MA106 – Linear Algebra

Assignment 6 **February 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 3** *and* **4** *only must be handed in by* **2.00 pm** *on* **THURSDAY 23 FEBRUARY** *(Thursday of the seventh week of term), or they will not be marked.*

1. Which of the following statements are true and which false? (If false, then give a counterexample. If true, then give a brief explanation of at most one sentence, referring to results in the lecture notes or previous assignment sheets if necessary. You will get no marks if you just write 'true' or 'false'.)

[1 mark for (i)–(v), 2 marks for (vi)]

(i) A sequence of n linearly dependent vectors in an n-dimensional vector space cannot span the space.

(ii) Any sequence of $n+1$ distinct non-zero vectors in an n-dimensional vector space must span the space.

(iii) If $T : U \to V$ is a linear map, then $\mathrm{Rank}(T) = (\dim(U) + \dim(V))/2$.

(iv) If A is an invertible $n \times n$ matrix, then $A^m \neq O_{n,n}$ for all natural m.

(v) If A is an $n \times n$ matrix with $A^k = I_n$ for some $k > 0$, then A has an inverse.

(vi) If A is an $n \times n$ matrix with $A^6 = I_n$, then either $A^3 = I_n$ or $A^3 = -I_n$.

2. Determine whether the following matrices over $\mathbb{R}$ have inverses and, if so, find them.

(i) $\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$; (ii) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$; (iii) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; (iv) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

3. For which values of $\alpha \in \mathbb{R}$ does the following matrix over $\mathbb{R}$ have an inverse?

$$\begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & 1 \\ 0 & \alpha & 2 \end{pmatrix}.$$

[3 marks]

Calculate the inverse when $\alpha = 1$. [3 marks]

4. Let A be the augmented $m \times (n+1)$ matrix of a system of m linear equations with n unknowns. Let B be the $m \times n$ matrix obtained from A by removing the last column. Let C be the matrix in row reduced form obtained from A by elementary row operations. Prove that the following four statements are equivalent.

(i) The linear equations have no solutions.

(ii) If $\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}$ are the columns of A, then $\mathbf{c}_{n+1}$ is not a linear combination of $\mathbf{c}_1, \ldots, \mathbf{c}_n$.

(iii) $\mathrm{Rank}(A) > \mathrm{Rank}(B)$.

(iv) The lowest non-zero row of C is $(0\ 0\ \cdots\ 0\ 0\ 1)$. [7 marks]

(*Note*: It is probably easiest to do this by proving (i)$\Leftrightarrow$(ii), (ii)$\Leftrightarrow$(iii) and (iii)$\Leftrightarrow$(iv).)