MA251 - Algebra I

Assignment 3 **Autumn 2013**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 11 NOVEMBER** *(Monday of the seventh week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Calculate the rank and signature of the quadratic forms corresponding to the matrices

$$\text{(i)} \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 \\ 1/2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

P2. Call two quadratic forms on the n-dimensional vector space V over field $\mathbb{K}$ *equivalent* if one can be obtained from the other by a change of coordinates. How many equivalence classes are there when (i) $\mathbb{K} = \mathbb{C}$ and $n = 4$; and (ii) $\mathbb{K} = \mathbb{R}$ and $n = 3$.

P3. What is the answer to parts (i) and (ii) of Question P2 for general n?

P4. (i) Show that any 2×2 real orthogonal matrix is equal to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some θ. (Hence the matrix represents a rotation about the origin or a reflection about a line through the origin in the 2-dimensional plane.)

(ii) Show that a 3×3 real orthogonal matrix A represents either a rotation about a line through the origin, or a reflection about a plane through the origin followed by a rotation (*Hint*: First show that A has an eigenvector $\mathbf{v}$, and change basis to include $\mathbf{v}$.)

P5. Part (ii) of this problem deals with a square root of a matrix. Part (iii) establishes *a polar decomposition* of a matrix A: it is analogous to writing a complex number in the polar form $re^{i\theta}$.

Let A be any invertible $n \times n$ matrix over $\mathbb{R}$.

(i) Show that AA^{T} is symmetric and positive definite.

(ii) Show that there is a symmetric positive definite matrix S with $S^2 = AA^{\mathrm{T}}$.

(iii) Show that there is a symmetric positive definite matrix S and an orthogonal matrix R such that $A = SR$. (*Hint*: same S as in (ii).)

P6. This question outlines a proof of the convergence of the exponential series for any matrix. For $A \in \mathbb{C}^{n,n}$ with entries $(a_{ij})_{1 \le i,j \le n}$, let $|A| = \sup_{i,j} |a_{ij}|$.

(i) Show that $|AB| \le n|A||B|$ for any $A, B \in \mathbb{C}^{n,n}$.

(ii) Hence deduce that $|A^k| \le n^{k-1}|A|^k$ for all $k \in \mathbb{N}$.

(iii) Prove that there exists $0 < r < 1$ and $C < \infty$ (depending on A) such that $|A^k|/k! < Cr^k$ for all $k \in \mathbb{N}$, and deduce that for each i, j, the sequence whose nth term is the (i, j) entry of $\sum_{k=0}^{n} A^k/k!$ is convergent.

P7. Find a 2×2 symmetric matrix over $\mathbb{C}$ which is not diagonalisable.

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Prove that the 2-dimensional quadratic from $q(x,y) = \alpha x^2 + \beta xy + \gamma y^2$ is positive definite if and only if $\alpha > 0$ and $\beta^2 - 4\alpha\gamma < 0$. [2 marks]

2. Find orthogonal matrices P such that $P^{\mathrm{T}}AP$ is diagonal for

$$\text{(i) } A = \begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix} \qquad \text{(ii) } A = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}.$$

[2,3 marks]

3. For the following Euclidean vector spaces (V, ω) and a basis $f_1, \ldots, f_n$, run Gram-Schmidt orthonormalisation process to arrive at an orthonormal basis.

(i) $V = \mathbb{R}^3$ with the the standard dot-product $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T\mathbf{w}$; $f_1 = (1,0,0)^T$, $f_2 = (1,1,1)^T$ and $f_3 = (1,-1,1)^T$. [2 marks]

(ii) $V = \mathbb{R}[X]_{\leq 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-1}^{1} f(X)g(X)dX \ ;$$

$f_i = X^{i-1}$, $i = 1,2,3,4,5$. [2 marks]

(iii) $V = \mathbb{R}[X]_{\leq 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-\infty}^{+\infty} e^{-X^2} f(X)g(X)dX \ ;$$

$f_i = X^{i-1}$, $i = 1,2,3,4,5$. (*Hint:* You will need the Gaussian integral $\int_{-\infty}^{+\infty} X^{2k}e^{-X^2}dX = 2^{-k}(2k-1)!!\sqrt{\pi}$ where $(2k-1)!!$ is the double factorial defined by $n!! = n \cdot (n-2)!!$ and $(-1)!! = 1$.) [2 marks]

4. Let $\tau : W \times V \to \mathbb{K}$ be a bilinear map, where V and W are vector spaces over a field $\mathbb{K}$. Recall that the dual space is denoted by V^*. Let U, U_i be subspaces of V.

(i) Prove that $U^\perp$, defined by

$$U^\perp = \{\mathbf{w} \in W \mid \tau(\mathbf{w}, \mathbf{u}) = \mathbf{0} \ \ \forall \mathbf{u} \in U\},$$

is a subspace of W. [1 marks]

(ii) Prove that $U \subseteq (U^\perp)^\perp$ [1 marks]

(iii) Prove that $U_1 \subseteq U_2$ implies $U_2^\perp \subseteq U_1^\perp$ and [1 marks]

(iv) Show that the map $T_U : W \to U^*$ defined by $T_U(\mathbf{w})(\mathbf{u}) = \tau(\mathbf{w}, \mathbf{u})$ for $\mathbf{w} \in W$, $\mathbf{u} \in U$ is a linear map from W to U^*, and that $\ker(T_U) = U^\perp$. [2 marks]

(v) Deduce that $\dim(U) + \dim(U^\perp) \geq \dim(W)$. [2 marks]