MA106 – Linear Algebra

Assignment 5 **February 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 3** *and* **4** *only must be handed in by* **2.00 pm** *on* **THURSDAY 16 FEBRUARY** *(Thursday of the sixth week of term), or they will not be marked.*

1. Let U and V be vector spaces of dimensions n and m over K, and let $\mathrm{Hom}_K(U, V)$ be the vector space over K of all linear maps from U to V. Find the dimension and describe a basis of $\mathrm{Hom}_K(U, V)$. (You may find it helpful to use the correspondence with $m \times n$ matrices over K.) [4 marks]

2. Reduce the following matrices to row-reduced form by using elementary row operations, and hence find their ranks.

(i) $\begin{pmatrix} 1 & 2 & -6 \\ -3 & -6 & 18 \\ 2 & 4 & -12 \end{pmatrix}$; (ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}$;

(iii) $\begin{pmatrix} 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 2 & 0 \\ 2 & -1 & 2 & -2 & 1 \end{pmatrix}$.

3. Write down the augmented matrices for the following systems of simultaneous linear equations (with coefficients in $\mathbb{R}$), reduce the matrices to row-reduced form, and then write down the set of all solutions of the equations in the form $\underline{\mathbf{x}} + \mathrm{Nullspace}(A)$ where A is the matrix for the original system; for example, one correct solution to (iii) is

$$(-1, 1, 0) + \{\alpha(2, -1, 0) + \beta(-3, 0, 1) \mid \alpha, \beta \in \mathbb{R}\},$$

but you should choose a different particular solution $\underline{\mathbf{x}}$ in your solution to (iii)!

[2 marks each]

(i) $5x - y = 4,\ \ x + y = 1,\ \ 3x - 3y = 2$;

(ii) $-y + 2z = 0,\ \ x - y - z = -2,\ \ -x - y + 5z = 2,\ \ -2x - y + 8z = 4$;

(iii) $x + 2y + 3z = 1,\ \ 2x + 4y + 6z = 2$;

(iv) $x - y + z = 0,\ \ x + y - z = -1,\ \ x + 5y - 5z = 3$.

4. Let $T : V \to V$ be a linear map, where V is a finite-dimensional vector space. Then T^2 is defined to be the composite TT of T with itself, and similarly $T^{i+1} = TT^i$ for all $i \geq 1$. Suppose that $\mathrm{Rank}(T) = \mathrm{Rank}(T^2)$.

(i) Prove that $\mathrm{Im}(T) = \mathrm{Im}(T^2)$. [2 marks]

(ii) For $i \geq 1$, let $U_i : \mathrm{Im}(T) \to \mathrm{Im}(T)$ be defined as the restriction of T^i to the subspace $\mathrm{Im}(T)$ of V. Show that U_i is nonsingular for all i. [3 marks]

(iii) Deduce that $\mathrm{Rank}(T) = \mathrm{Rank}(T^i)$ for all $i \geq 1$. [3 marks]

5. Show that there is only one possible row reduced form that can be obtained from a given matrix A by performing elementary row operations.