitation to topological robotics*, for detailed index calculation)