MA251 - Algebra I

Assignment 4 **Autumn 2013**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 25 NOVEMBER** *(Monday of the ninth week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Let A be a symmetric $n \times n$ real matrix. We consider a function $f\colon S \to \mathbb{R}$ where $f(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$ and $S = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}^T\mathbf{v} = 1\}$ is the unit sphere. The set S is compact and the function f is continuous. It is known in Analysis (Extreme Value Theorem or Weierstrass' Theorem) that there exists a point $\mathbf{v}_0 \in S$ such that $f(\mathbf{v})$ achieves its maximum at $\mathbf{v} = \mathbf{v}_0$. Prove that $\mathbf{v}_0$ is an eigenvector of A.

(Note: This is another proof of the fact that a symmetric matrix admits a real eigenvector, using methods of Analysis)

P2. In this problem we establish a so called QR-decomposition of a matrix.
(i) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Using Gram-Schmidt Orthonormalisation on columns of A, prove that we can write $A = QR$ where Q is orthogonal and R is upper-triangular.
(ii) Let $B \in \mathbb{R}^{n\times n}$. Prove that we can write $B = AS$ wheres A is invertible and S is upper-triangular.
(iii) Prove that we can write $B = QR$ where Q is orthogonal and R is upper-triangular.
(iv) Let $A = \begin{pmatrix} 3 & -2 & -7 \\ 6 & 2 & 4 \\ 6 & -1 & 4 \end{pmatrix}$. Find an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$.

P3. Prove that the eigenvalues of a complex Hermitian matrix are all real.

P4. Find a 2×2 complex matrix which is both Hermitian and unitary and whose entries are not all real numbers.

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. (i) Find the orders of all elements in the group $\mathbb{Z}_{12}$. [1 mark]

(ii) Write down and prove a formula for the order of an element $k \in \mathbb{Z}_n$ for general n and k. [2 marks]

2. Classify the following curves and surfaces (ellipse, parabola, etc.) and justify your answers:
(i) $x^2 - y^2 + 2xy - 1 = 0$ (2 dimensions); [1 mark]
(ii) $x^2 - y^2 + 2xy - 1 = 0$ (3 dimensions); [1 mark]
(iii) $x^2 + 2xy + y^2 + x + 1 = 0$ (2 dimensions); [1 mark]
(iv) $x^2 + y^2 - 2z^2 - x - y - 4z = 0$ (3 dimensions); [1 mark]
(v) $x^2 + y^2 + 2z^2 + 2xz - 2y + 2z + 2 = 0$ (3 dimensions); [1 mark]
(vi) $x^2 + y^2 - z^2 + 2xy - 2xz - 2yz - y = 0$ (3 dimensions). [1 mark]

3. Let $(V,<,>)$ be an Euclidean space. The Gram matrix of vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k \in V$ is

$$G(\mathbf{v}_1,\ldots,\mathbf{v}_k) = \begin{pmatrix} <\mathbf{v}_1,\mathbf{v}_1> & <\mathbf{v}_1,\mathbf{v}_2> & \cdots & <\mathbf{v}_1,\mathbf{v}_k> \\ \vdots & \vdots & & \vdots \\ <\mathbf{v}_k,\mathbf{v}_1> & <\mathbf{v}_k,\mathbf{v}_2> & \cdots & <\mathbf{v}_k,\mathbf{v}_k> \end{pmatrix}$$

(i) Prove that if $\mathbf{v}_1,\ldots,\mathbf{v}_k$ are linearly dependent then $\det(G(\mathbf{v}_1,\ldots,\mathbf{v}_k)) = 0$. [1 mark]

(ii) Prove that if $\mathbf{v}_1,\ldots,\mathbf{v}_k$ are linearly independent then $\det(G(\mathbf{v}_1,\ldots,\mathbf{v}_k)) > 0$. (*Hint:* Gram matrix is the matrix of $<,>$ restricted to the span of $\mathbf{v}_i$-s.) [2 marks]

(iii) The Gram inequality $\det(G(\mathbf{v}_1,\ldots,\mathbf{v}_k)) \geq 0$, which you have just proved, has a very important special case of $k = 2$. In this special case, it is called the Schwarz inequality. Write the Schwarz inequality explicitly (by using a formula for a 2×2-determinant) and explain how the Schwarz inequality can be used to define an angle between two vectors $x, y \in V$. [1 mark]

(iv) We define a Euclidean distance on V by $d(x,y) = \sqrt{<x-y,x-y>}$ Using the Schwarz inequality, prove the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$. [1 mark]

(v) Prove that a linear map $T\colon V \to V$ is orthogonal if and only if it is distance preserving, i.e. $d(a,b) = d(T(a),T(b))$ for all $a, b \in V$. [2 marks]

4. We are working with a finite dimensional Euclidean space $(V, <,>)$. Given vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n \in V$, one defines the parallelepiped

$$P(\mathbf{v}_1,\ldots,\mathbf{v}_n) = \{\sum \alpha_i \mathbf{v}_i \in V \mid \alpha_i \in [0,1]\}.$$

The n-dimensional volume of this parallelepiped is defined inductively:

$$\mathrm{Vol}_1(P(\mathbf{v}_1)) = ||\mathbf{v}_1|| = \sqrt{<\mathbf{v}_1,\mathbf{v}_1>},\ \mathrm{Vol}_n(P(\mathbf{v}_1,\ldots,\mathbf{v}_n)) = \mathrm{Vol}_{n-1}(P(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})))||\mathbf{w}||$$

where $\mathbf{w} = \mathbf{v}_n + \alpha_1\mathbf{v}_1 + \ldots + \alpha_{n-1}\mathbf{v}_{n-1}$ is the unique[1] vector of this form orthogonal to all $\mathbf{v}_i$ for $i \leq n-1$. Intuitively, we define the n-dimensional volume as a product of the $n-1$-dimensional volume of a base and the height. There is nothing to prove at this point.

(i) Using the Gram matrix from problem 3, prove that

$$\det(G(\mathbf{v}_1,\ldots,\mathbf{v}_n)) = \mathrm{Vol}_n(P(\mathbf{v}_1,\ldots,\mathbf{v}_n))^2.$$

[2 marks]

(ii) Now let n be the dimension of V. Let A be a square matrix whose columns are $\underline{\mathbf{v}}_i$, the coordinate vectors of $\mathbf{v}_i$ in some orthonormal basis. Using part (i), prove that

$$|\det(A)| = \mathrm{Vol}_n(P(\mathbf{v}_1,\ldots,\mathbf{v}_n)).$$

[2 marks]

[1]This follows from the direct sum decomposition $V = W \oplus W^{\perp}$ where W is the span of the first $n-1$ vectors. Then $\mathbf{w} = \pi(\mathbf{v}_n)$ where $\pi\colon V \to W^{\perp}$ is the projection map along W.