MA251 - Algebra I

Assignment 1 **Autumn 2013**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 14 OCTOBER** *(Monday of the third week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Find the eigenvalues of each of the following matrices A over the complex numbers $\mathbb{C}$. For each eigenvalue find one corresponding eigenvector, and then write down a matrix P such that $P^{-1}AP$ is diagonal.

(i) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, (iii) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, (iv) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

P2. Find the eigenvalues of the following pairs of matrices, and use them to decide which of the pairs are similar.

(i) $\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$;

(ii) $\begin{pmatrix} 8 & 3 & -6 \\ 2 & 1 & 0 \\ 10 & 4 & -7 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & 10 & -1 \end{pmatrix}$;

(iii) (harder) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix}$.

P3. Say whether the following statements are true or false. If true, briefly state why. If false, give a counterexample. In all parts, $A = (\alpha_{i,j})$ is an $n \times n$ matrix over a field $\mathbb{K}$ containing $\frac{1}{2}$, the trace of A is the sum of diagonal elements: $\mathrm{tr}(A) = \sum_{i=1}^{n} \alpha_{i,i}$.
(i) If A is skew-symmetric, that is $A^T = -A$, then $\mathrm{tr}(A) = 0$.
(ii) If n is odd and A is skew-symmetric, then $\det(A) = 0$.
(iii) If $A^3 = I_n$, then $\det(A) = 1$.
(iv) If $A^2 = A$, then A is singular.
(v) If $n = 2$, then $A^2 - \mathrm{tr}(A)A + \det(A)I_2 = A^2 - \mathrm{tr}(A)A - \frac{1}{2}\mathrm{tr}(A^2)I_2 + \frac{1}{2}\mathrm{tr}(A)^2 I_2 = 0$.

P4. Compute the following determinants. You may want to use elementary row and/or column operations to reduce the matrix to a simpler form first.

(i) $\begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \end{vmatrix}$; (iii) $\begin{vmatrix} 1 & 2 & 1 & -2 & -1 \\ 2 & 0 & 1 & 0 & 1 \\ -3 & 0 & 8 & 0 & -1 \\ 3 & 0 & 0 & -2 & 1 \\ -3 & -5 & -3 & 1 & 2 \end{vmatrix}$;

(iv) $\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$ where θ is some real number.

P5. Write down a matrix in JCF with minimal polynomial $(x+1)(x-2)^3(x+3)^2$ and characteristic polynomial $(1+x)^2(2-x)^4(3+x)^3$.

P6. Let $T: V \to V$ be a linear map, where $\dim(V) = n$, and suppose that $T^n = 0$ and that there exists a vector $v \in V$ with $T^{n-1}(v) \neq 0_V$. Prove that the vectors $v,\ T(v),\ T^2(v), \ldots,\ T^{n-1}(v)$ are linearly independent and that the nullity of T is 1.

P7. For the following matrices A, find minimal polynomials, matrices B, and invertible matrices P, such that $P^{-1}AP = B$ is in Jordan Canonical Form.

$$(i)\ \begin{pmatrix} 4 & 1 \\ -1 & 6 \end{pmatrix} \qquad (ii)\ \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$$

$$(iii)\ \begin{pmatrix} 10 & -4 & -8 \\ 8 & -2 & -8 \\ 8 & -4 & -6 \end{pmatrix} \qquad (iv)\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Let $V = \mathbb{R}_{\leq 2}[x]$ be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the bases $1, x, x^2$ and $1 + x^2,\ 1 + 2x,\ (1 - x)^2$ of V.

(i) Write down the matrices of linear maps $\mathbf{e}_i \mapsto \mathbf{e}'_i$ and $\mathbf{e}'_i \mapsto \mathbf{e}_i$. [1 mark]

(ii) Using the matrices found in (i), write the polynomial $3 - x + x^2$ as a linear combination of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. [1 mark]

(iii) Write the linear map $T: V \to V$ given by $T(f(x)) = f''(x)$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. [1 mark]

(iv) Using the matrices found in (i), write T in the basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. [1 mark]

2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$, where $n \geq 2$, $\mathbb{K}$ is a field. You will have to prove that

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^{n-1} \\ & \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

(i) Show it for $n = 2$. [1 mark]

(ii) Show it for $n = 3$. [1 mark]

(iii) Use induction to show it for general n. (*Hint:* Use elementary column operations to reduce the first row to $(1, 0, 0 \ldots)$, then expand by the first row.) [3 marks]

3. For the following matrices A, find minimal polynomials, matrices B, and invertible matrices P, such that $P^{-1}AP = B$ is in Jordan Canonical Form.

$$(i)\ \begin{pmatrix} 15 & -4 \\ 49 & -13 \end{pmatrix} \qquad (ii)\ \begin{pmatrix} -3 & 17 & -4 \\ -2 & 9 & -2 \\ -2 & 8 & -1 \end{pmatrix} \qquad (iii)\ \begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

[2 marks each]

4. Let $T: V \to V$ be a linear map for which $T^k = 0$ (the zero map) for some $k > 0$. (Linear maps with this property are called nilpotent.) Prove that 0 is an eigenvalue of T, and that it is the only eigenvalue of T. [5 marks]