MA106 – Linear Algebra

Assignment 9 **March 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 2, 3** *and* **5** *only must be handed in by* **2.00 pm** *on* **THURSDAY 15 MARCH** *(Thursday of the tenth week of term), or they will not be marked.*

$\boxed{\mathbf{1.}}$ Let $\mathbb{R}[x]_{\leq 2}$ be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the bases $1, x, x^2$ and $1, (1-x), (1+x)^2$ of $\mathbb{R}[x]_{\leq 2}$.

(i) Write down the basis change matrices from basis $\mathbf{e}'_i$ to basis $\mathbf{e}_i$ and from basis $\mathbf{e}_i$ to basis $\mathbf{e}'_i$. [2 marks]
(*Note*: Use the definition in the lecture notes, which says that the basis change matrix from basis $\mathbf{e}_i$ to basis $\mathbf{e}'_i$ has columns that represent $\mathbf{e}_i$ in terms of the $\mathbf{e}'_i$.)

(ii) Use (i) to write the polynomial $3 - 4x - 2x^2$ as a linear combination of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, and then check your answer by direct calculation. [2 marks]

$\boxed{\mathbf{2.}}$ Determine whether the following pairs of matrices are equivalent. [2 marks each]

(i) $\begin{pmatrix} 1 & 3 & -2 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & -2 \\ -2 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & -1 & -2 & 2 \\ -2 & -1 & 1 & -3 \\ -1 & -5 & -4 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & -1 & -2 \\ -1 & 1 & 3 & 1 \end{pmatrix}$.

$\boxed{\mathbf{3.}}$ Express the matrix $\begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}$ as a product of elementary matrices. [5 marks]

4. Find the eigenvalues of each of the following matrices A over the complex numbers $\mathbb{C}$. For each eigenvalue find one corresponding eigenvector, and then write down a matrix P such that $P^{-1}AP$ is diagonal.

(i) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, (ii) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, (iii) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

$\boxed{\mathbf{5.}}$ Find the eigenvalues of the following pairs of matrices, and use them to decide which of the pairs are similar.

(i) $\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$ and $\begin{pmatrix} 9 & 10 \\ -1 & 1 \end{pmatrix}$; [2 marks]

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & -3 & -2 \\ -4 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 4 & 10 & -1 \end{pmatrix}$; [2 marks]

(iii) $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{pmatrix}$. [3 marks]

6. Show that, if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of A^2. Is the converse true? (Give proof or counterexample.)

7. Let $T : V \to V$ be a linear map satisfying $T^2 = T$.

(i) Show that $\mathrm{Im}(T)$ and $\mathrm{Ker}(T)$ are complementary subspaces of V. (See Section 4 of the course for the definition of complementary subspaces.)

[*Hint*: Write each $\mathbf{v} \in V$ as $\mathbf{v} = T(\mathbf{v}) + (\mathbf{v} - T(\mathbf{v}))$.]

(ii) Show that a basis of $\mathrm{Im}(T)$ and a basis of $\mathrm{Ker}(T)$ together form a basis of V.

(iii) Deduce that T can be represented by the matrix

$$\begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ \mathbf{0}_{n-r,r} & \mathbf{0}_{n-r,n-r} \end{pmatrix}.$$

8. (past exam. question)
(i) State without proof two conditions, each of which implies that an $n \times n$ matrix is similar to a diagonal matrix.

(ii) Find the eigenvalues of the matrix $\begin{pmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}$.

(iii) For each eigenvalue find the set of all possible eigenvectors.

(iv) Determine whether this matrix is similar to a diagonal matrix, giving reasons for your conclusion.