# MA106 – Linear Algebra

**Assignment 8** **March 2011**

*Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems* **2, 4** *and* **5** *only must be handed in by* **2.00 pm** *on* **THURSDAY 8 MARCH** *(Thursday of the ninth week of term), or they will not be marked.*

**1.** An $n \times n$ matrix $B$ is called symmetric if $B = B^T$. For any $n \times n$ matrix $A$, show that both $AA^T$ and $A + A^T$ are symmetric matrices. (Hint: Prove that $(CD)^T = D^T C^T$ if $C$ and $D$ are $n \times n$ matrices.)

$\boxed{\textbf{2.}}$ Let $A$ be an $n \times n$ matrix such that $A^k = O_{n,n}$ (the $n \times n$ zero matrix) for some natural integer $k$. Show that $I_n + A$ is invertible. [7 marks]

**3.** Solve the following sets of linear equations using Cramer's rule.

(i) $x - y = -6, \quad 2x + y = 9.$

(ii) $x - y = 1, \quad y + 2z = 0, \quad 3x - y = -1.$

$\boxed{\textbf{4.}}$ Let $K$ be a subfield of a larger field $L$. (For example, $K, L$ could be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{R}, \mathbb{C}$.)

(i) Let $A$ be an $m \times n$ matrix with entries in $K$. Then $A$ can be regarded either as a matrix in $K^{m,n}$ or as a matrix in $L^{m,n}$. Show that the rank of $A$ is the same in either case. (*Hint*: Use Corollary 8.7 of lecture notes.) [4 marks]

(ii) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $K^n$. Show that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent in $K^n$ if and only if they are linearly independent in $L^n$. (*Hint*: Use (i).) [2 marks]

$\boxed{\textbf{5.}}$ Let $A$ be an $m \times n$ matrix. If $k \le m$ and $l \le n$, then a $k \times l$ matrix $B$ is said to be a *submatrix* of $A$, if $B$ can be obtained from $A$ by deleting some set of $m - k$ rows and $n - l$ columns of $A$.

(For example, if $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$, then $(6)$, $\begin{pmatrix} 3 \\ 11 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 10 & 12 \end{pmatrix}$ are all examples of submatrices of $A$.)

Define the *determinantal rank* of $A$ to be the largest $k$ for which $A$ has a $k \times k$ submatrix with nonzero determinant. Show that this is equal to the usual row or column rank of $A$. [7 marks]

**6.** Show directly (without using row and column operations on matrices), that if $T : U \to V$ is a linear map, then there exist bases of $U$ and $V$ such that the matrix of $T$ looks like

$$\begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ \mathbf{0}_{m-r,r} & \mathbf{0}_{m-r,n-r} \end{pmatrix}.$$

(*Hint*: Choose a basis of the kernel of $T$, extend it to a basis of $U$, reorder the resulting basis of $U$, and then use the images of this basis under $T$ as part of a basis of $V$.)