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Topology of robot motion planning

#1 - 27 June, morning

#2 - 28 June, morning

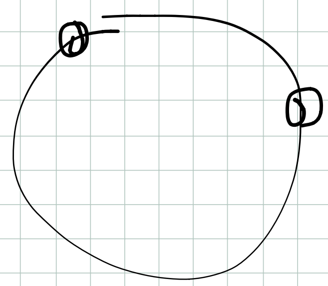
#3 - 28 June, evening

#1

configuration spaces - all states of a mechanical system

(1) robotic arm

(2) n robots on factory floor



$$B(Y,n) = \frac{F(Y,n)}{\Sigma_n}$$

$=$ Möbius Band.

(3) piano movers problem

no obstacles $\Rightarrow$ $\mathbb{R}^3 \times SO(3)$

with obstacles – a subset

Assume: X ANR = ANE

essentially locally contractible

(e.g CW complexes).

PX = pathspace in X

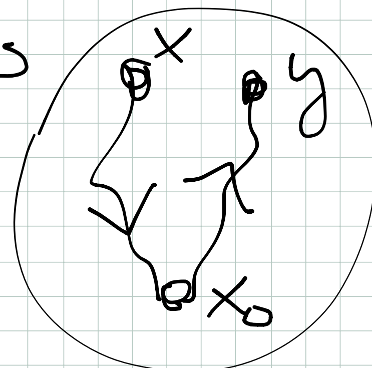
__Def__ A __motion planning algorithm__ (MPA) is a map

$s : X \times X \to PX$

__Prop__ If MPA exists $\iff$

X is contractible.

__Pf__ $\Leftarrow$ is



. //

__Def__ $TC(X) = \min(k)$

s.t. $\exists$ open cover

$U_0, \dots, U_k$ of $X \times X$ and

Continuous MPA for each

U_i : $s_i : U_i \to PX$

So

$TC(X) = 0 \iff X$ contractible

<u>Lemma 1</u> $F \underset{cl}{\subseteq} X \times X$

cont. MPA $s : F \to PX$,

$X \in ANE$

$\exists$ open $U \supseteq F$ and

$\bar{s} : U \to PX$ cont MPA

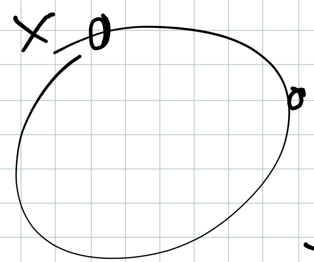
extending s

So you can use closed sets in the def. of TC.

Sasha's Student

Tulsi Srinivasan – can use any kind of sets.

Ex



y not antipodal then choose the shortest path

$$U_0 = \{(x,y) \in S^1 \mid x \neq -y\}$$

open

$$U_1 = \{(x,-x)\}$$

closed

Same argument for S^{2n+1}.

For flow use a nonzero vector

field which exists.

C-space of a robotic arm:
a state of a robot consists of positions of joints and the end effectors (as welding head, paint spray gun, finger tips, etc)

End effector space E

$e: X \to E$, $E \subseteq \mathbb{R}^3$.

If n effectors, $E \subseteq (\mathbb{R}^3)^n$.

or with obstructions

$$E \subseteq \frac{(\mathbb{R}^3)^n}{\Sigma_n}.$$

What is the MPA here?

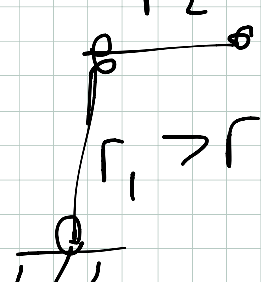
$\varphi : X \times E \longrightarrow PX$

with $\varphi(x,y)(0) = x$

& $\varphi(x,y)(1) = y$

<u>Ex</u> $X = S^1 \times S^1$

r_2

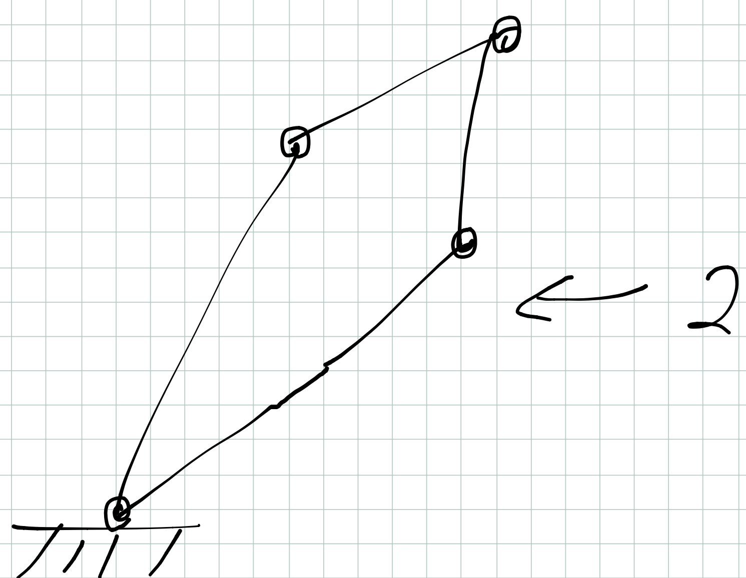


$r_1 > r_2$

$$E = A_{r_1+r_2}^{r_1-r_2}(0)$$

(annulus)

$|e^{-1}(y)| \leq 2$, usually 2:



Given $f: X \twoheadrightarrow Y$ then

$TC(f) = \min\{K \mid X \times Y =$

$U_0 \cup U_1 \cup U_2 \cup \ldots \cup U_K$ open cover

$\exists$ cont. $\varphi_i : U_i \to PX$ with

$\varphi_i(x,y)(0) = x,\ f\varphi_i(x,y) = y$

for all y.

<u>Notice</u> $TC(id_X) = TC(X)$

<u>Question</u> $TC(f) = 0 \Rightarrow$

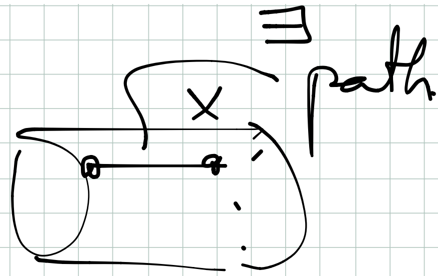
what is f?

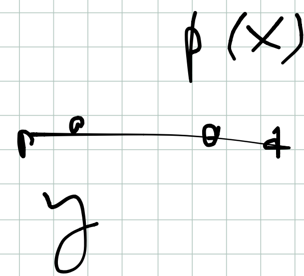
<u>Prop.</u> $TC(f) = 0 \Rightarrow f$ is

a retraction

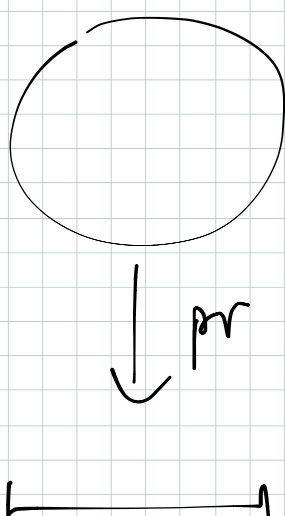
e.g. $pr : I \times F \to I$ is

always a retraction:





The reverse is not true:



$TC(f) > 0$.
(exercise)

<u>Lemma 2</u> $U \subseteq X \times X$ TFAE

1) U is admissible

2) $p_1|_U$ and $p_2|_U$ are $\simeq$.
where $p_i : X \times X \to X$ is proj onto the ith factor.

3) U can be deformed to the diagonal $\Delta X \subseteq X \times X$.

#2

$U \subseteq X \times X$ is <u>admissible</u> if $\exists$ motion planning algorithm on U.

<u>Lemma 2</u> $U \subseteq X \times X$ <u>TFAE</u>:

1) U is admissible
2) $p_1|_U \sim p_2|_U$
3) U can be continuously deformed to ΔX.

There is an analogue for maps $f: X \to Y$ given, $U \subseteq X \times X$ is admissible if $\exists$ cont. m.p.a. $\varphi: U \to PX$ st...

$$\varphi(x,y)(0) = x, \quad f\varphi(x,y)(1) = y$$

Lemma 2' $U \subseteq X \times Y$, $f: X \to Y$ is a fibration

TFAE 1) U is admissible

2) $f \circ p_1|_U \simeq f \circ p_2|_U$

3) U can be deformed to the graph of f, Γ_f

Def $f: X \to Y$ is a homotopy domination if $\exists$ section $s: Y \to X$

Prop If $f: X \to Y$ is a h. domination then $TC(X) \geq TC(Y)$.

Def $\operatorname{cat} X \leq K$ if $X = U_0 \cup \dots \cup U_K$ open sets which are contractible in X

Originally U_i were required closed, then Fox proved "open" given the same. Tulsi showed any sets give the same.

Schwarz genus

$p: X \to Y$ $\quad sg(p) = \min \{ K \mid \exists$ open cover $U_0 \cup \ldots \cup U_k = Y_k\}$

s.t. $\exists$ section $s_i : U_i \to X$ s.t. $f \circ s_i = 1_{U_i}\}$

Now

$$TC(X) = sg\,(p : PX \to X \times X)$$

$$cat(X) = sg\,(p_0 : P_0X \to X)$$

$$TC(f : X \to Y) = sg\,((1 \times f) \circ p : PX \to X \times Y)$$

here $x_0 \in X$, $P_0X \subseteq PX$

$$P_0X = \{\varphi : [0,1] \to X \mid \varphi(0) = x_0\}$$

$$p_0(\varphi) = \varphi(1)$$

Lower and upper bounds for cat

- $cat(X) \leq dim(X)$
- $TC(X) \leq cat(X \times X) \leq 2\,dim(X)$, where $dim(X \times X) = 2\,dim(X)$

Lower bounds

"cup length"

$\text{cat}\, X \geq c.l(X) \leftarrow \max\{k \mid \exists \alpha_i \in H^{k_i}(X, L_i),\ k_i > 0,\ \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}$

$TC(X) \geq z.c.l.(X \times X)$

"Zero cup length"

$$K = \ker\{H^*(X \times X) \xrightarrow{\cup} H^*(\Delta X)\}$$

then define

$$z.c.l. = \max\{k \mid \exists \alpha_i \in K,\ \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}.$$

Proof Assume $TC(X) < k$ then $\exists$ admissible cover $X \times X = V_1 \cup \cdots \cup V_k$ then there are classes $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1 \cup \cdots \cup \alpha_k = 0$. Pull back to $H^*(X \times X)$, contradiction.

Ex $TC(S^{2n}) = 2$

≤ 2 because of counting vector fields.

≥ 2 by cup length bounds.

Examples

$TC(M_g) = 4$ if $g > 1$

$TC(K) = ?$

K = Klein bottle.

Also $TC(X \times Y) \leq TC(X) + TC(Y)$

& Thm given a topological group G

then $TC(G) = cat(G)$

(actually true for H-spaces)

S. Yuzvinsky $\rightarrow$ $TC(P_n) = 3$

Bestvina - Schwarz class

#3

$\pi_1(X) = \pi$, M π-module

$H^*(X; M)$, $\tilde{X}$ univ. cover

$$C_n(\tilde{X}) \xrightarrow{\partial} C_{n-1}(\tilde{X})$$

$$\mathrm{Hom}_\pi(C_n, M) \longleftarrow \mathrm{Hom}_\pi(C_{n-1}, M)$$

$$H^*(B\pi, M) := H^*(\pi, M)$$

there is a cohomology class $\beta_\pi \in$
$H^1(B\pi, I(\pi))$, $I(\pi)$ augmentation ideal

$$H^0(B\pi, \mathbb{Z}) \xrightarrow{\delta} H^1(B\pi, I(\pi))$$

and by def. $\beta_\pi = \delta(1)$

Another interpretation is the first obstruction to lifting $B\pi \overset{p}{\longleftarrow} E\pi$

$g \longrightarrow sg$

if s exists then $sg - s \in I(\pi)$.

<u>Universality</u>: $\forall \alpha \in H^k(B\pi, M)$

$\exists \; I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to M$

such that

$\beta_\pi^k \longmapsto \alpha$

↖ k fold cup product.

<u>Monomorphism Lemma</u>

$X \in CW$. Suppose

$f: X \to B\pi$ induces a monom. of π_1

then $H^2(B\pi; M) \to H^2(X, M)$ is a mono.

<u>Corollary</u> $\text{cat}(X) = 1$ then $\pi_1(X)$ is free

Follows from Stallings - Swan thm

which says that if $cd(\pi) = 1$ then

π is free.

Pf of corollary: If $cd(\pi) > 1$ then $\beta_{\pi}^2 \neq 0$. Let $f: X \to B\pi$ will induce $\cong$ on π_1. Then $0 \neq \beta_X^2 \leftarrow \beta_{\pi}^2$

Then $c.l.(X) \geq 0 \Rightarrow cat(X) \geq 2$.

Theorem (Dranishnikov, Katz, Rudyak)

$cat(M^n) = 2$, $n > 2 \Rightarrow$

$\pi_1(M^n)$ is free.

<u>Proof</u> $f: M^n \to B\pi$ induces $\cong$ on π_1

Assume that π is not free $\Rightarrow$

$\beta_{\pi}^2 \neq 0 \Rightarrow \beta_M^2 \neq 0$.

then by P.D. $\exists\, \alpha \in H^{n-2}(M, L)$

such that $\alpha \cup \beta_M \neq 0 \Rightarrow$ cuplength

of $M \geq 3 \Rightarrow cat(M) \geq 3$. //

Theorem (Grant-Lupton-Oprea) X finite CW cx

If $TC(X) = 1$ then $X \simeq S^{2n+1}$ for some n.

Proof $\square$

Rudyak (conjecture) Given K, n and $K \leq n$

$\exists$ group π such that

$$TC(\pi) = K, \quad cd(\pi) = n.$$

[$cd(\pi) = cat(B\pi)$ ← known.]