MA106 – Linear Algebra

Assignment 4 **February 2011**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **2, 4, 5** *and* **6** *only must be handed in by* **2.00 pm** *on* **THURSDAY 9 FEBRUARY** *(Thursday of the fifth week of term), or they will not be marked.*

1. For the following pairs of matrices A, B, determine whether any of $A+B$, AB and BA are defined and calculate those that are defined.

1. $A = \begin{pmatrix} -1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$;
2. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix}$;
3. $A = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ -3 & 2 \end{pmatrix}$;
4. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

2. Are the following identities satisfied by all $n \times n$ matrices A and B, where $I = I_n$ is the $n \times n$ identity matrix? Give either a proof or counterexample. [2 marks each]

(i) $(A-B)(A+B) = A^2 - B^2$;
(ii) $(A-I)(A^2+A+I) = A^3 - I$.

3. Find the matrices representing the following linear maps $T : \mathbb{R}^3 \to \mathbb{R}^4$, with respect to the standard bases of $\mathbb{R}^3$, $\mathbb{R}^4$:

(i) $T(\alpha, \beta, \gamma) = (\alpha+\beta+\gamma, 2\beta-\gamma, \alpha-\beta, \alpha-\beta)$;

(ii) $T(\alpha, \beta, \gamma) = (0, \alpha+\gamma, \gamma, 2\alpha-3\gamma)$.

4. Find the matrices representing the following geometrical transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$, with respect to the standard basis.

(i) T rotates each point through an angle θ about the x-axis in the direction that moves the point $(0,0,1)$ towards the point $(0,-1,0)$. [2 marks]

(ii) T reflects each point in the (x,z)-plane. [1 marks]

5. (Effect of using a different basis.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $T(\alpha, \beta) = (\alpha+\beta, \alpha-\beta)$. Write down the matrix of T, firstly using the standard basis of $\mathbb{R}^2$ and secondly using the basis $(-1,1)$, $(0,-2)$ of $\mathbb{R}^2$.

[1+2 marks]

6. Let $S : U \to V$ and $T : V \to W$ be linear maps, where U, V and W are vector spaces over the same field K. Prove the following:

(i) $\mathrm{Rank}(TS) \le \mathrm{Rank}(T)$; [2 marks]
(ii) $\mathrm{Rank}(TS) \le \mathrm{Rank}(S)$; [3 marks]
(iii) If $U = V$ and S is nonsingular then $\mathrm{Rank}(TS) = \mathrm{Rank}(T)$; [2 marks]
(iv) If $V = W$ and T is nonsingular then $\mathrm{Rank}(TS) = \mathrm{Rank}(S)$. [3 marks]

[This question is quite difficult, but important. (i) and (iii) are easier than (ii) and (iv). As a hint for (ii), note that if $\mathrm{Rank}(S) = r$, then $\mathrm{Im}(S)$ is spanned by r vectors.]

7. Let $T : V \to W$ be a linear map, where $\dim(V)$ and $\dim(W)$ may be assumed to be finite. Prove:

(i) T is injective if and only if there is a linear map $S : W \to V$ with $ST = I_V$;
(ii) T is surjective if and only if there is a linear map $S : W \to V$ with $TS = I_W$.