MA106 – Linear Algebra

Assignment 7 **February 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 2, 4** *and* **5** *only must be handed in by* **2.00 pm** *on* **THURSDAY 1 MARCH** *(Thursday of the eighth week of term), or they will not be marked.*

$\boxed{\textbf{1.}}$ Write the following permutations of the set $\{1,2,3,4,5,6,7,8\}$ in cyclic form, and then express them as composites of transpositions, and hence decide whether they are even or odd permutations. [1 mark each; marks given for correct answers only]

(i) $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 8 & 4 & 1 & 7 & 5 & 3 & 2 & 6 \end{matrix}$; (ii) $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 & 8 \end{matrix}$; (iii) $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 6 & 5 & 1 & 8 & 7 & 2 & 3 \end{matrix}$.

$\boxed{\textbf{2.}}$ Evaluate the following determinants. You may want to use elementary row and/or column operations to reduce the matrix to a simpler form first.

[2 marks for (i), 3 marks for (ii).]

(i) $\begin{vmatrix} 1 & 4 & -1 \\ 2 & 1 & 1 \\ 3 & -4 & 8 \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & 2 & 0 & 6 \\ 2 & 5 & 3 & 0 \\ 1 & 6 & 0 & -2 \\ 0 & -2 & 5 & -1 \end{vmatrix}$.

3. Prove directly, that if A and B are 2×2 matrices, then $\det(AB) = \det(A)\det(B)$.

$\boxed{\textbf{4.}}$ Let K be the finite field with only the two elements 0 and 1, where $1+1=0$.

(i) How many 2×2 matrices with entries in K are there? [2 marks]

(ii) How many of these are non-singular? [4 marks]

$\boxed{\textbf{5.}}$ Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, where $n \geq 2$. Show that

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1\leq i<j\leq n} (\alpha_j - \alpha_i).$$

[*Suggestion*: Do it for $n=2$ and 3 and then try to use induction on n.] [6 marks]

6. (Continuation of Question 4.)
(i) Find a formula for the number of nonsingular $n\times n$ matrices with entries in the field K of order 2.
[*Hint*: Let the rows of the matrix A be $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Then A is nonsingular if and only if the $\mathbf{r}_i$ are linearly independent, which is the case if and only if, for each $i, \mathbf{r}_i$ is not in the subspace of the rowspace spanned by $\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}$. Use this to count the number of possibilities for $\mathbf{r}_i$, once $\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}$ have been chosen.]

(ii) Using a calculator, show that, if an $n\times n$ matrix over K is chosen at random then, as $n\to\infty$, the probability that the matrix is nonsingular approaches the limit (approximately) $0.288788\ldots$.