MA106 – Linear Algebra

Assignment 1 **January 2012**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1,3,4,5** *only must be handed in by* **2.00 pm** *on* **Thursday 19 January** *(Thursday of the second week of term), or they will not be marked.*

1. Some of the following expressions define one or more numbers. Decide whether these lie in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, or whether the expression does not define any numbers at all. (Of course, in some cases they will lie in more than one of these sets.)

$$0.62846284\ldots \quad \text{where the '6284' repeats infinitely often,}$$
$$\sqrt{-5}, \quad \sqrt{3249}, \quad e^{-2i\pi}, \quad 0^{-1}.$$

[5 marks]

2. Many of the familiar properties of numbers can be deduced from the axioms `A1-A4, M1-M4` and `D`. For example, the cancellation law of addition, $\alpha+\beta = \alpha+\gamma \Rightarrow \beta = \gamma$ is proved by adding $-\alpha$ to both sides of the equation and using `A2`; and the rule $0\alpha = 0$ for all α can be proved as follows:

$$0\alpha + 0 = 0\alpha \ \text{(by A3)} = (0+0)\alpha \ \text{(by A3)} = 0\alpha + 0\alpha \ \text{(by D)},$$

and then $0\alpha = 0$ follows from the cancellation law of addition.

Prove the following rules from the axioms (or from the two rules just proved above), for all numbers α, β:

(i) $\alpha(-\beta) = (-\alpha)\beta = -(\alpha\beta)$;

(ii) $(-\alpha)(-\beta) = \alpha\beta$;

(iii) $(-1)\alpha = -\alpha$.

3. Let α and β be elements of a field, and suppose that $\alpha\beta = 0$. Prove that at least one of α and β must equal 0. (You may assume the rule $0\alpha = 0$ mentioned in Question 2.) [3 marks]

4. Let K be the set of complex numbers of the form $a + ib\sqrt{7}$, where $a, b \in \mathbb{Q}$ (that is, a and b are rational numbers). Show that K is a "subfield" of $\mathbb{C}$, that is, $1 \in K$ and $\alpha - \beta, \alpha\beta, \gamma^{-1} \in K$ whenever $\alpha, \beta \in K$ and $\gamma \in K \setminus \{0\}$. (It follows that K is itself a field but you don't need to prove this.) [7 marks]

5. Define a *paddock* to be a set in which `A1-A4, M1-M4` and `D` hold, but instead of $0 \neq 1$, we have $0 = 1$. Find an example of a paddock. Prove that every paddock has just one element. Clearly the definition of paddock is of no use outside this exercise. [5 marks]

6. Let K be a field and $\alpha_1, \ldots, \alpha_n \in K$ where $n \geq 1$. Using the axioms of fields only, the sum

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{1}$$

is meaningless, not because of the dots, but because only a sum of two elements is known to be defined! In this exercise we repair this.

Define $\langle \alpha_1, \ldots, \alpha_n \rangle$ recursively by $\langle \alpha \rangle := \alpha$ and

$$\langle \alpha_1, \ldots, \alpha_n \rangle := \langle \alpha_1, \ldots, \alpha_{n-1} \rangle + \alpha_n \quad \text{if } n \geq 2.$$

(a) Prove

$$\langle \alpha_1, \ldots, \alpha_k \rangle + \langle \beta_1, \ldots, \beta_\ell \rangle = \langle \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell \rangle$$

for all $\alpha_i, \beta_j \in K$ and all $k, \ell \geq 1$. Where are you using which axiom for fields? Hint: use induction.

Of course (1) is one of the more usual notations for $\langle \alpha_1, \ldots, \alpha_n \rangle$.

(b) Let s be a bijective map from $\{1, \ldots, n\}$ to itself. Prove

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \langle \alpha_{s(1)}, \ldots, \alpha_{s(n)} \rangle.$$

7. Let V be a vector space over a field K. Deduce the following facts from the vector space axioms. (The first one is done for you!)

(i) $\alpha\mathbf{0} = \mathbf{0}$ for all $\alpha \in K$. Here $\mathbf{0}$ is the zero vector;
(Solution: $\alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0}$, so
$\mathbf{0} = -(\alpha\mathbf{0}) + \alpha\mathbf{0} = -(\alpha\mathbf{0}) + \alpha\mathbf{0} + \alpha\mathbf{0} = \mathbf{0} + \alpha\mathbf{0} = \alpha\mathbf{0}$.)

(ii) $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. Here 0 is the zero element of K;

(iii) $-(\alpha\mathbf{v}) = (-\alpha)\mathbf{v} = \alpha(-\mathbf{v})$, for all $\alpha \in K$ and $\mathbf{v} \in V$.

(iv) If $\alpha\mathbf{v} = \mathbf{0}$ with $\alpha \in K$, $\mathbf{v} \in V$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.