MA106 – Linear Algebra

Assignment 2   **January 2011**

*Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 3, 4** and **6** only must be handed in by **2.00 pm** on **THURSDAY 26 JANUARY** (Thursday of the third week of term), or they will not be marked.*

1. Determine whether the following sequences of vectors are linearly dependent or linearly independent. The field of scalars is $\mathbb{R}$ in each case, and V is $\mathbb{R}^n$ for the appropriate value of n.

(i) $(0, 0, 1)$.   [1 mark]

(ii) $(1, -1, 0)$, $(1, 0, -1)$, $(0, -1, 1)$.   [2 marks]

(iii) $(0, 1, -2)$, $(1, 2, 3)$, $(3, 0, -8)$.   [2 marks]

(iv) $(1, 0, 1, -1)$, $(2, -2, 1, 0)$, $(0, -2, 1, 2)$, $(1, -2, 2, 1)$.   [2 marks]

2. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ be vectors in a vector space, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_m$ be linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$. Suppose that $\mathbf{v}$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_m$. Prove that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$.

3. Determine whether the following sequences of vectors span $\mathbb{R}^n$. (*Hint*: One approach is to determine whether the standard basis vectors $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, etc. are linear combinations of the given vectors. If so, then use Qn. 2.)

(i) $(2, 1)$, $(-6, -3)$;   [2 marks]

(ii) $(2, 3, -2)$, $(1, 2, -2)$, $(1, 3, -4)$, $(2, 1, 0)$.   [3 marks]

4. Find a basis of $\mathbb{R}^5$ that contains the vectors $(1, -1, 1, -1, 0)$, $(-1, -1, 1, -1, 0)$, $(-1, 1, 1, -1, 0)$.

[3 marks]

5. The following sequences of vectors in $\mathbb{R}^4$ are *not* bases of $\mathbb{R}^4$. In each case give a reason why it is not a basis in one short sentence.

(i) $(1, 0, 1, 0)$, $(4, 3, 2, 1)$, $(0, 0, 0, 0)$, $(-2, -1.5, -11, 10^{100})$;

(ii) $(1, -1, \sqrt{3}, 5)$, $(5, 0, 2, 3)$, $(0, 3, -2, 0)$, $(1, 0, 1, 4)$, $(-4, 0, 0.5, \pi)$;

(iii) $(6, -3, -2, 5)$, $(1, 2, 1, 0)$, $(-1, -2, -1, 0)$, $(6, 3, 2, -5)$;

(iv) $(1, 0, 1, 2)$, $(3, 0, -1, -5)$, $(-4, 0, -2, 6)$, $(1, 0, -5, 7)$;

(v) $(2, 1, -1, 2^{0.01})$, $(0, 0, 0, -2.5)$, $(5, 5, 5, 5)$.

6. Find the dimension and a basis of the following vector spaces V over the given field K.   [1 mark for each part]

(i) V is the set of all vectors (α, β, γ) in $\mathbb{R}^3$ with $\alpha + 2\beta - 2\gamma = 0$; $K = \mathbb{R}$.

(ii) V is the set of all vectors (α, β, γ) in $\mathbb{R}^3$ with $\alpha - \beta = -\gamma$ and $\alpha + 2\beta = \gamma$; $K = \mathbb{R}$.

(iii) $V = \mathbb{C}^2$; $K = \mathbb{R}$.

(iv) V is the set of all polynomials over $\mathbb{R}$ of degree at most n, in which the sum of the coefficients is 0; $K = \mathbb{R}$.

(v) $K = \mathbb{R}$ and V is the set of functions from $\mathbb{R}$ to $\mathbb{R}$ which are solutions of the differential equation $\frac{d^2 f}{dx^2} - 5f = 0$.