Boyd's conjecture for a family of genus 2 curves

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1 Introduction

Let P be a nonzero laurent polynomial in n variables with coefficients in $\mathbb{C}$ (this means that P is an element of $\mathbb{C}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$). Then we define its *(logarithmic) Mahler measure* $m(P)$ as follows:

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| dt_1 \cdots dt_n. \tag{1}$$

Because all the singularities of the integrand are logarithmic, this integral converges and is well-defined. In general, it is very hard to compute the Mahler measure of a polynomial, there are only very few cases for which explicit formulas are known. In this thesis we will be concerned about polynomials in 2 variables with integer coefficients.

This Mahler measure looks like a strange integral but in fact it is a generalization of the heigth function in algebraic number theory: if α is an algebraic number and P is its minimal polynomial over $\mathbb{Z}$, then the number $m(P)/\deg(P)$ is known as the *height* of α and is denoted by $h(\alpha)$, which can also be defined as

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathrm{val}(K)} \max\left(0, \left[K_v : \mathbb{Q}_{v|_\mathbb{Q}}\right] \log |\alpha|_v\right),$$

where one can take for K an arbitrary number field containing α.

Often it seems that we can relate the Mahler measure of a polynomial to the special value of an L-series. Let's first look at the L-series of a dirichlet character χ of $\mathbb{Z}$. It is defined as follows:

$$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p) p^{-s}}. \tag{2}$$

This L-series converges absolutely and uniformly on compacts sets for $\Re s > 1$ but it has analytic continuation to the entire complex plane together with a functional equation relating $L(\chi, s)$ and $L(\overline{\chi}, 1-s)$.

Let m be a positive integer congruent to 0 or 3 modulo 4. Suppose that χ is an odd real-valued Dirichlet character, with a given minimal period m (a charachter is called odd if $\chi(-1) = -1$). It's known that only one such nontrivial χ exists, namely the Jacobi-symbol $\chi(n) = \left(\frac{-m}{n}\right)$, see for example [26] (even if you don't want to check this, it's a good book to read). Let's denote this character by χ_{-m}. There is a functional equation for $L(\chi, s)$ from which it follows that

$$L'(\chi_{-m}, -1) = \frac{m\sqrt{m}}{4\pi} L(\chi_{-m}, 2), \tag{3}$$

where $L'(\chi, s)$ simply denotes $\frac{d}{ds} L(\chi, s)$.

In [4] the authors show how the Mahler measure of a polynomial P can be expressed in terms of the so called dilogarithm function (for its definition, look at section 5) if P is of the form $A(x)y + B(x)$. In some cases this dilogarithm can be expressed in terms of $L(\chi, 2)$ and thus in terms of $L'(\chi, -1)$. They prove, among many other examples, that

$$m(y + x + 1) = L'(\chi_{-3}, -1) \quad \text{and} \quad m(y + x^2 + x + 1) = \frac{2}{3} L'(\chi_{-4}, -1).$$

Another example of an L-series which seems to occur in the computation of Mahler measures is that of an elliptic curve: if E is an elliptic curve over $\mathbb{Q}$, defined by a minimal equation with coeffincients in $\mathbb{Z}$ then we can define the L-series of E as follows:

$$L(E,s) = \prod_{p|\Delta(E)} \frac{1}{1-a_p p^{-s}} \cdot \prod_{p\nmid\Delta E} \frac{1}{1-a_p p^{-s}+p^{1-2s}},$$

where p runs through all prime numbers and $a_p = p+1-\#E(\mathbb{F}_p)$. This product converges for $\Re s > \frac{3}{2}$. There is also a functional equation relating $L(E,s)$ and $L(E,2-s)$. In fact this follows from the celebrated theorem of Wiles, Taylor and others that every elliptic curve is modular, implying Fermat's Last Theorem.

The first conjectural identity involving an L-series of an elliptic curve was found by Deninger, who gave a cohomological interpretation of $m(P)$, see [7]. He found this way the following formula to hold numerically:

$$m(x+\frac{1}{x}+1+y+\frac{1}{y}) = L'(E,0),$$

where E is the elliptic curve defined by the polynomial $x+\frac{1}{x}+1+y+\frac{1}{y}$ of which the Mahler measure is taken. Inspired by this example, in [4], David Boyd gives several lists of polynomials P of which he conjectures the following:

$$m(P) = rL'(E,0),$$

where $r \in \mathbb{Q}$ and E is an elliptic curve. In most cases, the elliptic curve E is isomorphic to (the jacobian variety of) the zero set of P. For these cases, Fernando Rodriguez Villegas gives an interesting method to tackle the problem (see [17]). But Boyd also gives some families of P which define curves of genus 2. In this thesis we will try to tackle the problem for that case.

The precise family of polynomials that we will study is

$$P_k(x,y) = y^2 + (x^4+kx^3+2kx^2+kx+1)y + x^4.$$

Although the zero set of this polynomial is generically a curve of genus 2, it seems that $m(P_k)$ is a rational multiple of $L'(E,0)$ for $k \in \mathbb{Z}_{\leq 4} - \{-1\}$ with E a certain elliptic curve that is a factor of the jacobian of the genus 2 curve. Also it appears that $m(P_k)$ is a rational multiple of $L'(\chi,-1)$ for $k \in \{-1,8\}$ with χ a certain real-valued Dirichlet character. In this thesis we will prove the following:

Theorem 1 *Let $P_k(x,y) = y^2 + (x^4+kx^3+2kx^2+kx+1)y + x^4$. Then*

$$\begin{aligned} m(P_2) &= L'(E,0) \quad \textit{with } E: y^2 = x^3+1, \\ m(P_{-1}) &= 2L'(\chi_{-3},-1), \\ m(P_8) &= 4L'(\chi_{-4},-1). \end{aligned}$$

These three identities can be found back in propositions 8, 7 and 6 respectively.

Although I succeeded in proving these identities, I still don't understand why such identities *must* be valid. These proofs give a deduction which one can verify step-by-step but they seem to exist coincidentally. It is not at all clear how the arithmetic structure of the polynomials is related to the arithmetic structure of the L-series. We will rewrite the integral to so-called *dilogarithm sums* and these we will rewrite in terms of L-series. It is an open problem whether this is always possible. In the elliptic curve case it is not clear at all how to prove a relation between the dilogarithm sums and the L-series, except for a few instances.

It is based on the *Bloch-Beilinson conjecture.* To be able to understand this conjecture, one first has to read sections 4 and 6 first. Let C be a complete nonsingular curve over $\mathbb{Q}$ and let g be its genus. Let $K_2^T(C)$ be the subgroup of $K_2(\mathbb{Q}(C))$ consisting of those elements for which the tame symbol vanishes at every $\overline{\mathbb{Q}}$-rational point of C. Furthermore, let $H_1(C(\mathbb{C}),\mathbb{Z})^-$ be the subgroup of $H_1(C(\mathbb{C}),\mathbb{Z})$ consisting of the homology classes that are anti-invariant under the action of complex conjugation. It is easy to see that this is a free abelian group of rank g. Define the following pairing:

$$\langle\cdot,\cdot\rangle : H_1(C(\mathbb{C}),\mathbb{Z})^- \times K_2^T(C)/\operatorname{torsion} \to \mathbb{R} : \langle\gamma,\alpha\rangle \mapsto \frac{1}{2\pi}\int_\gamma \eta(\alpha).$$

The original conjecture of Beilinson states the following:

Conjecture 1 *The abelian group $K_2^T(C)/\operatorname{torsion}$ is free of rank g and the given pairing is non-degenerate. If we choose bases for the groups $K_2^T(C)/\operatorname{torsion}$ and $H_1(C(\mathbb{C}),\mathbb{Z})^-$ then the determinant of the pairing is a rational multiple of $\pi^{-2g}L(C,2)$, where $L(C,s)$ denotes the L-series of C.*

However, this conjecture turned out to be incorrect. The rank of $K_2^T(C)/\operatorname{torsion}$ can be higher than 1 in the case $g=1$, as is pointed out in [3]. The computer experiments done in that article inspired Bloch to do a reformulation of the Beilinson conjecture. Instead of looking solely at the curve C over $\mathbb{Q}$, we now take a model $\mathcal{C}$ of C over $\mathbb{Z}$, which is regular and proper. Then not only for each point P of $C=\mathcal{C}_\mathbb{Q}$ but also for each prime number p and each irreducible component D of the fiber $\mathcal{C}_p$, we have a tame symbol $K_2(\mathbb{Q}(C)) \to \mathbb{F}_p(D)^*$. We define $K_2^T(\mathcal{C})$ to be the subgroup of $K_2(\mathbb{Q}(C))$ consisting of those elements for which all the tame symbols, including the new ones, vanish. The reformulated conjecture is now

Conjecture 2 *The abelian group $K_2^T(\mathcal{C})/\operatorname{torsion}$ is free of rank g and the given pairing is nondegenerate. If we choose bases for the groups $K_2^T(\mathcal{C})/\operatorname{torsion}$ and $H_1(C(\mathbb{C}),\mathbb{Z})^-$ then the determinant of the pairing is a rational multiple of $\pi^{-2g}L(C,2)$.*

For another experimental verification of this conjecture, see [9]. To my knowledge, the conjecture is at present far from solved. The rank of $K_2^T(\mathcal{C})/\operatorname{torsion}$ hasn't been proven to be finite. What has been proven is that in the case that E is an elliptic curve with complex multiplication there exists an element of $K_2^T(\mathcal{E})/\operatorname{torsion}$, representable as a sum of elements $\{f,g\}$ whose divisors are fully supported on torsion points, for which the pairing (with any element of $H_1(C(\mathbb{C}),\mathbb{Z})^-$ of course) gives a rational multiple of $\pi^{-2}L(E,2)$, see [2], [8] and also section 7 of this thesis. Also, for several modular curves C, the L-series part of the conjecture follows from the rank part, see [19].

2 Some elementary properties of the Mahler measure

Let n be a positive integer. Then we define the n-torus T^n as follows:

$$T^n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| = |z_2| = \cdots = |z_n| = 1\}.$$

We have a parametrization $[0, 1[^n \to T^n$ by

$$(\theta_1, \ldots, \theta_n) \mapsto (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}).$$

If we make a change of variables using this parametrization, then (1) goes over into

$$m(P) = \frac{1}{(2\pi i)^n} \int \cdots \int_{T^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

If $n < m$, then $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, so if P is a polynomial in n variables we can compute its mahler measure as an n-fold integral, but also as an m-fold integral. It's clear that these two integrals have the same value.

Also note the following trivial identity:

$$m(PQ) = m(P) + m(Q). \tag{4}$$

Lemma 1 *If $n = 1$, then we can compute $m(P)$ with* Jensen's formula*: Write $P = ax^{-k}(x - \alpha_1) \cdots (x - \alpha_m)$, then*

$$m(P) = \log |a| + \sum_{k=1}^{m} \log^+ |\alpha_k|, \quad \textit{where } \log^+ x = \max(0, \log x).$$

Proof: Of course, we may suppose that $P(z) = z - \alpha$. There are 3 cases to consider here: the number α lies inside, outside or on the unit circle.

Let's first consider the case that α lies inside the unit circle. We can rewrite the integral:

$$\int_0^1 \log |e^{2\pi it} - \alpha| dt = \int_0^1 \log |e^{2\pi it}| dt + \int_0^1 \log |1 - \alpha e^{-2\pi it}| dt = -\Re \int_0^1 \log(1 - \alpha e^{2\pi it}) dt$$
$$= -\Re \left(\frac{1}{2\pi i} \int_\gamma \frac{\log(1 - \alpha z) dz}{z} \right),$$

where $\gamma : [0, 1] \to \mathbb{C} : t \mapsto e^{2\pi it}$ walks over the unit circle. Now, we can define a branch of the function $\log(1 - \alpha z)$ on the disc $D(0, \frac{1}{|\alpha|})$ so it follows from the Cauchy integral formula that our integral is equal to $-\Re \log 1 = 0$.

In the case that α lies outside the unit circle we can define a branch of the function $\log(z - \alpha)$ on the disc $D(0, |\alpha|)$. Hence the result follows immediately from the Cauchy integral formula and the fact that $\log |x| = \Re \log x$ for any complex number x.

So it remains to give a proof in case the zero α of P lies on the unit circle. After a rotation over an angle of $-\arg(\alpha)$, we may without loss of generality assume that $\alpha = 1$. Choose $0 < \varepsilon < 1$. From the previous part of this proof it follows that

$$\int_0^1 \log |(1 - \varepsilon)e^{2\pi it} - 1| dt = \log(1 - \varepsilon). \tag{5}$$

We will show now that the integral $\int_0^1 \log|P(e^{2\pi it})|dt$ converges absolutely and that its value is 0. It's clear that one can find a constants $c1, c2 > 0$ such that, for $t \in]-1/2, 1/2]$, the inequality $c_1|t| < |e^{2\pi it} - 1| < c_2|t|$ holds, because the Taylor series of $e^{2\pi it} - 1$ at $t = 0$ starts with $0 + 2\pi it + \cdots$. Hence there is a constant $c > 0$ such that

$$\left|\log|e^{2\pi it} - 1|\right| \leq c + |\log|t|| \,. \tag{6}$$

As everyone knows that $\int_{-1/2}^{1/2} |\log|t||dt$ converges absolutely, it follows that the integral defining $m(P)$ converges absolutely as well. Now we still have to compute its value. From (6) one can easily deduce that $\int_{-T}^{T} \log|e^{2\pi it} - 1|dt = \mathcal{O}(T|\log T|)$ for $T \searrow 0$. Let's now choose $T \in]0, \frac{1}{2}[$ and $\varepsilon = T^2$. We want to compute the difference between the integral of (5) and $\int_0^1 \log|e^{2\pi it} - 1|dt$. If $t \in [-\frac{1}{2}, \frac{1}{2}] - [-T, T]$ then

$$\left|\log|1-(1-\varepsilon)e^{2\pi it}| - \log|1 - e^{2\pi it}|\right| = \left|\log\left|\frac{1-(1-\varepsilon)e^{2\pi it}}{1-e^{2\pi it}}\right|\right| = \left|\log\left|1 + \frac{T^2 e^{2\pi it}}{1-e^{2\pi it}}\right|\right|$$

We can further estimate this expression by realizing that $|1 - e^{2\pi it}| > c_1 T$, using this one easily finds that

$$\left|\log|1-(1-\varepsilon)e^{2\pi it}| - \log|1 - e^{2\pi it}|\right| = \mathcal{O}(T).$$

It's also clear that $\int_{-T}^{T} |\log|1-(1-\varepsilon)e^{2\pi it}|dt = \mathcal{O}(T|\log T|)$. Hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \log|1-(1-\varepsilon)e^{2\pi it}|dt - \int_{-\frac{1}{2}}^{\frac{1}{2}} \log|1-e^{2\pi it}|dt = \mathcal{O}(T) + \mathcal{O}(T|\log T|),$$

as one can now see by splitting up the integration interval $[-\frac{1}{2}, \frac{1}{2}]$ into the two pieces $[-\frac{1}{2}, \frac{1}{2}] - [-T, T]$ and $[-T, T]$. Letting $T \searrow 0$ and using (5) we see that $m(P) = 0$. □

Another elementary but important identity is the following:

Lemma 2 *Let $A = (a_{ij})_{i,j}$ be a $n \times n$-matrix with integer coefficients and nonzero determinant. Then*

$$m(P(x_1, \ldots, x_n)) = m(P(x_1^{a_{11}} \cdots x_n^{a_{1n}}, \ldots x_1^{a_{n1}} \cdots x_n^{a_{nn}}))$$

Proof: It's clear that A defines a $|\det A|$-fold cover of $(\mathbb{R}/\mathbb{Z})^n$ by itself. Using this and some elementary calculus we can easily verify that

$$\begin{aligned}
m(P) &= \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi it_1}, \ldots, e^{2\pi it_n})|dt_1 \cdots dt_n \\
&= \frac{1}{|\det A|} \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i(a_{11}t_1 + \cdots + a_{1n}t_n)}, \ldots, e^{2\pi i(a_{n1}t_1 + \cdots + a_{nn}t_n)})| \\
&\quad d(a_{11}t_1 + \cdots + a_{1n}t_n) \cdots d(a_{n1}t_1 + \cdots + a_{nn}t_n) \\
&= \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i(a_{11}t_1 + \cdots + a_{1n}t_n)}, \ldots, e^{2\pi i(a_{n1}t_1 + \cdots + a_{nn}t_n)})|dt_1 \cdots dt_n \\
&= m(P(x_1^{a_{11}} \cdots x_n^{a_{1n}}, \ldots x_1^{a_{n1}} \cdots x_n^{a_{nn}})).
\end{aligned}$$

□

2.1 Polynomials of the form $A(x)y^2 + B(x)y + C(x)$

Suppose that P is of the following special form:

$$P(x, y) = A(x)y^2 + B(x)y + C(x).$$

Suppose furthermore that

$$|C(x)/A(x)| = 1 \quad \text{for all } |x| = 1. \tag{7}$$

Then, for fixed $x = x_0$ of absolute value equal to 1, the equation $A(x_0)y^2 + B(x_0)y + C(x_0) = 0$ will generally have two roots for y (unless $A(x_0) = 0$), whose product is of absolute value 1. This means that either both roots are on the unit circle or one of them lies inside the unit circle and the other one outside the unit circle. If one root lies outside the unit circle and one inside, then denote the root outside the unit circle by $y_+(x_0)$ and we see that from lemma 1 it follows that

$$m(P(x_0, y)) = \log|A(x_0)| + \log|y_+(x_0)|.$$

If both roots are on the unit circle, then for the value of the Mahler measure it doesn't really matter which root we take to be $y_+(x_0)$, since $\log|y_+(x_0)| = 0$ for both roots; we have to make a 'smart' choice for $y_+(x_0)$ in this case. So we can compute $m(P)$ as follows:

$$m(P) = m(A(x)) + \int_0^1 \log|y_+(e^{2\pi it})|dt = m(A(x)) + \frac{1}{2\pi i}\int_{T^1} \log|y_+(x)|\frac{dx}{x} \tag{8}$$

To be able to tackle the computation of (8), we want y_+ to satisfy the following condition:

Condition 1 *The function y_+, which sends x to a zero of $A(x)y^2 + B(x)y + C(x)$ with absolute value at least one is a continuous complex-valued function on the unit circle, which can be extended to a function that is holomorphic on an open set U which contains all points of the unit circle except possibly a finite number of points where the discriminant $D(x) = B(x)^2 - 4A(x)C(x)$ vanishes.*

Let C be the curve over $\mathbb{C}$ defined as the zero set of P. If condition 1 is satisfied, then the function

$$\gamma : [0, 1] \mapsto (\exp(2\pi it), y_+(\exp(2\pi it))) \tag{9}$$

defines a closed, piecewise smooth path on C. If furthermore $A(x)$ is a product of a power of x and some cyclotomic polynomials, then from lemma 1 it follows that $m(A(x)) = 0$ and we can compute $m(P)$ as follows:

$$m(P) = \frac{1}{2\pi i}\int_\gamma \log|y|\frac{dx}{x} = \frac{1}{2\pi i}\int_\gamma \log|y|\frac{dx}{x} - \log|x|\frac{dy}{y}.$$

Suppose now that P has real coefficients. Then we can put an extra condition on y_+, namely

$$y_+(\overline{x}) = \overline{y_+(x)} \quad \text{for all } |x| = 1 \text{ and } x \in U. \tag{10}$$

The path γ has the interesting property that $\gamma(1 - t) = \overline{\gamma(t)}$. In other words, if we reverse γ we get the path $\overline{\gamma}$. This means that y_+ satisfies (10). We will see later (lemma 22) that this is very important in order to compute $m(P)$. Because $m(P)$ is real we might as well take the imaginary part of the integrand to compute it:

$$m(P) = \frac{1}{2\pi}\int_\gamma \log|y|d\arg x - \log|x|d\arg y. \tag{11}$$

The differential $\log|y|d\arg x - \log|x|d\arg y$ is studied in more detail in section 6.

3 Elliptic curves

This section will be used to state and refresh some general and well-known results and definitions about elliptic curves, which we will need. Most things here will be obtained from [21], so proofs will be omitted here when they can be found there. Moreover, it might be worth mentioning that implemented algorithms that can compute various properties of elliptic curves can be found at [6].

Let K be a perfect field. An *elliptic curve* E over K is a nonsingular projective curve over K of genus one, with a given K-rational point $\mathcal{O}$ on it. For example all nonsingular projective cubic plane curves with a given rational point have this property. We can always put this curve in *Weierstrass form*:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad \mathcal{O} = [0:1:0],$$

with $a_1, \ldots, a_6 \in K$. Note that there is no a_5. We have additional quantities

$$\begin{aligned}
&b_2 = a_1^2 + 4a_2,\ b_4 = 2a_4 + a_1a_3,\ b_6 = a_3^2 + 4a_6,\ b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,\\
&c_4 = b_2^2 - 24b_4,\ c_6 = -b_2^3 + 36b_2b_4 - 216b_6,\\
&\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6,\ j = c_4^2/\Delta,\\
&\omega = dx/(2y + a_1x + a_3) = dy/(3x^2 + 2a_2x + a_4 - a_1y).
\end{aligned}$$

The number Δ is called the *discriminant* of the Weierstrass normal form of E, the number j is called the *j-invariant* of E and ω is called the *invariant differential* of the Weierstrass normal form of E. Two elliptic curves are isomorphic over $\overline{K}$ if and only if they have the same j-invariant.

The points of E (you can take the L-rational points of any field extension of K if you like) form an abelian group, in the following way (we suppose that E has a given Weierstrass normal form): the point $\mathcal{O}$ is the zero element of E. If $P = (x, y) \in E$, then

$$-P = (x, -y - a_1x - a_3).$$

Now let $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$. Suppose that $P_1 \neq -P_2$ (otherwise we already know that $P_1 + P_2 = \mathcal{O}$). If $x_1 \neq x_2$, let

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1},\ \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1},$$

otherwise let

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3},\ \nu = \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}.$$

Then we define $P_1 + P_2 = (x_3, y_3)$ by

$$x_3 = \lambda^2 + a_1\lambda - a_2 - x_1 - x_2,\ y_3 = -(\lambda + a_1)x_3 - \nu - a_3.$$

There is also a geometric interpretation of this group operation: draw a line though P and Q and define $P * Q$ as the third intersection point of this line with E. Then $P + Q$ is the point that we get if we draw a vertical line trough $P * Q$ and check what the second intersection point of this line with E is.

If we have a Weierstrass equation with discriminant equal to 0, then this does not define an elliptic curve but it defines a curve of genus 0 with singular point S, say. Let E_{ns} be the set of nonsingular points of E. Then the above formulas still define a group law on E_{ns}. There are 2 possibilities for the singularity: it could be either a cusp or a node. If it is a cusp, which is the case if and only if $c_4 = 0$, then we have a tangent line $y = \alpha x + \beta$ to E at S. The map

$$E_{ns} \to \overline{K} : (x, y) \mapsto \frac{x - x(S)}{y - \alpha x - \beta}$$

will be an isomorphism of groups. This type of singularity is called *additive*. The other type of singularity, which we call *multiplicative*, appears when it is a node, in that case $c_4 \neq 0$. There are two tangent lines, $y = \alpha_1 x + \beta_1$ and $y = \alpha_2 x + \beta_2$ say. We have the following isomorphism of abelian groups:

$$E_{ns} \to \overline{K}^* : (x, y) \mapsto \frac{y - \alpha_1 x - \beta_1}{y - \alpha_2 x - \beta_2}.$$

In the additive case, the isomorphism is always defined over K. In the multiplicative case, there are 2 possibilities again. If $\alpha_1, \alpha_2 \in K$, then the isomorphism is defined over K and we speak of a *split* case. Otherwise, α_1, α_2 are roots of an irreducible quadratic polynomial over K hence the isomorphism is defined over the quadratic extension $K(\alpha_1, \alpha_2)$ of K and we speak of a *non-split* case.

3.1 Elliptic curves over $\mathbb{Q}$

Let for the moment K be equal to $\mathbb{Q}$. Then we can always get a Weierstrass equation with coefficients in $\mathbb{Z}$. We call such an equation *minimal* if $|\Delta|$ is minimal. This is the case if and only if the valuation $\mathrm{ord}_p(\Delta)$ is minimal for each prime number p. For $p \neq 2, 3$, the number $\mathrm{ord}_p(\Delta)$ is minimal if and only if $\mathrm{ord}_p(\Delta) < 12$ and $\mathrm{ord}_p(c_4) < 4$. And for all p, if $\mathrm{ord}_p(\Delta) < 12$ then $\mathrm{ord}_p(\Delta)$ is minimal.

Suppose now that we are given such a minimal Weierstrass equation for E. Let p be a prime. Then we can take this equation modulo p. This equation defines a curve over $\mathbb{F}_p$ which we call the *reduction* of E modulo p, which we will denote by $E/\mathbb{F}_p$. If $p \nmid \Delta$, then $E/\mathbb{F}_p$ will be an elliptic curve and we say that E has *good* reduction modulo p. However, if $p \mid \Delta$, then $\Delta(E/\mathbb{F}_p) = 0$ so $E/\mathbb{F}_p$ will be singular. As described above, we have different types of singular reduction, namely additive, split multiplicative and non-split multiplicative.

There is also a number $N = N(E)$, called the *conductor* of E. Its precise definition is not of importance here but we can write is as a product

$$N(E) = \prod_p p^{f_p},$$

where f_p is related to the reduction behavior of E modulo p. In any case,

$$f_p \leq \mathrm{ord}_p \Delta(E).$$

If E has good reduction, then $f_p = 0$. If the reduction is multiplicative, then $f_p = 1$. If the reduction is additive then $f_p \geq 2$, where equality holds if $p \neq 2, 3$.

3.2 Elliptic curves over $\mathbb{C}$

Now, let E be an elliptic curve over $\mathbb{C}$. In this case we can use complex analytic methods to study E. Namely, we can always find a lattice $\Lambda \subset \mathbb{C}$ such that $E \cong \mathbb{C}/\Lambda$ (as curves and as groups). We can find Λ as follows: let ω be a holomorphic differential on E, then

$$\Lambda = \left\{ \int_\gamma \omega : \gamma \text{ closed piecewise smooth path on } E \right\}. \tag{12}$$

Let $P \in E$ and choose a path γ from $\mathcal{O}$ to P on E. Then

$$P \mapsto \int_\gamma \omega \tag{13}$$

defines an isomorphism $E \to \mathbb{C}/\Lambda$. Every algebraic endomorphism of E can via the identification be written as multiplication by an element $\alpha \in \mathbb{C}$ such that $\alpha\Lambda \subset \Lambda$. For most curves, only $\alpha \in \mathbb{Z}$

suffice this condition. If there is an $\alpha \notin \mathbb{Z}$ that suffices this condition, then we say that E has *complex multiplication*. In this case, if $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, then $\mathbb{Q}(\tau)/\mathbb{Q}$ is a quadratic extension. Note that we can always scale Λ in such a way that it becomes of the form $\mathbb{Z} + \tau\mathbb{Z}$.

If E is defined over $\mathbb{R}$ then we can say even more about Λ:

Lemma 3 *Let E be an elliptic curve over $\mathbb{R}$. Then there is a canonical Λ of the form $\mathbb{Z}+\tau\mathbb{Z}$ such that E is isomorphic to $\mathbb{C}/\Lambda$ over $\mathbb{R}$, where we give $\mathbb{C}/\Lambda$ an $\mathbb{R}$-structure via complex conjugation. In particular, $\overline{\Lambda} = \Lambda$.*

Proof: Let ω be a holomorphic differential form on E defined over $\mathbb{R}$. Then

$$\Omega := \left\{ \int_\gamma \omega : \gamma \text{ closed piecewise smooth path on } E \right\}$$

is a lattice in $\mathbb{C}$. There is a closed path on E of which every point is defined over $\mathbb{R}$ so Ω contains a smallest positive real number, ω_1 say. Define $\Lambda = \frac{1}{\omega_1}\Omega$. We can construct an isomorphism of E with $\mathbb{C}/\Lambda$ as follows. Let P be a point of E and let γ_P be an arbitrary piecewise smooth path on E which start at $\mathcal{O}$ and ends at P. Define

$$\phi(P) := \frac{1}{\omega_1} \int_{\gamma_P} \omega \mod \Lambda.$$

Then ϕ defines an isomorphism of E with $\mathbb{C}/\Lambda$. To prove that ϕ is defined over $\mathbb{R}$ we have to verify that $\phi(\overline{P}) = \overline{\phi(P)}$ for all $P \in E$. This is immediate, as we can take for $\gamma_{\overline{P}}$ the conjugate of the path γ_P and ω is defined over $\mathbb{R}$. It is now also clear that $\overline{\Lambda} = \Lambda$, otherwise complex conjugation is not well-defined on $\mathbb{C}/\Lambda$. $\square$

Lemma 4 *Let E be an elliptic curve over $\mathbb{R}$. Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be the lattice that belongs to E according to lemma 3. If $\Delta(E) < 0$ then we can choose τ of the form $\tau = 1/2 + it$ for some $t \in \mathbb{R}_{>0}$. And if $\Delta(E) > 0$ then we can choose τ of the form $\tau = it$ for some $t \in \mathbb{R}_{>0}$.*

Proof: Obviously, as $\overline{\Lambda} = \Lambda$, we can always find τ of one of the two given forms. Now, $\Delta(E) > 0$ if and only if all the 2-torsion points are defined over $\mathbb{R}$. This is the case if and only if the corresponding points in $\mathbb{C}/\Lambda$ satisfy $z = \overline{z}$, which in turn is the case if and only if Λ is of the form $\mathbb{Z} + it\mathbb{Z}$. $\square$

We can also do the converse: given a lattice $\Lambda \subset \mathbb{C}$, construct an elliptic curve that has its lattice equal to Λ. We use the *Weierstrass $\wp$-function*, defined as follows:

$$\wp = \wp_\Lambda : \mathbb{C} - \Lambda \to \mathbb{C} : z \mapsto \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

Then actually $\wp(z+\lambda) = \wp(z)$ for all $z \in \mathbb{C} - \Lambda$ and $\lambda \in \Lambda$ and all singularities are poles (of order 2) so that $\wp$ defines a meromorphic function on $\mathbb{C}/\Lambda$. The derivative $\wp'(z)$ of $\wp(z)$ satisfies

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}.$$

Now define the *Eisenstein series of weight 2k* as

$$G_{2k} = G_{2k}(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{\lambda^{2k}}.$$

Then we have an isomorphism from $\mathbb{C}/\Lambda$ to

$$E : y^2 = 4x^3 - 60G_4(\Lambda) - 140G_6(\Lambda)$$

defined by

$$z \mapsto \begin{cases} [0:1:0] & \text{if } z \in \Lambda, \\ (\wp(z), \wp'(z)) & \text{otherwise.} \end{cases} \tag{14}$$

This isomorphism is inverse to the isomorphism defined by (13) if we choose $\omega = dx/y$. If $\Lambda = \overline{\Lambda}$, then $\wp$, E, and the isomorphism from $\mathbb{C}/\Lambda$ to E are clearly defined over $\mathbb{R}$.

3.3 Divisors on an elliptic curve

If C is an arbitary projective non-singular curve over an algebraically closed field K, then a *divisor* on C is a formal sum of points, often written as

$$D = \sum_{P \in C} n_P[P] \quad \text{with } n_P \in \mathbb{Z} \text{ for all } P.$$

These divisors form a free abelian group which we denote by $\mathrm{Div}(C)$. We define the degree of D by

$$\deg D = \sum_{P \in C} n_P,$$

this is a group homomorphism from $\mathrm{Div}(C)$ to $\mathbb{Z}$, whose kernel we denote by $\mathrm{Div}^0(C)$.

Let P be a point of C. The ring $\mathcal{O}_{C,P}$ of functions that are defined at P is a local ring with maximal ideal m_P say. For a function f that is defined at P we define $\mathrm{ord}_P(f)$ as the biggest integer n such that $f \in m_P^n$. Furthermore we define $\mathrm{ord}_P(f/g) = \mathrm{ord}_P(f) - \mathrm{ord}_P(g)$ so that we have a discrete valuation $\mathrm{ord}_P : K(C)^* \to \mathbb{Z}$. We can also define $\mathrm{ord}_P(\omega)$ of a differential ω. Let t be a uniformizer at P, then $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(\omega/dt)$. If $\mathrm{char}(K) = 0$ and $\mathrm{ord}_P(f) > 0$, then $\mathrm{ord}_P(df) = \mathrm{ord}_P(f) - 1$. By successively applying d to f we can compute $\mathrm{ord}_P(f)$ with this relation.

Now, if $f \in K(C)^*$, then we define the divisor (f) as follows:

$$(f) = \sum_{P \in C} \mathrm{ord}_P(f)[P].$$

A divisor which can be written as (f) for some f is called a *principal* divisor. Similarly, for a differential ω we define (ω) as follows:

$$(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega)[P]$$

and every divisor of this form is called a *canonical* divisor. Every principal divisor has degree 0, and clearly the principal divisors form a group since $f \mapsto (f)$ is a group homomorphism from $K(C)^*$ to $Div(C)$. So the group PDiv(C) of principal divisors is a subgroup of $\mathrm{Div}^0(C)$. We call the quotient group $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\,\mathrm{PDiv(C)}$ the *Picard* group of C.

If now E is an elliptic curve something remarkable happens: the group structure of E is isomorphic with $\mathrm{Pic}^0(E)$ and the following map is an isomorphism:

$$E \to \mathrm{Pic}^0(E) : P \mapsto [[P] - [\mathcal{O}]].$$

In particular, a divisor $D = \sum n_P[P]$ on E is principal if and only if $\sum n_P = 0$ and $\sum n_P P = \mathcal{O}$, where the last sum is taken in the group structure of E.

4 Steinberg symbols and K_2

In this section we will give some very basic K-theory, which is needed for the rest of this thesis. No knowledge of homological algebra (which is used very often if one wants to go deeper into K-theory) is needed to understand this section.

Let F be a field. A *Steinberg symbol* on F is a bilinear map

$$c : F^* \times F^* \to A,$$

where A is an abelian group whose operation we write multiplicatively for the moment, which satisfies the following condition:

$$c(a, 1-a) = 1 \quad \text{for all } a \in F^* - \{1\}.$$

Note that bilinear here means that $c(xy, z) = c(x, z)c(y, z)$ and $c(x, yz) = c(x, y)c(x, z)$ since the group F^* is also written multiplicatively.

Directly linked to these Steinberg symbols is the group $K_2(F)$, which we define as follows:

$$K_2(F) := (F^* \otimes F^*)/I,$$

where I is the abelian subgroup of $F^* \otimes F^*$ generated by all elements of the form $a \otimes (1-a)$. We will write the coset represented by $x \otimes y$ as $\{x, y\}$ and we will write the group operation multiplicatively. Furthermore, we will let an automorphism σ of F act on $K_2(F)$ in the following way:

$$\sigma(\{x, y\}) := \{\sigma(x), \sigma(y)\},$$

multiplicatively extending this to all of $K_2(F)$. This is well-defined because every automorphism σ maps I to itself.

It might be worth noting that not only $\{a, 1-a\}$ is always equal to 1 but also $\{a, -a\} = 1$:

$$\{a, -a\} = \{a, 1-a\}\left\{a, \frac{a-1}{a}\right\} = \left\{\frac{1}{a}, \frac{a-1}{a}\right\}^{-1} = \left\{\frac{1}{a}, 1 - \frac{1}{a}\right\}^{-1} = 1.$$

From this it follows that $\{a, a\}^2 = 1$ because $\{a, a\}^2 = (\{a, -a\}\{a, -1\})^2 = \{a, -1\}^2 = \{a, 1\} = 1$. Also, the symbol $\{-, -\}$ is skew-symmetric:

$$\{a, b\}\{b, a\} = \{a, -a\}\{a, b\}\{b, a\}\{b, -b\} = \{a, -ab\}\{b, -ab\} = \{ab, -ab\} = 1. \tag{15}$$

It is immediate that the Steinberg symbols on F with values in A are in 1-1 correspondence with the homomorphisms $K_2(F) \to A$, where c corresponds to the function $\{x, y\} \mapsto c(x, y)$. As a slight abuse of notation we will denote this homomorphism also with c. Note also that simply sending (x, y) to $\{x, y\}$ gives a (universal) Steinberg symbol on F.

A very important example of a Steinberg symbol is the so-called *tame symbol*: let v be a discrete valuation on F (i.e. a surjective homomorphism $v : F^* \to \mathbb{Z}$ which satisfies $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in F^*$). We see that $\mathcal{O}_v := \{x \in F : v(x) \geq 0\} \cup \{0\}$ is a local ring, with maximal ideal equal to $m_v := \{x \in F : v(x) > 0\} \cup \{0\}$ and residue field equal to $k_v := \mathcal{O}_v/m_v$. We define the tame symbol on F at v as follows:

$$(x, y)_v := (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \bmod m_v. \tag{16}$$

It is not so difficult to check that this is indeed a Steinberg symbol on F with values in k_v^*. The only not completely trivial part is to check that $(x, 1-x)_v = 1$ for all $x \in F$. If $v(x) > 0$ then $v(1-x) = 0$ so

$$(x, 1-x)_v \equiv \frac{1}{1-x} \equiv 1 \bmod m_v,$$

if $v(1-x) = v(x)$, which is for example the case when $v(x) < 0$, then

$$(x, 1-x)_v \equiv (-1)^{v(x)^2} \left(\frac{x}{1-x}\right)^{v(x)} \equiv (-1)^{v(x)} \left(\frac{1}{\frac{1}{x} - 1}\right)^{v(x)} \equiv 1 \bmod m_v$$

and if $v(x) = 0$ then either $v(1-x) = 0$ or $v(1-x) > 0$ and each of these two cases is equivalent to one of the above after interchanging x and $1-x$. The homomorphism from $K_2(F)$ to k_v^* belonging to the tame symbol $(-,-)_v$ will be denoted by

$$\partial_v : K_2(F) \to k_v^*.$$

If F is the function field of a complete non-singular algebraic curve C over an algebraically closed field K, then the discrete valuations on F are in 1-1 correspondence with the closed points of C, where P corresponds to $f \mapsto \operatorname{ord}_P(f)$. So the tame symbol becomes now equal to

$$(f, g)_P = (-1)^{\operatorname{ord}_P(f) \operatorname{ord}_P(g)} \left.\frac{f^{\operatorname{ord}_P(g)}}{g^{\operatorname{ord}_P(f)}}\right|_P \tag{17}$$

Let $\phi : C_1 \to C_2$ be a non-constant morphism of algebraic curves. For $P \in C_1$, we denote by $e_P(\phi)$ the ramification index of ϕ at P. From the fact that $\operatorname{ord}_P(\phi^* f) = e_P(\phi) \operatorname{ord}_{\phi(P)}(f)$ for all $f \in K(C)^*$ one can immediately compute that

$$\partial_P\{\phi^* f, \phi^* g\} = \left(\partial_{\phi(P)}\{f, g\}\right)^{e_P(\phi)} \tag{18}$$

If S is a finite set of points, then we define $K_2(C, S)$ as the subgroup of $K_2(F)$ consisting of those elements $\{f, g\}$ for which $(f, g)_P$ is a root of unity for all $P \in C - S$, or in other words $\{f, g\}$ should be in the kernel of the map

$$K_2(F) \to \left(\coprod_{P \in C-S} k_P^*\right) \otimes \mathbb{Q} : \{f, g\} \mapsto \left(\coprod_{P \in C-S} (f, g)_P\right) \otimes 1.$$

Now let E/F be a finite extension of fields. We will now try to study a bit how $K_2(F)$ and $K_2(E)$ are related. One can consider the canonical homomorphism

$$\operatorname{res}_{E/F} : K_2(F) \to K_2(E) : \{x, y\} \mapsto \{x, y\},$$

which we call the *restriction* homomorphism. In general, this map is not injective. However its kernel contains only torsion elements so it is not that far from being injective. This will follow immediately from the first condition in proposition 1.

Besides the restriction homomorphism, there also exists a norm homomorphism. Its precise definition is rather lenghty and we do not need it in this thesis. However we do need its existence and certain properties:

Proposition 1 *For each finite extension of fields there is a homomorphism $N_{E/F} : K_2(E) \to K_2(F)$ that satisfies the following conditions:*

$$\begin{aligned} N_{E/F}\left(\operatorname{res}_{E/F}(x)\right) &= x^{[E:F]} && \text{for all } x \in K_2(E), \\ N_{E/F}\{x,y\} &= \{x, N_{E/F}(y)\} && \text{if } x \in F \text{ and } y \in E, \\ \operatorname{res}_{E/F}\left(N_{E/F}(x)\right) &= \prod_{\sigma \in \operatorname{Gal}(E/F)} \sigma(x) && \text{if } E/F \text{ is a Galois extension,} \\ N_{E/F} \circ N_{L/E} &= N_{L/F} && \text{if } L/E \text{ is another finite extension.} \end{aligned}$$

Proof: In [1] Bass and Tate give a construction of the norm function using a set of generators for E/F. They prove the first and second equality. They also prove the fourth equality if this function is independent of the chosen set of generators. Kato shows in [14] that the function constructed in [1] really is independent of this set of generators so that the fourth equality follows. It remains to prove that $\operatorname{res}_{E/F}\left(N_{E/F}(x)\right) = \prod_{\sigma \in \operatorname{Gal}(E/F)} \sigma(x)$ if E/F is a Galois extension. From the commutative diagram on page 39 in [1], this equality immediately follows in the case that E/F is a simple extension. However, a Galois extension is always separable and a finite separable extension is always simple (see sections V.3 and V.6 of [13]). $\square$

Now, we will discuss a manipulation trick which is very useful if one wants to compute norms. Let $E = F(\alpha)$. Suppose we want to compute $N_{E/F}\{x,y\}$ where $x = b_1\alpha - a_1$ and $y = b_2\alpha - a_2$, with $\alpha_i, \beta_i \in F$. Put $a = y/b_2 - x/b_1 \in F$. Then $y/(ab_2) + x/(-ab_1) = 1$ so $\{x/(-ab_1), y/(ab_2)\} = 1$. If we work this out, we see that this is equal to $\{x,y\}\{ab_2,x\}\{-ab_1,y\}^{-1}\{-ab_1,ab_2\}$. Hence

$$\{x,y\} = \{-ab_1, y\}\{ab_2, x\}^{-1}\{-ab_1, ab_2\}^{-1} \tag{19}$$

and proposition 1 now shows how to compute $N_{E/F}\{x,y\}$.

5 Dilogarithms and Eisenstein-Kronecker-Lerch series

In this section we will introduce the dilogarithm function as well as the elliptic dilogatithm functions. As is pointed out in [5], in some special cases the Mahler measure of a polynomial P is closely related to values of the dilogarithm, in case the polynomial P defines a curve of genus 0. In [17] it is explained how the mahler measure of a polynomial P relates to the elliptic dilogarithm in case P defines a curve of genus 1.

5.1 The dilogarithm

This subsection will be used to introduce the Bloch-Wigner dilogarithm and some elementary properties will be given. Many of these properties will be proven in section 6, also see [2], [24] and [25].

The *Bloch-Wigner dilogarithm* is defined as

$$D(z) := \log|z|\arg(1-z) - \Im\left(\int_0^z \log(1-t)\frac{dt}{t}\right), \tag{20}$$

where we take the principal branch of the arg function and the path of integration is a straight line segment from 0 to z. Initially, this is defined on $\{z \in \mathbb{C} : 0 < |z| < 1\}$ but it can be extended to a real-valued continuous function on $\mathbb{C} \cup \{\infty\}$ which is real-analytic on $\mathbb{C} - \{0, 1\}$.

The dilogarithm satisfies the following relation:

$$D(1/z) = D(\overline{z}) = -D(z) = D(1-z) \quad \text{for all } z \in \mathbb{C} \cup \{\infty\} \tag{21}$$

and $D(0) = D(1) = D(\infty) = 0$. From (21) it follows immediately that

$$D(x) = 0 \quad \text{if } x \in \mathbb{R}. \tag{22}$$

We can use relation (21) to compute $D(z)$ for $z \in \mathbb{C}$ of absolute value greater than 1. To compute $D(z)$ for z of absolute value equal to 1, one can use the following formula:

$$D(e^{i\theta}) = -\int_0^\theta \log|1 - e^{it}|dt.$$

We can also define the dilogarithm in terms of a power series:

$$D(z) = \log|z|\arg(1-z) + \Im(Li_2(z)), \quad \text{where } Li_2(z) = \textstyle\sum_{n=1}^\infty \frac{z^n}{n^2}.$$

By writing down the power series for $\log(1-z)$ it is immediate that this definition is equivalent with (20). The power series defining Li_2 converges for all z with $|z| \leq 1$ so it follows that

$$D(e^{i\theta}) = \Im\left(\sum_{n=1}^\infty \frac{e^{in\theta}}{n^2}\right) = \sum_{n=1}^\infty \frac{\sin(n\theta)}{n^2}. \tag{23}$$

The function $\theta \mapsto \sum_{n=1}^\infty \frac{\sin(n\theta)}{n^2}$ is known as Clausen's function and denoted by $Cl_2(\theta)$.

Other interesting identities of the dilogarithm function are the following:

$$\frac{1}{n}D(z^n) = \sum_{k=0}^{n-1} D(e^{2\pi i\frac{k}{n}}z) \tag{24}$$

and

$$D(x) + D(y) + D(1-xy) + D(\frac{1-x}{1-xy}) + D(\frac{1-y}{1-xy}) = 0. \tag{25}$$

5.2 The elliptic dilogarithm

In this section we will introduce the elliptic dilogarithm, which is an analogue of the ordinary dilogarithm, but in this case the function is defined on an elliptic curve instead of a rational curve. For proofs of the properties we'll give here, see again [2] and [24].

Let E be an elliptic curve over $\mathbb{C}$. Choose $\tau \in \mathfrak{h}$ such that $E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and set $q = e^{2\pi i\tau}$. A complex point on E corresponds to an element $\zeta + (\mathbb{Z}+\mathbb{Z}\tau)$ of $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Write $z = e^{2\pi i\zeta}$. Then we define the *elliptic dilogarithm* as follows:

$$D_q(\zeta) := \sum_{l\in\mathbb{Z}} D(zq^l),$$

which we view as a function from $E(\mathbb{C})$ to $\mathbb{R}$ (note that its value does not depend on the choice of the representant ζ because of the normal convergence of the summation: we see from (20) that $D(z) = \mathcal{O}(|z|\log|z|)$ for $z \to 0$ and by (21) that $D(z) = \mathcal{O}(|z|^{-1}\log|z|))$ for $z \to \infty$. Be aware of the fact that this elliptic dilogarithm does depend not on the choice of τ but only on the lattice $\mathbb{Z}+\mathbb{Z}\tau$.

An immediate consequence of (21) is the following identity:

$$D_q(-\zeta) = -D_q(\zeta) \tag{26}$$

5.3 Eisenstein-Kronecker-Lerch series

Let $\Lambda = \mathbb{Z}+\mathbb{Z}\tau \subset \mathbb{C}$ be a lattice. Define a function $(\,,)$ from $\mathbb{C}\times\Lambda$ to $\mathbb{C}^*$ by

$$(\zeta,\lambda) = (\zeta,\lambda)_\Lambda := \exp\left(\frac{2\pi i}{\tau-\overline{\tau}}(\zeta\overline{\lambda}-\overline{\zeta}\lambda)\right). \tag{27}$$

One can easily see that this definition does not depend on the choice of $\tau \in \mathfrak{h}$. This function is bilinear and satisfies $|(\zeta,\lambda)| = 1$ for all ζ, λ. It is easy to see that $(1,\lambda) = 1$ and $(\tau,\lambda) = 1$ so the value of (ζ,λ) only depends on the image of ζ in $\mathbb{C}/\Lambda$. Hence if E is the elliptic curve which is isomorphic to $\mathbb{C}/\Lambda$ over $\mathbb{C}$, then $(\,,)$ defines a pairing $E\times\Lambda \to \mathbb{C}^*$. One can immediately verify that this pairing is perfect.

Now, we define for $a \in \mathbb{Z}$ and $s \in \mathbb{C}$ a series by

$$K_{a,\Lambda}(x,\zeta,s) := \sum_{\lambda\in\Lambda-\{-x\}} (\zeta,\lambda)(\overline{x+\lambda})^a|x+\lambda|^{-2s}. \tag{28}$$

This type of series is called an *Eisenstein-Kronecker-Lerch series.*

Theorem 2 *As a function of the variable s, the series defining $K_{a,\Lambda}(x,\zeta,s)$ converges absolutely and uniformly on compact subsets for $\Re s > \frac{a}{2}+1$. There is a meromorphic continuation of $K_{a,\Lambda}(x,\zeta,s)$ to the entire s-plane and it satisfies the following functional equation:*

$$\Gamma(s)K_{a,\Lambda}(x,\zeta,s) = (-1)^a(\zeta,x)\left(\frac{\tau-\overline{\tau}}{2\pi i}\right)^{a+1-2s}\Gamma(a+1-s)K_{a,\Lambda}(\zeta,x,a+1-s).$$

A pole can only occur if $a=0$. Then if $\zeta\in\Lambda$ there is a simple pole at $s=1$ and if $x\in\Lambda$ there is a simple pole at $s=0$. No other poles can occur than these ones.

Proof: See chapter 8 of [23]. $\square$

One has the relation

$$K_{-a,\Lambda}(x,\zeta,s) = \overline{K_{a,\Lambda}(x,-\zeta,\overline{s}+a)}, \tag{29}$$

which we can easily check for $\Re(s) \gg 0$ where the series converge and according to theorem 2 it must hold for any s.

We will specialize to $x = 0$ now.

Lemma 5 *The following identity holds:*

$$K_{a,\Lambda}(0,\zeta,s) = (-1)^a K_{a,\Lambda}(0,-\zeta,s).$$

Proof: First suppose that $\Re s \gg 0$ is high enough so that the series defining K converges nicely. Since $\Lambda = -\Lambda$, it is immediate that

$$\sum_{\lambda\in\Lambda-\{0\}} (\zeta,\lambda)\overline{\lambda}^a|\lambda|^{-2s} = \sum_{\lambda\in\Lambda-\{0\}} (\zeta,-\lambda)\overline{-\lambda}^a|-\lambda|^{-2s} = (-1)^a \sum_{\lambda\in\Lambda-\{0\}} (-\zeta,\lambda)\overline{\lambda}^a|\lambda|^{-2s},$$

so the result follows in the case that $\Re s \gg 0$ and hence by theorem 2 for any s. $\square$

Let's compute some partial derivatives of $K_{a,\Lambda}(0,\zeta,s)$:

$$\frac{\partial K_{a,\Lambda}(0,\zeta,s)}{\partial\zeta} = \frac{2\pi i}{\tau-\overline{\tau}}K_{a+1,\Lambda}(0,\zeta,s) \quad \text{and} \quad \frac{\partial K_{a,\Lambda}(0,\zeta,s)}{\partial\overline{\zeta}} = -\frac{2\pi i}{\tau-\overline{\tau}}K_{a-1,\Lambda}(0,\zeta,s-1). \tag{30}$$

This can be directly verified from (28) if $\Re s \gg 0$ when the series converge and again because of the meromorphic continuation it holds for any s.

We will be mainly interested in the case $a = 1, x = 0, s = 2$. The series is then equal to

$$K_{1,\Lambda}(0,\zeta,2) = \sum_{\lambda\in\Lambda-\{0\}} \frac{(\zeta,\lambda)}{\lambda^2\overline{\lambda}}. \tag{31}$$

We define a new function $M : \mathfrak{h}\times\mathbb{C}\to\mathbb{C}$ by

$$M(\tau,\zeta) := (\Im\tau)^2 K_{1,\Lambda}(0,\zeta,2) = (\Im\tau)^2 \sum_{\lambda\in\Lambda-\{0\}} \frac{(\zeta,\lambda)}{\lambda^2\overline{\lambda}}. \tag{32}$$

We will often write $M_\tau(\zeta)$ instead of $M(\tau,\zeta)$ and in case it's clear what τ is we will sometimes write $M(\zeta)$. At first sight this series might seem completely random but we will see later that we can very often express the Mahler measure of a polynomial in terms of M.

If $D = \sum n_i[P_i]$ is a divisor on E, then we define

$$M_\tau(D) := \sum n_i M_\tau(P_i),$$

Let's define an equivalence relation $\equiv$ on $\mathrm{Div}(E)$ as follows: $[-P] \equiv -[P]$ for all P and $[P] \equiv 0$ if $2P = \mathcal{O}$ and extend this linearly to $\mathrm{Div}(E)$. Next, we define a function $\diamond : \mathrm{Div}(E)\times\mathrm{Div}(E) \to \mathrm{Div}(E)$: if we write

$$D_1 = \sum_i n_i[P_i] \quad \text{and} \quad D_2 = \sum_i m_i[P_i],$$

then

$$D_1 \diamond D_2 := \sum_{i,j} (n_i m_j)[P_i - P_j]. \tag{33}$$

Since clearly $M_\tau(\zeta) = M_\tau(-\zeta)$ it follows that the value of $M_\tau(D)$ only depends on the image of D in $\mathrm{Div}(E)^- := \mathrm{Div}(E)/\equiv$. This M function now satisfies a steinberg relation as well:

$$M(\tau, (f) \diamond (1-f)) = 0 \quad \text{for all } f \in \mathbb{C}(E)^* - \{1\}.$$

A proof of this will be given in section 6.

A very interesting result which connects these series to the elliptic dilogarithm is the following:

Theorem 3 *Suppose $\tau \in \mathfrak{h}$ and write $q = e^{2\pi i\tau}$. Let $\zeta \in \mathbb{C}$. Then*

$$\Re M_\tau(\zeta) = -\pi D_q(e^{2\pi i\zeta}).$$

Proof: See theorem 1 in [24]. In fact Zagier proves a much more general identity there, for any a and s. $\square$

6 Regulators

In this section we will introduce a real meromorphic differential form η on the complex manifold defined by a polynomial $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. We will see that the Mahler measure of P can sometimes be related to a period of η. We will also see that if P defines an elliptic curve, the periods of η can be related to values of an Eisenstein-Kronecker-Lerch series and hence to the elliptic dilogarithm.

Let X be a complete non-singular algebraic curve over $\mathbb{C}$. Suppose that we have 2 functions $f, g \in \mathbb{C}(X)^*$. Define $\Sigma(x)$ to be the set of zeroes and poles of an element x of $\mathbb{C}(X)^*$ and set $\Sigma(f,g) = \Sigma(f) \cup \Sigma(g)$. Set $S = \Sigma(f,g)$. We define a real-meromorphic differential 1-form on $X(\mathbb{C}) - S$ by

$$\eta(f,g) := \log|f| d\arg g - \log|g| d\arg f. \tag{34}$$

So η defines a function from $\mathbb{C}(X)^* \times \mathbb{C}(X)^*$ to the $\mathbb{R}$-vector space of real-meromorphic differential 1-forms on $X(\mathbb{C})$, which we denote by $M(X)$. It is easy to see that that η is bilinear. It is also easy to see that η satisfies $\eta(f,f) = 0$ and $\eta(f,g) = -\eta(g,f)$. As $M(X)$ is a vector space over $\mathbb{Q}$, we can view η as a map

$$\eta : \bigwedge^2 \mathbb{C}(X)^* \otimes \mathbb{Q} \to M(X), \tag{35}$$

where $(f \wedge g) \otimes 1$, which we shortly write as $f \wedge g$, is sent to $\eta(f,g)$. Also, the following interesting relation between η and the dilogarithm can be immediately verified from (20):

$$dD(z) = \eta(z, 1-z). \tag{36}$$

We have that

$$\begin{aligned} d\eta(f,g) &= \Im\, d\left(\log|f|\frac{dg}{g} - \log|g|\frac{df}{f}\right) = \Im\left(\frac{1}{2}\left(\frac{df}{f} + \overline{\left(\frac{df}{f}\right)}\right) \wedge \frac{dg}{g} + \frac{df}{f} \wedge \frac{1}{2}\left(\frac{dg}{g} + \overline{\left(\frac{dg}{g}\right)}\right)\right) \\ &= \Im\left(\frac{df}{f} \wedge \frac{dg}{g}\right). \end{aligned}$$

Written out in local coordinates, this is equal to

$$\Im\left(\frac{f'(z)}{f(z)}dz \wedge \frac{g'(z)}{g(z)}dz\right) = 0.$$

so it follows that $\eta(f,g)$ is a closed differential form for each f and g. This implies that if $S = \Sigma(f,g)$ and γ is a closed path in $X(\mathbb{C}) - S$, then the value of the integral $\int_\gamma \eta(f,g)$ only depends on the homology class of γ in $H_1(X(\mathbb{C}) - S, \mathbb{Z})$, where we give $X(\mathbb{C})$ the usual complex-analytic topology. Hence we can define a map $r(f,g) : H_1(X(\mathbb{C}) - S, \mathbb{Z}) \to \mathbb{R}$ by

$$r(f,g) = r_{X(\mathbb{C})-S}(f,g) : [\gamma] \mapsto \int_\gamma \eta(f,g), \tag{37}$$

called the *regulator map*. From (36) it immediately follows that $r(f, 1-f)([\gamma]) = 0$ for any closed path γ so that

$$r(f, 1-f) = 0.$$

Now, for each finite susbet S of $X(\mathbb{C})$, the cohomology group $H^1(X(\mathbb{C}) - S)$ is naturally isomorphic to the $\mathbb{R}$-vector space of linear maps $H_1(X(\mathbb{C}) - S), \mathbb{Z}) \to \mathbb{R}$. And for $S_1 \subset S_2$ there is a natural map

$$H^1(X(\mathbb{C}) - S_1, \mathbb{R}) \to H^1(X(\mathbb{C}) - S_2, \mathbb{R}).$$

Because the set of zeroes and poles of functions on X can get arbitrarily large, we must take the direct limit of these cohomology groups if we want to view r as a Steinberg symbol on $\mathbb{C}(X)$:

$$r : K_2(\mathbb{C}(X)) \otimes \mathbb{Q} \to \varinjlim H^1(X(\mathbb{C}) - S, \mathbb{R}).$$

Now, let P be a point of $X(\mathbb{C})$ and suppose ω is a closed real-meromorphic differential 1-form on X. Since $X(\mathbb{C})$ is a Riemann surface, we can take an open set $U \ni P$ which is complex-analytically isomorphic to a disc in $\mathbb{C}$. Let γ be a circle in U around P, which is oriented positively (i.e. counterclockwise). If γ is so small that the interior of the circle defined by γ does not contain any poles of ω, except possibly P, then we define the *residue* of ω at P to be

$$\operatorname{Res}_P \omega := \frac{1}{2\pi}\int_\gamma \omega.$$

Note that this value does not depend on the choice of γ. Note also that $\operatorname{Res}_P \omega = 0$ if P is not a pole of ω. The following identity is well-known:

$$\sum_{P \in X(\mathbb{C})} \operatorname{Res}_P(\omega) = 0.$$

There is a relation between these residues and the tame symbols:

Lemma 6 *Let $\alpha = \{f, g\} \in K_2(\mathbb{C}(X))$, then*

$$\operatorname{Res}_P \eta(\alpha) = \log|\partial_P(f,g)|, \tag{38}$$

where ∂_P is the tame symbol defined in section 4.

Proof: It is clear that it suffices to show that $\operatorname{Res}_P \eta(f,g) = \log|(f,g)_P|$ for all f and g that are defined on an open neighborhood of P. Only a sketch of a proof of that is given in [17], let's complete it here. Since Res_P is a linear map, both sides of (38) are bilinear. Because of 15 and the fact that $\mathbb{R}$ is torsion-free, both sides of (38) are skew-symmetric as well. So we only need to check the identity in de cases $(\operatorname{ord}_P(f), \operatorname{ord}_P(g)) = (0,0), (0,1), (1,1)$. The second case is done in [17] so let's only do the first and the third case.

Let's assume that $\operatorname{ord}_P(f) = \operatorname{ord}_P(g) = 0$. We see that $\log|(f,g)_P| = 0$ so we must prove that $\operatorname{Res}_P \eta(f,g) = 0$, but this is immediate since P is not a zero or pole of f or g so P is not a pole of $\eta(f,g)$.

Now suppose that $\operatorname{ord}_P(f) = \operatorname{ord}_P(g) = 1$. Take a local coordinate z at P. Then we can write $f = az + \mathcal{O}(z^2)$ and $g = bz + \mathcal{O}(z^2)$ for some nonzero $a, b \in \mathbb{C}$ and z small enough. We see that $f/g = a/b + \mathcal{O}(z)$ for z small enough so $\log|(f,g)_P| = \log|a| - \log|b|$. We can also compute $\eta(f,g)$ in terms of this local coordinate:

$$\log|f|\frac{dg}{g} = \log|az + \mathcal{O}(z^2)|\frac{b + \mathcal{O}(z)}{bz + \mathcal{O}(z^2)}dz = \log|az + \mathcal{O}(z^2)|(\frac{1}{z} + \mathcal{O}(1))dz$$

and a similar formula holds if we exchange f and g. So

$$\begin{aligned}\eta(f,g) = \log|f|d\arg g - \log|g|d\arg f &= \Im\left(\log|f|\frac{dg}{g} - \log|g|\frac{df}{f}\right)\\ &= \Im\left((\log|az + \mathcal{O}(z^2)| - \log|bz + \mathcal{O}(z^2)|)(\frac{1}{z} + \mathcal{O}(1))dz\right)\\ &= \Im\left((\log|\frac{a}{b} + \mathcal{O}(z)|)(\frac{1}{z} + \mathcal{O}(1))dz\right).\end{aligned}$$

It follows from the Cauchy residue theorem that if we integrate η along a circle with sufficiently small radius then the value of this integral is $2\pi\log|\frac{a}{b}|$. This gives $\operatorname{Res}_P(\eta(f,g)) = \log|\frac{a}{b}|$, as desired. □

Let $\Sigma_0(f,g)$ be the set of points $P \in X(\mathbb{C})$ for which $(f,g)_P$ is not a root of unity. It's clear that $\Sigma_0(f,g) \subset \Sigma(f,g)$. It's also easy to see that if two paths γ_1 and γ_2 are homologous in $X(\mathbb{C}) - \Sigma_0(f,g)$, then

$$\int_{\gamma_1} \eta(f,g) = \int_{\gamma_2} \eta(f,g).$$

In all cases that we will study in this thesis, it will appear that $\Sigma_0(f,g) = \emptyset$ so that $\int_\gamma \eta(f,g)$ only depends on the homology class of γ in $H_1(X(\mathbb{C}),\mathbb{Z})$.

We will now prove two theorems which relate the values of these integrals to values of the dilogarithm in case $X = P^1(\mathbb{C})$ and to values of the Eisenstein-Kronecker-Lerch series and the elliptic dilogaritm in the case that X is an elliptic curve. Let's start with the theorem for $X = \mathbb{P}^1$:

Theorem 4 *Let $f, g \in \mathbb{C}(z)$. Suppose that the supports of the divisors of f and g are in $\mathbb{C}^*$. Then*

$$\int_0^\infty \eta(f,g) - T(f,g) = \sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g) D\left(\frac{a}{b}\right),$$

where

$$T(f,g) = \sum_{P\in\mathbb{C}^*} \log|(f,g)_P| d\arg(z-P).$$

Proof: We proved above that $\eta(f,g)$ is closed. Because of lemma 6, subtracting the $T(f,g)$ ensures that all the residues of the integrand are 0. This implies that the integral is well-defined and independent of a chosen path from 0 to ∞. The fact that f and g are supported outside $\{0,\infty\}$ makes sure that the fractions a/b are well-defined if $\operatorname{ord}_a(f) \neq 0$ and $\operatorname{ord}_b(g) \neq 0$.

Define $\eta'(f,g) = \eta(f,g) - T(f,g)$. As T is a steinberg symbol, $\eta'(f,1-f) = \eta(f,1-f)$. From (36) and the bilinearity of η' it follows now that

$$dD\left(\frac{z-b}{z-a}\right) = \eta'\left(\frac{z-b}{z-a}, \frac{b-a}{z-a}\right) = \eta'(z-b,b-a) - \eta'(z-b,z-a) + \eta'(z-a,z-a) - \eta'(z-a,b-a).$$

A direct calculation shows that

$$\eta'(z-b,b-a) = \eta'(z-a,b-a) = 0.$$

Of course, $\eta'(f,g) = -\eta'(g,f)$ and $\eta'(f,f) = 0$ so we see that

$$dD\left(\frac{z-b}{z-a}\right) = \eta'(z-a,z-b).$$

Let's now write

$$f(z) = c\prod_{a\in\mathbb{C}^*}(z-a)^{\operatorname{ord}_a(f)} \quad \text{and} \quad g(z) = d\prod_{b\in\mathbb{C}^*}(z-b)^{\operatorname{ord}_b(g)}.$$

Then

$$\begin{aligned}\eta'(f,g) &= \eta'(c,d) + \eta'(c,g/d) + \eta'(f/c,d) + \sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g)\eta'(z-a,z-b) \\ &= \sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g)\eta'(z-a,z-b) = d\left(\sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g) D\left(\frac{z-b}{z-a}\right)\right),\end{aligned}$$

so using (21) we see that

$$\int_0^\infty \eta'(f,g) = \sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g)\left(D(1) - D\left(\frac{b}{a}\right)\right) = \sum_{a,b\in\mathbb{C}^*} \operatorname{ord}_a(f)\operatorname{ord}_b(g) D\left(\frac{a}{b}\right).$$

$\square$

And now comes the theorem for elliptic curves:

Theorem 5 *Let E be an elliptic curve over $\mathbb{C}$. Pick a $\tau \in \mathfrak{h}$ such that E is isomorphic to $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Let f and g be rational functions on E. We can view f and g as elliptic meromorphic functions on $\mathbb{C}$ with period lattice Λ. Then the following identity holds:*

$$\left((\Im\tau)\pi \int_0^1\int_0^1 \log|f(s+t\tau)|\frac{g'(s+t\tau)}{g(s+t\tau)} - \log|g(s+t\tau)|\frac{f'(s+t\tau)}{f(s+t\tau)}\right) ds\,dt = M_\tau((f)\diamond(g)).$$

To prove this theorem we will first require some lemmas.

Lemma 7 *The function*

$$z \mapsto \log|z| + \frac{\Im\tau}{2\pi} K_{1,\Lambda}(0,z,1),$$

where $K_{1,\Lambda}(0,z,1)$ is defined by (31) and theorem 2 defines a harmonic function from $(\mathbb{C}-\Lambda)\cup\{0\}$ to $\mathbb{R}$.

Proof: From theorem 1 and proposition 2 in [24] it follows that

$$\begin{aligned}\frac{\Im\tau}{\pi}K_{1,\Lambda}(0,z,1) = &-2\sum_{l=0}^{\infty}\log|1-\exp(2\pi i(z+l\tau))| - 2\sum_{l=1}^{\infty}\log|1-\exp(2\pi i(-z+2l\tau))| \\ &- 2\pi(\Im\tau)\left(\left(\frac{\Im z}{\Im\tau}\right)^2 - \frac{\Im z}{\Im\tau} + \frac{1}{6}\right).\end{aligned}$$

In this summation $-2\log|1-\exp(2\pi i(z+l\tau))|$ is the only term that is not harmonic at $z=0$. By expanding $\exp(z)$ in its Taylor series one sees that $1-\exp(2\pi i(z+l\tau)) = -2\pi i z + \mathcal{O}(z^2)$ from which it is immediate that $2\log|z| - 2\log|1-\exp(2\pi i(z+l\tau))|$ is harmonic at 0 so the result follows. $\square$

Corollary 1 *If f is an elliptic meromorphic function on $\mathbb{C}$ with period lattice Λ, then*

$$\log|f(z)| = -\frac{\Im\tau}{2\pi}\sum_{a\in\mathbb{C}/\Lambda}\mathrm{ord}_a(f)K_{1,\Lambda}(0,z-a,1) + C_f,$$

where C_f is a constant which does not depend on z but only on f and Λ.

Proof: Note that the sum is finite. From lemma 7 it follows immediately that the function

$$C_f(z) := \log|f(z)| + \frac{\Im\tau}{2\pi}\sum_{a\in\mathbb{C}/\Lambda}\mathrm{ord}_a(f)K_{1,\Lambda}(0,z-a,1)$$

is a harmonic function from $\mathbb{C}$ to $\mathbb{R}$ because locally around each point $P\in\mathbb{C}$ the function $\log|f(z-P)| - \mathrm{ord}_P(f)\log|z-P|$ is harmonic. Furthermore, we see that $C_f(z)$ is periodic with period lattice Λ. As $\mathbb{C}/\Lambda$ is compact it follows that $C_f(z)$ has a maximum value somewhere. From theorem 7.2.2 in [11] it follows now that $C_f(z)$ is constant. $\square$

Corollary 2 *The following formula holds:*

$$\int_{\mathbb{C}/\Lambda}\log|f|d\log|g|\wedge dz = \frac{(\Im\tau)^2}{2\pi i}\sum_{a,b\in\mathbb{C}/\Lambda}\mathrm{ord}_a(f)\,\mathrm{ord}_b(g)K_{-1,\Lambda}(0,a-b,1).$$

Proof: First of all, $\sum_{a\in\mathbb{C}/\Lambda}\operatorname{ord}_a(g)=0$, so that from Stokes' formula it follows that

$$\int_{\mathbb{C}/\Lambda} d\log|g|\wedge dz=0.$$

Using this and corollary 1 we see that

$$\int_{\mathbb{C}/\Lambda}\log|f|d\log|g|\wedge dz=\left(\frac{\Im\tau}{2\pi}\right)^2\sum_{a,b\in\mathbb{C}/\Lambda}\operatorname{ord}_a(f)\operatorname{ord}_b(g)\int_{\mathbb{C}/\Lambda}K_{0,\Lambda}(0,z-a,1)dK_{0,\Lambda}(0,z-b,1).$$

Applying (30) we see that this is equal to

$$-\frac{\Im\tau}{4\pi}\sum_{a,b\in\mathbb{C}/\Lambda}\operatorname{ord}_a(f)\operatorname{ord}_b(g)\int_{\mathbb{C}/\Lambda}K_{0,\Lambda}(0,z-a,1)K_{-1,\Lambda}(0,z-b,0)d\overline{z}\wedge dz.$$

We will compute the more general integral

$$\int_{\mathbb{C}/\Lambda}K_{0,\Lambda}(0,z-a,s)K_{-1,\Lambda}(0,z-b,s-1)d\overline{z}\wedge dz. \tag{39}$$

Let's first suppose that $\Re s>2$. In this case all sums converge nicely if we work out (39) by substituting the sum in (28). We obtain that (39) is equal to

$$\begin{aligned}&\sum_{\lambda_1,\lambda_2\in\Lambda-\{0\}}\lambda_2|\lambda_1|^{-2s}|\lambda_2|^{-2s}\int_{\mathbb{C}/\Lambda}(z-a,\lambda_1)(z-b,\lambda_2)d\overline{z}\wedge dz\\ &=\sum_{\lambda_1,\lambda_2\in\Lambda-\{0\}}|\lambda_1|^{-2s}\lambda_2|\lambda_2|^{-2s}\int_{\mathbb{C}/\Lambda}(z,\lambda_1+\lambda_2)(a,\lambda_1)^{-1}(b,\lambda_2)^{-1}d\overline{z}\wedge dz.\end{aligned}$$

If $\lambda_1=-\lambda_2$, then

$$\int_{\mathbb{C}/\Lambda}(z,\lambda_1+\lambda_2)d\overline{z}\wedge dz=\int_{\mathbb{C}/\Lambda}d\overline{z}\wedge dz=2i\int_{\mathbb{C}/\Lambda}dx\wedge dy=2i\Im\tau.$$

If $\lambda_1\neq-\lambda_2$, then by substituting $z=s+t\tau$ with $s,t\in[0,1]$ and $\lambda_1+\lambda_2=m+n\tau$ with $m,n\in\mathbb{Z}$, one obtains

$$\int_{\mathbb{C}/\Lambda}(z,\lambda_1+\lambda_2)d\overline{z}\wedge dz=(\tau-\overline{\tau})\int_0^1\int_0^1\exp(2\pi i(tm-sn))ds\,dt$$

and this integral clearly vanishes as m and n are both integers but not both zero. Putting this together we see that (39) is equal to

$$2i\Im\tau\sum_{\lambda\in\Lambda-\{0\}}(a-b,\lambda)\lambda|\lambda|^{-4s},$$

hence

$$\int_{\mathbb{C}/\Lambda}K_{0,\Lambda}(0,z-a,s)K_{-1,\Lambda}(0,z-b,s-1)d\overline{z}\wedge dz=2i\Im\tau K_{-1,\Lambda}(0,a-b,2s-1).$$

We assumed here that $\Re s>2$ but because of theorem 2 this formula should hold for any s, in particular for $s=1$ and the result follows at once. $\square$

Proof of theorem 5: As $\log|g|=\frac{1}{2}(\log g+\log\overline{g})$, it is easy to see that

$$\int_{\mathbb{C}/\Lambda}\log|f|d\log|g|\wedge dz=\frac{1}{2}\int_{\mathbb{C}/\Lambda}\log|f|\overline{\left(\frac{g'}{g}\right)}d\overline{z}\wedge dz.$$

After substitution of $z = s + t\tau$ this becomes equal to

$$(i\Im\tau)\int_0^1\int_0^1 \log|f(s+t\tau)|\overline{\left(\frac{g'(s+t\tau)}{g(s+t\tau)}\right)}ds\,dt.$$

So applying corollary 2 one derives that

$$\int_0^1\int_0^1 \log|f(s+t\tau)|\overline{\left(\frac{g'(s+t\tau)}{g(s+t\tau)}\right)}ds\,dt = \frac{\Im\tau}{2\pi}\sum_{a,b\in\mathbb{C}/\Lambda}\operatorname{ord}_a(f)\operatorname{ord}_b(g)K_{-1,\Lambda}(0,a-b,1).$$

Exchanging f and g and using lemma 5 we get that

$$\begin{aligned}&\int_0^1\int_0^1\left(\log|f(s+t\tau)|\overline{\left(\frac{g'(s+t\tau)}{g(s+t\tau)}\right)} - \log|g(s+t\tau)|\overline{\left(\frac{f'(s+t\tau)}{f(s+t\tau)}\right)}\right)ds\,dt\\ &= \frac{\Im\tau}{\pi}\sum_{a,b\in\mathbb{C}/\Lambda}\operatorname{ord}_a(f)\operatorname{ord}_b(g)K_{-1,\Lambda}(0,a-b,1).\end{aligned}$$

The result of the theorem follows now immediately from (29) and (32). □

Let's give an example of how we can use theorem 5 to express the regulator maps defined by (37) in terms of the Eisenstein-Kronecker-Lerch series. Let again be given $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathfrak{h}$ and an identification of $E(\mathbb{C})$ with $\mathbb{C}/\Lambda$. In this example we assume for simplicity that all tame symbols $(f,g)_P$, where f and g are fixed and P varies over $E(\mathbb{C})$, are roots of unity. It follows from (38) that $\eta(f,g)$ has all its residues equal to 0 so that the value of the integral only depends on the homology class $[\gamma]$ of γ in $H_1(E(\mathbb{C}),\mathbb{Z})$. Let

$$\gamma_1 : [0,1] \to \mathbb{C} : t \mapsto t \quad \text{and} \quad \gamma_2 : [0,1] \to \mathbb{C} : t \mapsto t\tau$$

be the straight paths in $\mathbb{C}$ from 0 to 1 and τ respectively. It's clear that the corresponding paths on $E(\mathbb{C})$, which we denote also by γ_1 and γ_2 are closed and their homology classes form a basis for $H_1(E(\mathbb{C}),\mathbb{Z})$. So it is sufficient to compute $r(f,g)(c)$ for $c = [\gamma_1]$ and $c = [\gamma_2]$.

First we compute $r(f,g)([\gamma_1])$. For each $t \in [0,1]$, let $\delta_t : [0,1] \to \mathbb{C} : s \mapsto s + t\tau$ be the straight path from $t\tau$ to $1 + t\tau$. We see that all the corresponding paths on $E(\mathbb{C})$ are homologous to γ_1, so that $\int_{\delta_t}\eta(f,g) = \int_{\gamma_1}\eta(f,g) = r(f,g)([\gamma_1])$ for all t Since $\eta(f,g) = \Im(\log|f|\frac{dg}{g} - \log|g|\frac{df}{f})$, it follows from theorem 5 that

$$\begin{aligned}r(f,g)([\gamma_1]) &= \int_0^1\left(\int_{\delta_t}\eta(f,g)\right)dt\\ &= \Im\int_0^1\int_0^1\left(\log|f(s+t\tau)|\frac{g'(s+t\tau)}{g(s+t\tau)} - \log|g(s+t\tau)|\frac{f'(s+t\tau)}{f(s+t\tau)}\right)ds\,dt\\ &= \frac{\Im M_\tau((f)\diamond(g))}{(\Im\tau)\pi}.\end{aligned} \tag{40}$$

Similarly it follows that

$$\begin{aligned}r(f,g)([\gamma_2]) &= \Im\int_0^1\int_0^1\left(\log|f(s+t\tau)|\frac{g'(s+t\tau)}{g(s+t\tau)} - \log|g(s+t\tau)|\frac{f'(s+t\tau)}{f(s+t\tau)}\right)\tau dt\,ds\\ &= \frac{\Im\left(\tau M_\tau((f)\diamond(g))\right)}{(\Im\tau)\pi}.\end{aligned} \tag{41}$$

In particular, if τ is purely imaginary it follows from theorem 3 that $r(f,g)([\gamma_2]) = -D_q((f)\diamond(g))$. Using the identification of $E(\mathbb{C})$ with $\mathbb{C}/\Lambda$ we can identify $H_1(E(\mathbb{C}),\mathbb{Z})$ with Λ, where we identify $[\gamma]$ with $\int_\gamma dz$, then the following formula follows immediately from (40) and (41):

$$r(f,g)([\gamma]) = \frac{\Im\left([\gamma]M_\tau((f)\diamond(g))\right)}{(\Im\tau)\pi}. \tag{42}$$

6.1 Dilogarithm identities

Using the theory of this section we can prove the dilogarithm identities from section 5. We start with an interesting lemma:

Lemma 8 *Let $f_1, \dots, f_n \in \mathbb{C}(z) - \{0,1\}$. Suppose that*

$$\sum_{k=1}^{n} f_i \wedge (1-f_i) = 0 \quad \textit{in } \bigwedge^2 \mathbb{C}(z)^* \otimes \mathbb{Q}.$$

Then

$$\sum_{k=1}^{n} D(f_i(z))$$

is a constant function of z.

Proof: From (35) and (36) it follows immediately that the derivative of $\sum_{k=1}^{n} D(f_i(z))$ is zero.□

In fact we can see (36), together with $D(0) = 0$ as a definition of $D(z)$ because lemma 6 shows that all the residues of $\eta(z, 1-z)$ are equal to 0.

Let's prove some identities now, using this lemma. We begin with (21). We see that

$$z \wedge (1-z) + \frac{1}{z} \wedge \left(1 - \frac{1}{z}\right) = z \wedge (1-z) - z \wedge \frac{z-1}{z} = z \wedge (1-z) - z \wedge (z-1) + z \wedge z = 0,$$

so that $D(z) + D(1/z)$ is constant. Substituting $z = 1$ we see that $D(z) = -D(1/z)$ for all z. The identities $D(z) = -D(1-z)$ and $D(z) = -D(\overline{z})$ can be directly verified from (36).

Let's now prove (24). We can immediately verify that

$$z^n \wedge (1-z^n) = n\,(z \wedge (1-z^n)) = n\left(z \wedge \prod_{k=0}^{n-1}\left(1 - e^{\frac{2\pi i k}{n}} z\right)\right) = n \sum_{k=0}^{n-1} z \wedge \left(1 - e^{\frac{2\pi i k}{n}} z\right)$$
$$= n \sum_{k=0}^{n-1} e^{\frac{2\pi i k}{n}} z \wedge \left(1 - e^{\frac{2\pi i k}{n}} z\right),$$

where in the last equality we make use of the fact that $\zeta \wedge f = 0$ for all roots of unity ζ. We see that $D(z^n)/n - \sum_{k=0}^{n-1} D(\exp(2\pi i k/n) z)$ is constant and after substituting $z = 0$ we see that it is 0.

Let's finally do the last one, (25). If $x \in \{0, 1, \infty\}$ then it is trivial, so let's suppose that x is a fixed complex number, not equal to 0 or 1. Then $z, 1-xz, (1-x)/(1-xz), (1-z)/(1-xz)$ are all nonconstant functions. We can see that

$$\frac{1-x}{1-xz} \wedge \left(1 - \frac{1-x}{1-xz}\right) = \frac{1-x}{1-xz} \wedge \frac{x(1-z)}{1-xz} = (1-xz) \wedge \frac{1-x}{x(1-z)} + (1-x) \wedge x(1-z)$$

and similarly

$$\frac{1-z}{1-xz} \wedge \left(1 - \frac{1-z}{1-xz}\right) = (1-xz) \wedge \frac{1-z}{z(1-x)} + (1-z) \wedge z(1-x).$$

Adding these two identities gives, after working out and cancelling out certain terms,

$$\frac{1-x}{1-xz} \wedge \left(1 - \frac{1-x}{1-xz}\right) + \frac{1-z}{1-xz} \wedge \left(1 - \frac{1-z}{1-xz}\right) = -(1-xz) \wedge xz + (1-x) \wedge x + (1-z) \wedge z.$$

From this it follows that

$$x \wedge (1-x) + z \wedge (1-z) + (1-xz) \wedge xz + \frac{1-x}{1-xz} \wedge \left(1 - \frac{1-x}{1-xz}\right) + \frac{1-z}{1-xz} \wedge \left(1 - \frac{1-z}{1-xz}\right) = 0.$$

It follows that $D(x) + D(z) + D(1-xz) + D((1-x)/(1-xz)) + D((1-z)/(1-xz)$ is a constant function of z. Similarly it is a constant function of x. By continuity it is a constant function of x and z for all $x, z \in \mathbb{P}^1(\mathbb{C})$. Specializing to $x = z = 0$ we see that the constant value is 0.

7 Elliptic curves with complex multiplication

In this section, we let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication. We'll show that we can relate the the L-series of E to an Eisenstein-Kronecker-Lerch series.

Choose $\tau \in \mathfrak{h}$ such that E is isomorphic to $\mathbb{C}/\Lambda$ over $\mathbb{C}$ where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Because E has complex multiplication, the field $K := \mathbb{Q}(\tau)$ is quadratic over $\mathbb{Q}$. Since $\tau \in \mathbb{C}$, we can view K as an embedded subfield of $\mathbb{C}$. We will suppose that the class number of K is equal to 1, which is always the case when E is defined over $\mathbb{Q}$, as is a well-known result.

To compute the L-series of E we have to define what a Hecke character is. Let F be a number field and let $\sigma_1, \ldots, \sigma_r$ be the embeddings of F into $\mathbb{C}$. Suppose that $\mathfrak{f}$ is an integral ideal of F. Denote by $I(\mathfrak{f})$ the multiplicative group of fractional ideals of F which are coprime to $\mathfrak{f}$. A *Hecke character* of F is a group homomorphism

$$\psi : I(\mathfrak{f}) \to \mathbb{C}^*$$

for which there exist integers $n_1, \ldots, n_r$ such that

$$\psi(\alpha\mathcal{O}_F) = \prod_{i=1}^{r} \sigma_i(\alpha)^{n_i}$$

for every $\alpha \in F$ with $\alpha \equiv 1 \bmod \mathfrak{f}$. We call $(n_1, \ldots, n_r)$ the *infinite type* of ψ. We assume that ψ is nontrivial, which means that it is not identically 1 on $I(\mathfrak{f})$. The ideal $\mathfrak{f}$ is called a *modulus* of ψ. It's clear that the sum of 2 moduli is also a modulus so that there exist a largest modulus of ψ (because $\mathcal{O}_K$ is noetherian), which is called the *conductor* of ψ.

Given such a Hecke character ψ, we define its L-function by

$$L_F(\psi, s) = \sum_{\mathfrak{a} \text{ ideal of } \mathcal{O}_F} \psi(\mathfrak{a})N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_F)} \frac{1}{1-\psi(\mathfrak{p})N(\mathfrak{p})^{-s}},$$

where we put $\psi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime with the conductor $\mathfrak{f}$ of ψ.

Let's now go back to our elliptic curve E and our field K. Because $h(K) = 1$, every ideal of K is principal. If ψ is a Hecke character of K of infinite type (n_1, n_2) and conductor (f), then it is immediate that $\chi(\alpha) := \frac{\psi((\alpha))}{\alpha^{n_1}\overline{\alpha}^{n_2}}$ induces a character of the group $(\mathcal{O}_K/f\mathcal{O}_K)^*$, which we will also denote by χ. Hence we can write

$$\psi((\alpha)) = \chi(\alpha)\alpha^{n_1}\overline{\alpha}^{n_2}.$$

The following theorem is well-known:

Theorem 6 *If E is an elliptic curve defined over $\mathbb{Q}$ with complex multiplication by an order in a quadratic number field K, then the class number of K is 1 and then there is a Hecke character of infinite type $(0,1)$ of K such that*

$$L(E, s) = L_K(\psi, s).$$

Furthermore, the corresponding character χ takes values in μ_K, the group of roots of unity in K. Also, the conductors N of E and $\mathfrak{f}$ of ψ are related in the following way:

$$N = N(\mathfrak{f})|\Delta_K|,$$

where Δ_K is the discriminant of K.

Proof: See [20]. $\square$

If ϵ is a unit of $\mathcal{O}_K$, then we see that

$$\chi(\epsilon) = \frac{\psi((\epsilon))}{\overline{\epsilon}} = \psi((1))\epsilon = \epsilon. \tag{43}$$

It follows now immediately that

$$L_K(\psi, s) = \frac{1}{|\mu_K|} \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\chi(\alpha)\overline{\alpha}}{|\alpha|^{2s}}, \tag{44}$$

Lemma 9 *Let p be a prime number. If p ramifies in K then $p \mid \Delta(E)$. If $p \mid \Delta(E)$ then $a_p = 0$. If $p \nmid \Delta(E)$ and p is inert in K then $a_p = 0$. And if $p \nmid \Delta(E)$ and p splits in K, say $p = \pi\overline{\pi}$, then*

$$a_p = \chi(\pi)\overline{\pi} + \chi(\overline{\pi})\pi.$$

Furthermore, this relation determines $\chi(\pi)$ uniquely if a_p is known.

Proof: We can use theorem 6 to express a_p in terms of χ. In any case,

$$\prod_{\mathfrak{p}|p}(1 - \psi(\mathfrak{p})N(\mathfrak{p})^{-s}) = \begin{cases} 1 - a_p p^{-s} + p^{1-2s} & \text{if } p \nmid \Delta(E); \\ 1 - a_p p^{-s} & \text{if } p \mid \Delta(E). \end{cases} \tag{45}$$

There are 3 cases to examine: p can ramify, split or be inert in K. If p ramifies, say $(p) = (\pi)^2$, then we do not get a p^{1-2s} term if we work out the right hand side of (45). If $\chi(\pi) \neq 0$, then we can see that $a_p = \chi(\pi)\overline{\pi}$ cannot be an integer, so $\chi(\pi) = 0$. If p is inert, we can easily deduce that and $a_p = 0$ and also that $\chi(p) = 0$ or $\chi(p) = -1$, depending on whether $p \mid \Delta(E)$ or $p \nmid \Delta(E)$ respectively.

So it remains to consider the case that p splits in K. Let $(\pi) \mid p$, with (π) a prime ideal of K. Then $p = \pi\overline{\pi}$. We can split this up in two subcases again: either $p \mid \Delta(E)$ or $p \nmid \Delta(E)$. If $p \mid \Delta(E)$, then we see by working out the right hand side in (45) that $\chi(\pi)\chi(\overline{\pi}) = 0$ and that $\chi(\pi)\overline{\pi} + \chi(\overline{\pi})\pi = a_p$. Without loss of generality we can assume that $\chi(\pi) = 0$ (otherwise exchange π with $\overline{\pi}$). As $\chi(\overline{\pi})\pi$ must be an integer, we see that also $\chi(\overline{\pi}) = 0$ and $a_p = 0$. Now let's switch to the case $p \nmid \Delta(E)$. By working out the right hand side in (45) we see that $\chi(\pi)\chi(\overline{\pi}) = 1$, from which it follows that $\chi(\overline{\pi}) = 1/\chi(\pi)$ and $a_p = \chi(\pi)\overline{\pi} + \chi(\overline{\pi})\pi$. It follows that $\overline{\pi}\chi(\pi)^2 - a_p\chi(\pi) + \pi$. This is a quadratic equation for $\chi(\pi)$ whose roots have product equal to $\pi/\overline{\pi}$. As $(\pi) \neq (\overline{\pi})$ the fraction $\pi/\overline{\pi}$ cannot be in μ_K, so the roots cannot both lie in μ_K. From theorem 6 it follows however that $\chi(\pi)$ must be an element of μ_K, so at most 1 root of the quadratic equation can equal $\chi(\pi)$. Now it follows from $\chi(\pi)\chi(\overline{\pi}) = 1$ that $\chi(\overline{\pi}) = \overline{\chi(\pi)}$. $\square$

Lemma 10 *The character χ satisfies*

$$\chi(\overline{\alpha}) = \overline{\chi(\alpha)}$$

for all $\alpha \in \mathcal{O}_K$ and the conductor ideal $\mathfrak{f}$ satisfies

$$\overline{\mathfrak{f}} = \mathfrak{f}.$$

Proof: See the proof of lemma 9: for all roots of unity we have the relation $\chi(\overline{\alpha}) = \overline{\chi(\alpha)}$ and also for all prime elements. The statement about $\mathfrak{f}$ is now immediate. $\square$

The series of (44) involves a multiplicative character. We want to transform it into a sum of series involving the additive character defined by (27), where we initially take our lattice to be $\mathcal{O}_K$ and later switch over to a lattice Λ such that E is $\mathbb{R}$-isomorphic to $\mathbb{C}/\Lambda$. Let's develop some tools to do that. The tools here are mainly based on chapter 11 of [2] and section 4 of [8]. Suppose that $f \in \mathcal{O}_K$ is a generator of $\mathfrak{f}$. Let C be any (preferably the smallest) positive integer in $\mathfrak{f}$ and suppose that $g \in \mathcal{O}_K$ is such that $C = fg$. Also, choose $\delta \in \mathfrak{h}$ such that $\mathcal{O}_K = \mathbb{Z}[\delta]$. We define a pairing $\mathcal{O}_K \times \mathcal{O}_K \to \mathbb{C}^*$ as follows:

$$\langle x, y\rangle := \exp\left(2\pi i \frac{x\overline{y} - \overline{x}y}{C(\delta - \overline{\delta})}\right). \tag{46}$$

It is clear that this defines a pairing $(\mathcal{O}_K/C\mathcal{O}_K)\times(\mathcal{O}_K/C\mathcal{O}_K) \to \mathbb{C}^*$. Also, $\langle x, y\rangle = (x/C, y)$. Let's start with proving some elementary properties of this pairing. From (46) it follows immediately that

$$\langle x, -y\rangle = \langle -x, y\rangle = \langle y, x\rangle = \overline{\langle x, y\rangle}.$$

Another immediate identity is

$$\langle xy, z\rangle = \langle x, \overline{y}z\rangle,$$

for all $x, y, z \in \mathcal{O}_K$. We also have the following identity:

Lemma 11 *Let $y \in \mathcal{O}_K$. Then*

$$\sum_{x\in(\mathcal{O}_K/g\mathcal{O}_K)} \langle x, \overline{f}y\rangle = \begin{cases} N(g) & \text{if } \overline{g} \mid y \\ 0 & \text{otherwise,} \end{cases}$$

where it doesn't matter which representatives we take for x.

Proof: If $\overline{g} \mid y$, it is clear since every term in our summation is equal to 1 and we sum $N(g)$ terms. So suppose that $\overline{g} \nmid y$. It suffices to show that $\sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)}\langle x, \overline{f}y\rangle = 0$, since this sum is a multiple of the original sum. Hence it suffices to show that $\sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)}\langle x, y\rangle = 0$ when $C \nmid y$. If we write $y = a\delta + b$, then

$$\sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \langle x, y\rangle = \sum_{k=1}^{C}\sum_{l=1}^{C} \exp\left(\frac{2\pi i(bk - al)}{C}\right).$$

This sum is clearly equal to 0 since at least one of the numbers a and b is not divisible by C. $\square$

The following identity can be proved in the same way:

$$\sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \langle x, y\rangle = \begin{cases} C^2 & \text{if } C \mid x \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

To transform the character χ to a sum of symbols $\langle x, y\rangle$, we will define some kind of discrete analogue of a Fourier transformation. Let $\phi : (\mathcal{O}_K/C\mathcal{O}_K) \to \mathbb{C}$ be an arbitrary function. Then we define its Fourier transform $\hat{\phi}$ as follows:

$$\hat{\phi}(x) = \frac{1}{C} \sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} \phi(y)\langle y, x\rangle.$$

Lemma 12 *Let $\phi : (\mathcal{O}_K/C\mathcal{O}_K) \to \mathbb{C}$ be an arbitrary function. Then*

$$\phi(x) = \frac{1}{C} \sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} \hat{\phi}(y)\langle y, x\rangle = \frac{1}{C} \sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} \hat{\phi}(y)\langle x, -y\rangle.$$

Proof: First of all, let $\phi(x) = \langle x, y\rangle$, with y fixed. Using (47), one sees that

$$\hat{\phi}(x) = \frac{1}{C}\sum_{z\in(\mathcal{O}_K/C\mathcal{O}_K)} \langle z, y\rangle\langle z, x\rangle = \frac{1}{C}\sum_{z\in(\mathcal{O}_K/C\mathcal{O}_K)} \langle z, x+y\rangle = \begin{cases} C & \text{if } x = -y \\ 0 & \text{otherwise,}\end{cases}$$

and this proves the lemma for these functions ϕ.

By linearity, we can also prove the lemma for $\phi(x) = \sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} c_y\langle x, y\rangle$, where $c_y \in \mathbb{C}$ are arbitrary coefficients. The vector space of functions $(\mathcal{O}_K/C\mathcal{O}_K) \to \mathbb{C}$ has dimension C^2. To prove the lemma it now suffices to prove that the functions $x \mapsto \langle x, y\rangle$, with y fixed, span this vector space. As the number of such functions is C^2, we have to show that they are linearly independent. Suppose that $\phi(x) = \sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} c_y\langle x, y\rangle = 0$. Then on one hand, $\hat{\phi}(x) = 0$ but on the other hand, $\hat{\phi}(x) = c_{-x}$ for all x. So $c_y = 0$ for each y, which proves linear independence. □

We will now restrict to $\phi = \chi$. We need a few extra identities before we can completely rewrite the summation. The first one shows that, if we know $\hat{\chi}(x)$ for some x, then we can compute $\hat{\chi}(xy)$ for all y coprime to $\overline{f}$:

$$\hat{\chi}(xy) = \overline{\chi}(\overline{y})\hat{\chi}(x). \tag{48}$$

We can prove it in the following way:

$$\hat{\chi}(xy) = \frac{1}{C}\sum_{z\in\mathcal{O}_K/C\mathcal{O}_K} \chi(z)\langle z, xy\rangle = \frac{1}{C}\sum_{z\in(\mathcal{O}_K/C\mathcal{O}_K)} \overline{\chi}(\overline{y})\chi(\overline{y}z)\langle\overline{y}z, x\rangle = \overline{\chi}(\overline{y})\hat{\chi}(x).$$

The next identity is:

$$\hat{\chi}(x) = 0 \quad \text{if } \overline{g} \nmid x. \tag{49}$$

We can prove this using lemma 11:

$$\begin{aligned}
\hat{\chi}(x) &= \frac{1}{C}\sum_{y\in(\mathcal{O}_K/C\mathcal{O}_K)} \chi(y)\langle y, x\rangle = \frac{1}{C}\sum_{y_1\in(\mathcal{O}_K/f\mathcal{O}_K)}\sum_{y_2\in(\mathcal{O}_K/g\mathcal{O}_K)} \chi(y_1+fy_2)\langle y_1+fy_2, x\rangle \\
&= \frac{1}{C}\sum_{y_1\in(\mathcal{O}_K/f\mathcal{O}_K)} \chi(y_1)\langle y_1, x\rangle \sum_{y_2\in(\mathcal{O}_K/g\mathcal{O}_K)} \langle fy_2, x\rangle \\
&= \frac{1}{C}\sum_{y_1\in(\mathcal{O}_K/f\mathcal{O}_K)} \chi(y_1)\langle y_1, x\rangle \sum_{y_2\in(\mathcal{O}_K/g\mathcal{O}_K)} \langle y_2, \overline{f}x\rangle = 0,
\end{aligned}$$

where of course it doesn't matter which representatives we take for y_1 and y_2 as long as we keep the choices fixed in each summation. There is another case where $\hat{\chi}(x) = 0$:

Lemma 13 *Suppose that x is not coprime with $\mathfrak{f}$. Then*

$$\hat{\chi}(\overline{g}x) = 0.$$

Proof: Suppose that $(x, \overline{f}) = (d)$ and write $\overline{f} = df'$. The claim is that there is a $y \in \mathcal{O}_K$, coprime with $\overline{f}$ such that $\chi(y) \neq 1$ and $y \equiv 1 \bmod f'$. Suppose that for $\chi(y) = 1$ for all y with $y \equiv 1 \bmod f'$ and y coprime to $\overline{f}$. Then one can easily deduce that (f') is a modulus for χ, which is in contradiction with the fact that $\mathfrak{f}$ is the conductor. For y which suffices the claim, one can easily see that $\overline{g}xy \equiv \overline{g}x \bmod C$ so that $\hat{\chi}(\overline{g}xy) = \hat{\chi}(\overline{g}x)$. On the other hand, (48) shows that $\hat{\chi}(\overline{g}xy) \neq \hat{\chi}(\overline{g}x)$ except when $\hat{\chi}(\overline{g}x) = 0$. □

We are now ready to rewrite (44). Assume, to ensure normal convergence, that $\Re s > 3/2$. First we apply lemma 12:

$$L_K(\psi, s) = \frac{1}{|\mu_K|}\sum_{\alpha\in\mathcal{O}_K-\{0\}} \frac{\chi(\alpha)\overline{\alpha}}{|\alpha|^{2s}} = \frac{1}{C|\mu_K|}\sum_{\alpha\in\mathcal{O}_K-\{0\}}\sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \hat{\chi}(x)\frac{\langle x, \alpha\rangle\overline{\alpha}}{|\alpha|^{2s}}.$$

By (49), we can drop all terms with x not divisible by $\overline{g}$, so that

$$L_K(\psi, s) = \frac{1}{C|\mu_K|} \sum_{x \in (\mathcal{O}_K/\overline{f}\mathcal{O}_K)} \hat{\chi}(\overline{g}x) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle \overline{g}x, \alpha \rangle \overline{\alpha}}{|\alpha|^{2s}}.$$

Because of lemma 13 we can drop all terms with x not coprime to $\overline{f}$, therefore

$$\begin{aligned} L_K(\psi, s) &= \frac{1}{C|\mu_K|} \sum_{x \in (\mathcal{O}_K/\overline{f}\mathcal{O}_K)^*} \hat{\chi}(\overline{g}x) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle \overline{g}x, \alpha \rangle \overline{\alpha}}{|\alpha|^{2s}} \\ &= \frac{\hat{\chi}(\overline{g})}{C|\mu_K|} \sum_{x \in (\mathcal{O}_K/\overline{f}\mathcal{O}_K)^*} \overline{\chi}(\overline{x}) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle \overline{g}x, \alpha \rangle \overline{\alpha}}{|\alpha|^{2s}}, \end{aligned}$$

where this last equality follows from (48). Now,

$$\frac{\langle \overline{g}x, \alpha \rangle \overline{\alpha}}{|\alpha|^{2s}} = \frac{\langle x, g\alpha \rangle \overline{\alpha}}{|\alpha|^{2s}} = \frac{|g|^{2s}}{\overline{g}} \frac{\langle x, g\alpha \rangle \overline{g\alpha}}{|g\alpha|^{2s}} = |g|^{2s-2} g \frac{\langle x, g\alpha \rangle \overline{g\alpha}}{|g\alpha|^{2s}},$$

therefore

$$L_K(\psi, s) = \frac{\hat{\chi}(\overline{g})|g|^{2s-2} g}{C|\mu_K|} \sum_{x \in (\mathcal{O}_K/\overline{f}\mathcal{O}_K)^*} \overline{\chi}(\overline{x}) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle x, g\alpha \rangle \overline{g\alpha}}{|g\alpha|^{2s}}.$$

Applying lemma 10 we see that this is equal to

$$L_K(\psi, s) = \frac{\hat{\chi}(\overline{g})|g|^{2s-2} g}{C|\mu_K|} \sum_{x \in (\mathcal{O}_K/f\mathcal{O}_K)} \chi(x) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle x, g\alpha \rangle \overline{g\alpha}}{|g\alpha|^{2s}},$$

where we dropped the condition that x has to be a unit in $\mathcal{O}_K/f\mathcal{O}_K$ as for the non-units $\chi(x) = 0$. From lemma 10 it also follows that divisibility by $\overline{g}$ is equivalent with divisibility by g. So it follows now from lemma 11 that

$$\begin{aligned} \sum_{x \in (\mathcal{O}_K/C\mathcal{O}_K)} \chi(x)\langle x, \alpha \rangle &= \sum_{x_1 \in (\mathcal{O}_K/f\mathcal{O}_K)} \chi(x_1) \sum_{x_2 \in (\mathcal{O}_K/g\mathcal{O}_K)} \langle x_1 + f x_2, \alpha \rangle \\ &= \sum_{x_1 \in (\mathcal{O}_K/f\mathcal{O}_K)} \chi(x_1)\langle x_1, \alpha \rangle \sum_{x_2 \in (\mathcal{O}_K/g\mathcal{O}_K)} \langle x_2, \overline{f}\alpha \rangle \\ &= \begin{cases} N(g) \sum_{x_1 \in (\mathcal{O}_K/f\mathcal{O}_K)} \chi(x_1)\langle x1, \alpha \rangle & \text{if } g \mid \alpha \\ 0 & \text{if } g \nmid \alpha, \end{cases} \end{aligned}$$

so that we can conclude that

$$L_K(\psi, s) = \frac{\hat{\chi}(\overline{g})|g|^{2s-2}}{C|\mu_K|\overline{g}} \sum_{x \in (\mathcal{O}_K/C\mathcal{O}_K)} \chi(x) \sum_{\alpha \in \mathcal{O}_K - \{0\}} \frac{\langle x, \alpha \rangle \overline{\alpha}}{|\alpha|^{2s}}. \tag{50}$$

Let's now for simplicity assume that our elliptic curve has complex multiplication by the full ring of integers $\mathcal{O}_K$. Let Λ be the canonical lattice associated to E according to lemma 3. As $\mathcal{O}_K\Lambda \subset \Lambda$ and Λ has rank 2 as a free $\mathbb{Z}$-module, we see that Λ is a projective $\mathcal{O}_K$-module of rank 1. But $\mathcal{O}_K$ has class number 1 so it follows that Λ is isomorphic to $\mathcal{O}_K$. The number 1 is an element of Λ so there is a $\lambda \in \mathcal{O}_K$ such that

$$\Lambda = \lambda^{-1}\mathcal{O}_K.$$

Let's rewrite (50) as a sum over Λ. We want to rewrite it in terms of the function defined by (28) from section 5. Note that, for $x \in \mathcal{O}_K, \alpha \in \Lambda$ we have the following identity:

$$\begin{aligned} \left(\frac{x}{\lambda C}, \alpha\right)_\Lambda &= \exp\left(\frac{\pi}{C \det(\Lambda)} \left(\frac{x\overline{\alpha}}{\lambda} - \frac{\overline{x}\alpha}{\overline{\lambda}}\right)\right) = \exp\left(\frac{\pi|\lambda|^2}{C \det(\mathcal{O}_K)} \left(\frac{x\overline{\alpha}}{\lambda} - \frac{\overline{x}\alpha}{\overline{\lambda}}\right)\right) \\ &= \exp\left(\frac{\pi}{C \det(\mathcal{O}_K)} \left(\overline{\lambda}x\overline{\alpha} - \lambda\overline{x\alpha}\right)\right) = \langle x, \lambda\alpha \rangle. \end{aligned}$$

We can use this to derive that

$$\begin{aligned}
L_K(\psi,s) &= \frac{\hat{\chi}(\overline{g})|g|^{2s-2}}{C|\mu_K|\overline{g}} \sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \chi(x) \sum_{\alpha\in\Lambda-\{0\}} \frac{\langle x,\lambda\alpha\rangle\overline{\lambda\alpha}}{|\lambda\alpha|^{2s}} \\
&= \frac{\hat{\chi}(\overline{g})|g|^{2s-2}\overline{\lambda}}{C|\mu_K|\,\overline{g}\,|\lambda|^{2s}} \sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \chi(x) \sum_{\alpha\in\Lambda-\{0\}} \frac{\left(\frac{x}{\lambda C},\alpha\right)_\Lambda\overline{\alpha}}{|\alpha|^{2s}} \\
&= \frac{\hat{\chi}(\overline{g})|g|^{2s-2}\overline{\lambda}}{C|\mu_K|\,\overline{g}\,|\lambda|^{2s}} \sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} \chi(x) K_{1,\Lambda}\left(0,\frac{x}{\lambda C},s\right).
\end{aligned}$$

Now let $\zeta\in\mu_K$. Then multiplication with ζ sends Λ to itself, hence

$$K_{1,\Lambda}(0,\zeta x,s) = \sum_{\alpha\in\Lambda-\{0\}} \frac{(\zeta x,\alpha)\overline{\alpha}}{|\alpha|^{2s}} = \sum_{\alpha\in\Lambda-\{0\}} \frac{(x,\overline{\zeta}\alpha)\overline{\alpha}}{|\alpha|^{2s}} = \sum_{\alpha\in\Lambda-\{0\}} \frac{(x,\alpha)\overline{\zeta\alpha}}{|\zeta\alpha|^{2s}} = \overline{\zeta}\,K_{1,\Lambda}(0,x,s). \tag{51}$$

Furthermore, lemma 10 shows that χ takes values in μ_K, so that we have arrived at a proof of

Theorem 7 *Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the full ring of integers in the imaginary quadratic field K. Let χ, f, g, C and λ be as above. Then*

$$L(E,s) = \frac{\hat{\chi}(\overline{g})|g|^{2s-2}\overline{\lambda}}{C|\mu_K|\,\overline{g}\,|\lambda|^{2s}} \sum_{x\in(\mathcal{O}_K/C\mathcal{O}_K)} K_{1,\Lambda}\left(0,\frac{\overline{\chi}(x)x}{\lambda C},s\right).$$

7.1 The curve $y^2 = x^3 + 1$

Let's use theorem 7 to express the L-series of the elliptic curve

$$E : y^2 = x^3 + 1$$

in terms of the Eisenstein-Kronecker-Lerch series. We will meet this curve later when calculating a Mahler measure.

The discriminant $\Delta(E)$ is equal to $-432 = -2^4\cdot 3^3$, so results from section 3 show that the given Weierstrass equation for E is minimal.

Let's now show that $K = \mathbb{Q}(\zeta_3)$ and that E has complex multiplication by the full ring $\mathcal{O}_K$. This actually immediately follows from the fact that $G_4(\mathcal{O}_K) = 0$, which is easy to see because $G_4(\mathcal{O}_K) = G_4(\zeta_3\mathcal{O}_K) = G_4(\mathcal{O}_K)\zeta_3^{-4}$, so the curve $\mathbb{C}/\mathcal{O}_K$ can be given the Weierstrass equation $y^2 = 4x^3 - 140G_6(\mathcal{O}_K)$, which is (over $\mathbb{C}$) clearly isomorphic to E.

However, it is also easy to see that two curves of the form $y^2 = x^3 + a_6$ are isomorphic over $\mathbb{R}$ if and only if the signs of their coefficient a_6 are equal. We will show now that $G_6(\mathcal{O}_K) > 0$ but $G_6((\sqrt{3}i)^{-1}\mathcal{O}_K) < 0$ (it is clear that both numbers are real). The group μ_K acts freely on the set $\mathcal{O}_K - \{0\}$ and each orbit has a representative of the form $a + b\zeta_6$ with $a \geq 1$ and $b \geq 0$. So we see that

$$\begin{aligned}
G_6(\mathcal{O}_K) &= \sum_{\alpha\in\mathcal{O}_K-\{0\}} \frac{1}{\alpha^6} = \sum_{a\geq 1,b\geq 0}\sum_{k=1}^{6} \frac{1}{((a+b\zeta_6)\zeta_6^k)^6} = 6\sum_{a\geq 1,b\geq 0} \frac{1}{(a+b\zeta_6)^6} \\
&= 6\left(1 + \sum_{\substack{a\geq 1,b\geq 0\\(a,b)\neq(1,0)}} \frac{1}{(a+b\zeta_6)^6}\right).
\end{aligned}$$

To prove that $G_6(\mathcal{O}_K) > 0$, we will show that the remaining sum has absolute value less than 1:

$$\begin{aligned}\sum_{\substack{a\geq 1, b\geq 0\\(a,b)\neq(1,0)}} \frac{1}{|(a+b\zeta_6)|^6} &= \sum_{\substack{a\geq 1, b\geq 0\\(a,b)\neq(1,0)}} \frac{1}{(a^2+ab+b^2)^3} = \frac{1}{3^3} + \sum_{\substack{a\geq 1, b\geq 0\\ \max(a,b)\geq 2}} \frac{1}{(a^2+ab+b^2)^3}\\ &\leq \frac{1}{3^3} + \sum_{\substack{a\geq 1, b\geq 0\\ \max(a,b)\geq 2}} \frac{1}{\max(a,b)^6} = \frac{1}{3^3} + \sum_{c=2}^{\infty} \sum_{\substack{a\geq 1, b\geq 0\\ \max(a,b)= c}} \frac{1}{c^6}\\ &= \frac{1}{3^3} + \sum_{c=2}^{\infty} \frac{2c}{c^6} = \frac{1}{3^3} + 2(\zeta(5)-1) < 1.\end{aligned}$$

So we see that $G_6(\mathcal{O}_K) > 0$, from which it follows that $G_6((\sqrt{3}i)^{-1}\mathcal{O}_K) = -3^3 G_6(\mathcal{O}_K) < 0$. So over $\mathbb{R}$, the curve E is isomorphic to $\mathbb{C}/\Lambda$ where $\Lambda = (\sqrt{3}i)^{-1}\mathcal{O}_K$. This means that we can take $\lambda = \sqrt{3}i$.

Now we will determine the character χ and its conductor ideal. The discriminant $\Delta(E)$ consists only of the prime factors 2 and 3. In K, the prime (2) is the only one that lies above 2 and $(\sqrt{3}i)$ is the only prime that lies above 3. So f, a generator for the conductor of χ must divisible by $2\sqrt{3}i$. We also know that $N(E) \mid |\Delta(E)| = 2^4 \cdot 3^3$ As $\Delta_K = -3$ we conclude from theorem 6 that $f \mid 2^2 \cdot (\sqrt{3}i)^2$. This means in any case that the ideal $(2^2 \cdot (\sqrt{3}i)^2)$ is a modulus for χ. To prove that $(2\sqrt{3}i)$ is the conductor of χ we must verify that $\chi(x) = 1$ for all x in the kernel of the map $(\mathcal{O}_K/((2\sqrt{3}i)^2))^* \to (\mathcal{O}_K/(2\sqrt{3}i))^*$. This kernel consists of elements represented by 1, $1+\sqrt{3}i$, $-2+\sqrt{3}i$ and $-2-\sqrt{3}i$. The identity $\chi(1) = 1$ is trivial. We can use lemma 9 to compute the value of χ for the other elements. To verify that $\chi(1+\sqrt{3}i) = 1$, we must show that $a_7 = -4$, which can be verified easily by simply counting points. And to verify that $\chi(-2+\sqrt{3}i) = \chi(-2-\sqrt{3}i) = 1$ we must show that $a_{13} = 2$ which we can also count by hand. In conclusion, the character χ has conductor (f), where

$$f = 2\sqrt{3}i.$$

The group μ_K is exactly a set of representatives for $(\mathcal{O}_K/(f))^*$ so that (43) shows that χ is determined by $\chi(x) = x$ for all $x \in \mu_K$.

We want to express $L(E,s)$ in terms of $K_{1,\Lambda}(0,x,s)$. As $f = 2\sqrt{3}i$, we can take $g = -\sqrt{3}i$ and $C = 6$. A straightforward calculation shows that

$$\frac{\hat{\chi}(\overline{g})|g|^{2s-2}\overline{\lambda}}{C|\mu_K|\,\overline{g}\,|\lambda|^{2s}} = \frac{\sqrt{3}i}{108}.$$

We can also straightforwardly let x run through all elements of $\mathcal{O}_K/C\mathcal{O}_K$ that are coprime with f and see that the expression $\overline{\chi}(x)x$ meets each of the values 1, $4+\sqrt{3}i$ and $4-\sqrt{3}i$ (of course $\bmod 6\Lambda$) exactly 6 times. So applying theorem 6 we conclude that

$$\begin{aligned}L(E,s) &= \frac{\sqrt{3}i}{108}\left(K_{1,\Lambda}\left(0,\frac{1}{6\sqrt{3}i},s\right) + K_{1,\Lambda}\left(0,\frac{4+\sqrt{3}i}{6\sqrt{3}i},s\right) + K_{1,\Lambda}\left(0,\frac{4-\sqrt{3}i}{6\sqrt{3}i},s\right)\right)\\ &= \frac{\sqrt{3}i}{108}\left(K_{1,\Lambda}\left(0,\frac{-\sqrt{3}i}{18},s\right) + K_{1,\Lambda}\left(0,\frac{3-4\sqrt{3}i}{18},s\right) + K_{1,\Lambda}\left(0,\frac{-3-4\sqrt{3}i}{18},s\right)\right).\end{aligned}$$

We are almost satisfied with this formula but not entirely because the coefficient $\sqrt{3}i/3$ is not a rational number. We can use (51) to do something about this. Write $\sqrt{3}i = \zeta_3 - \overline{\zeta_3}$, then we see that

$$\begin{aligned}\sqrt{3}iK_{1,\Lambda}\left(0,\frac{-\sqrt{3}i}{18},s\right) &= (\zeta_3-\overline{\zeta_3})K_{1,\Lambda}\left(0,\frac{-\sqrt{3}i}{18},s\right)\\ &= K_{1,\Lambda}\left(0,\frac{-\overline{\zeta_3}\sqrt{3}i}{18},s\right)-K_{1,\Lambda}\left(0,\frac{-\zeta_3\sqrt{3}i}{18},s\right) \qquad (52)\\ &= K_{1,\Lambda}\left(0,\frac{-3+\sqrt{3}i}{36},s\right)-K_{1,\Lambda}\left(0,\frac{3+\sqrt{3}i}{36},s\right).\end{aligned}$$

Similarly,

$$\sqrt{3}iK_{1,\Lambda}\left(0,\frac{3-4\sqrt{3}i}{18},s\right)=K_{1,\Lambda}\left(0,\frac{-15+\sqrt{3}i}{36},s\right)-K_{1,\Lambda}\left(0,\frac{9+7\sqrt{3}i}{36},s\right) \qquad (53)$$

and

$$\sqrt{3}iK_{1,\Lambda}\left(0,\frac{-3-4\sqrt{3}i}{18},s\right)=K_{1,\Lambda}\left(0,\frac{-9+7\sqrt{3}i}{36},s\right)-K_{1,\Lambda}\left(0,\frac{15+\sqrt{3}i}{36},s\right). \qquad (54)$$

Adding (52), (53) and (54) leads to an expression for $L(E,s)$ in terms of $K_{1,\Lambda}(0,x,s)$ with rational coefficients. If we put

$$\tau=\frac{3+\sqrt{3}i}{6},$$

then $1,\tau$ is a basis for Λ and we can write out everything in this basis. If we also keep in mind that $K_{1,\Lambda}(0,-x,s)=-K_{1,\Lambda}(0,x,s)$ and $K_{1,\Lambda}(0,x,s)=K_{1,\Lambda}(0,y,s)$ when $x-y\in\Lambda$, then we see that

$$\begin{aligned}L(E,s)=\frac{1}{18}\Bigg(&K_{1,\Lambda}\left(0,\frac{-1+\tau}{6},s\right)+K_{1,\Lambda}\left(0,\frac{-\tau}{6},s\right)+K_{1,\Lambda}\left(0,\frac{3+\tau}{6},s\right)\\ &+K_{1,\Lambda}\left(0,\frac{2-\tau}{6},s\right)+K_{1,\Lambda}\left(0,\frac{1+\tau}{6},s\right)+K_{1,\Lambda}\left(0,\frac{-2-\tau}{6},s\right)\Bigg).\end{aligned}$$

We can even further simplify this. Define for $x,y\in\Lambda$:

$$S(x,y)=\sum_{k=0}^{5}\left(\frac{x}{6},\zeta_6^k y\right)\zeta_6^{-k}.$$

As μ_K acts freely on $\Lambda-\{0\}$, we can see that

$$K_{1,\Lambda}\left(0,\frac{x}{6},s\right)=\frac{1}{6}\sum_{\lambda\in\Lambda-\{0\}}S(x,\lambda)\overline{\lambda}|\lambda|^{-2s},$$

since each term in the sum defining $K_{1,\Lambda}$ is added 6 times now. Also define

$$T(y)=S(-1+\tau,y)+S(-\tau,y)+S(3+\tau,y)+S(2-\tau,y)+S(1+\tau,y)+S(-2-\tau,y),$$

then

$$L(E,s)=\frac{1}{108}\sum_{\lambda\in\Lambda-\{0\}}T(\lambda)\overline{\lambda}|\lambda|^{-2s}.$$

Now it is clear from the definition of $(x/6,y)$ that $S(x,y)$ and $T(y)$ only depend on $y \bmod 6\Lambda$. The following formula can be proven by verifying it straightforwardly for all posibilities of $x \bmod 6\Lambda$ and $y \bmod 6\Lambda$:

$$T(y)=\begin{cases}0 & \text{if } y\notin\sqrt{3}i\Lambda;\\ -3\sqrt{3}iS(1,x) & \text{if } y=\sqrt{3}ix \text{ with } x\in\Lambda.\end{cases}$$

Note here that $\sqrt{3}i\Lambda = \mathbb{Z} + 3\tau\mathbb{Z}$, and that whether $y \in \sqrt{3}i\Lambda$ does not depend on the representant of $y \bmod 6\Lambda$. So one simply verifies that $T(y) = 0$ for all $y \bmod 6\Lambda$ not of the form $a + 3b\tau$ and that $T(\sqrt{3}ix) = -3\sqrt{3}iS(1,x)$ for all $x \bmod 6$. We can put everything together now:

$$\begin{aligned} L(E,s) &= \frac{1}{108}\sum_{\lambda\in\Lambda-\{0\}} T(\lambda)\overline{\lambda}|\lambda|^{-2s} = \frac{1}{108}\sum_{\lambda\in\Lambda-\{0\}} T(\sqrt{3}i\lambda)\overline{\sqrt{3}i\lambda}|\sqrt{3}i\lambda|^{-2s} \\ &= \frac{-3\sqrt{3}i}{108}\sum_{\lambda\in\Lambda-\{0\}} S(\lambda)\overline{\sqrt{3}i\lambda}3^{-s}|\lambda|^{-2s} = \frac{-1}{2^2 3^{s+1}}\sum_{\lambda\in\Lambda-\{0\}} S(\lambda)\overline{\lambda}|\lambda|^{-2s} \\ &= \frac{-1}{2\cdot 3^s}K_{1,\Lambda}\left(0,\frac{1}{6},s\right). \end{aligned} \tag{55}$$

Also, it can be verified easily that $S(2,y) = 0$ for all $y \in \Lambda$ so that

$$K_{1,\Lambda}\left(0,\frac{1}{3},s\right) = 0, \tag{56}$$

an identity which will appear to be of use later. We assumed that $\Re s > 3/2$ but by theorem 2 these two formulas hold for any $s \in \mathbb{C}$ (though it should not be too difficult to see the analytic continuation of the latter one without knowing this theorem).

Now that we know that $L(E,s)$ has an analytic continuation, we can also try to determine a functional equation for it, relating $L(E,s)$ and $L(E,2-s)$ to each other. Theorem 2 gives the following functional equation:

$$\Gamma(s)K_{1,\Lambda}\left(0,\frac{1}{6},s\right) = -(2\sqrt{3}\pi)^{2s-2}\Gamma(2-s)K_{1,\Lambda}\left(\frac{1}{6},0,2-s\right). \tag{57}$$

So we should take a closer look at $K_{1,\Lambda}(1/6,0,s)$. Assume that $\Re s > 3/2$. Set

$$u : \Lambda \to \mathbb{C} : \lambda \mapsto \begin{cases} 1 & \text{if } \lambda \equiv 1 \bmod 6\Lambda; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad U : \Lambda - \{0\} \to \mathbb{C} : \lambda \mapsto \sum_{k=0}^{5} u(\zeta_6^k\lambda)\zeta_6^{-k}.$$

Then

$$\begin{aligned} K_{1,\Lambda}\left(\frac{1}{6},0,s\right) &= \sum_{\lambda\in\Lambda}\overline{\left(\frac{1}{6}+\lambda\right)}\left|\frac{1}{6}+\lambda\right|^{-2s} = 6^{2s-1}\sum_{\lambda\in\Lambda}\overline{(1+6\lambda)}|1+6\lambda|^{-2s} \\ &= 6^{2s-1}\sum_{\lambda\in\Lambda} u(\lambda)\overline{\lambda}|\lambda|^{-2s} = 6^{2s-2}\sum_{\lambda\in\Lambda-\{0\}} U(\lambda)\overline{\lambda}|\lambda|^{-2s}. \end{aligned}$$

It can be verified straightforwardly, as above, that

$$U(x) = \begin{cases} 0 & \text{if } x \notin \sqrt{3}i\Lambda; \\ \frac{-\sqrt{3}i}{6}S(1,y) & \text{if } x = \sqrt{3}iy \text{ with } y \in \Lambda. \end{cases}$$

So, using also (55), we see that

$$\begin{aligned} K_{1,\Lambda}\left(\frac{1}{6},0,s\right) &= 6^{2s-2}\sum_{\lambda\in\Lambda-\{0\}} U(\lambda)\overline{\lambda}|\lambda|^{-2s} = -6^{2s-3}\sum_{\lambda\in\Lambda-\{0\}}\sqrt{3}iS(1,\lambda)\overline{\sqrt{3}i\lambda}|\sqrt{3}i\lambda|^{-2s} \\ &= -2^{2s-3}3^{s-2}\sum_{\lambda\in\Lambda-\{0\}} S(1,\lambda)\overline{\lambda}|\lambda|^{-2s} = 6^{2s-1}L(E,s). \end{aligned}$$

Using (55) and (57) and the analytic continuation we see now that

$$\Gamma(s)L(E,s) = 3^{2-2s}\pi^{2s-2}\Gamma(2-s)L(E,2-s)$$

We know that $\Gamma(s)$ has a simple pole of residue 1 at $s = 0$. From this it follows now that $L(E,0) = 0$ and

$$L'(E,0) = \frac{9}{\pi^2}L(E,2). \tag{58}$$

8 Hyperelliptic curves

This section will be used to state some general properties of hyperelliptic curves. A hyperelliptic curve is an algebraic curve C over a field K of genus at least 2 for which there exists a morphism $\pi : C \mapsto \mathbb{P}^1$ of degree 2. To serve generality, we will allow genus 0 and genus 1 curves that have this property as well and abusively call them hyperelliptic. The function field $K(C)$ is a quadratic extension of $K(x)$. From this it follows immediately that there is automorphism ι of C with $\iota^2 = \mathrm{id}$ and $\pi\iota = \pi$. We call ι the *hyperelliptic involution* of C. This ι need not be unique when $g(C) \leq 1$ but we will see in corollary 3 that it is unique when $g(C) \geq 2$.

Let's assume that $\mathrm{char}(K) = 0$. Then it follows from the fact that $[K(C) : K(x)] = 2$ that we can define C by an affine equation of the form

$$y^2 = f(x), \tag{59}$$

where $f(x)$ is a polynomial in $K[x]$ with no multiple roots. Now the morphisms π and ι are defined as follows:

$$\pi : (x, y) \mapsto x \quad \text{and} \quad \iota : (x, y) \mapsto (x, -y).$$

Let d be the degree of f and set $g = \lfloor \frac{d-1}{2} \rfloor$.

Lemma 14 *A complete nonsingular model of C consists of two affine pieces: the curve C_0 defined by (59) and the curve C_1 defined by*

$$w^2 = v^d f(1/v) \text{ if } d \text{ is even} \quad \text{and} \quad w^2 = v^{d+1} f(1/v) \text{ if } d \text{ is odd},$$

where C_0 and C_1 are glued to each other by the morphisms

$$\phi : C_0 - \{(0, y)\} \to C_1 : (x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right) \qquad \psi : C_1 - \{(0, w)\} \to C_0 : (v, w) \mapsto \left(\frac{1}{v}, \frac{w}{v^{g+1}}\right).$$

Proof: It's easy to see that the glued curve is nonsingular as both affine pieces are. It's also irreducible because C_0 and C_1 are irreducible and $C_0 \cap C_1$ is nonempty and open. So let's prove that the curve is projective. The map $\pi : C \to \mathbb{P}^1$ is defined as follows:

$$\pi(x, y) = [x : 1] \text{ if } (x, y) \in C_0 \quad \text{and} \quad \pi(v, w) = [1 : v] \text{ if } (v, w) \in C_1.$$

This is clearly a well-defined morphism of degree 2. Let's compute the degrees of the fibres. These degrees are at most 2 and we want to show that they are all equal to 2. Let $[x_0 : 1]$ be a point of $\mathbb{P}^1$. Then the fibre is contained in C_0 and consists of the points $(x_0, \pm y_0)$ that lie on C_0. If $y \neq 0$ then this degree is certainly 0 and if $y = 0$ then π is ramified at (x_0, y_0) because y is a uniformiser of the local ring at $(x_0, 0)$ and $x - x_0 \in (y)^2$ in that local ring. So also in the case $y = 0$ the degree of the fibre is 2, since it is more than 1 and at most 2. Now consider the point $[1 : 0]$ of $\mathbb{P}^1$. Its fibre is contained in C_1 and for analogous reasons as above, this fibre has degree 2. From the fact that all fibres of our model for C have full degree it follows immediately that this model is complete. $\square$

Lemma 15 *The curve C has genus g.*

Proof: Let's distinguish between the cases d even and d odd. Consider the model of C given in lemma 14. If d is even then the map π is ramified in the points $(x_0, 0)$ of C_0 where x_0 is a zero of f. The ramification index is 2 for each of those points and hence the result follows from the Riemann-Hurwitz formula. If d is odd there is an extra ramification point above $[1 : 0]$, namely $(v, w) = (0, 0)$. Apply Riemann-Hurwitz again. $\square$

Proposition 2 *A basis for the vector space of holomorphic differentials on C can be given by*

$$\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y}.$$

Proof: We use the model from lemma 14 again. The given differentials are clearly linearly independent and $\dim_K(H^0(C, \Omega^1_{C/K})) = g$ so the only thing left to prove is that they are holomorphic. On C_0, the differentials are certainly holomorphic at the points with $y \neq 0$. At a point $(x_0, 0)$, the local ring has uniformizer y and $x - x_0$ has valuation 2. So $dx/y = d(x - x_0)/y$ has valuation 0 so it is holomorphic there as well. Now let's go to C_1. The given differentials go over into

$$\frac{v^{g-1}dv}{w}, \ldots, \frac{dv}{w}$$

and the same argument as above applies. $\square$

Corollary 3 *If $g(C) \geq 2$, then the hyperelliptic involution ι is unique.*

Proof: We may suppose here that K is algebraically closed and use the model of lemma 14 again. If $g(C) \geq 2$, then from proposition 2 it immediately follows that the subfield of $K(C)$ generated by all quotients ω_1/ω_2 where ω_1 and ω_2 are nonzero holomorphic differentials is equal to $K(x)$, which is the fixed field of ι^*, where $\iota^* : K(C) \to K(C)$ is defined by $g \mapsto g \circ \iota$. So apparently, the fixed field of ι^* does not depend on ι at all, only on the space of holomorphic differentials. Of course there can be only 1 field automorphism of order 2 with a given fixed field so the result follows immediately. $\square$

The next thing we will prove is

Proposition 3 *Every curve of genus 2 is hyperelliptic.*

Proof: Let ω_1, ω_2 be a basis for the vector space of holomorphic differentials. Define a map

$$\pi : C \to \mathbb{P}^1 : P \mapsto [\omega_1(P) : \omega_2(P)].$$

Locally, we can choose a coordinate x such that we can write $\omega_1(P) = f_1(P)dx, \omega_2(P) = f_2(P)dx$ hence $\pi(P) = [f_1(P) : f_2(P)]$. The ratio $f_1(P) : f_2(P)$ does not depend on the choice of x so that π is a well-defined morphism. If we can prove that π has degree 2 then we are done.

The divisor (ω) of any differential ω on a curve of genus g has degree $2g - 2$. So in this particular case ω_1 and ω_2 both have 2 zeroes (counted with multiplicities) and no poles since they are holomorphic. If we can show that they have no zeroes in common then we're done since that would mean that the fibre of π over $[0 : 1] \in \mathbb{P}^1$ has degree 2, hence π is of degree 2. If ω_1 and ω_2 have 2 zeroes in common, then ω_1/ω_2 would be a constant function, contradicting the fact that ω_1 and ω_2 form a basis for $H^0(C, \Omega^1/C)$. If ω_1 and ω_2 have 1 zero in common, then ω_1/ω_2 has exactly 1 zero, which impies that the degree of π is 1, so π is an isomorphism which is clearly impossible. So indeed ω_1 and ω_2 have no zeroes in common, which implies that π has degree 2.$\square$

8.1 Homology of hyperelliptic curves

From now on we let K be the field $\mathbb{C}$. Let f be a polynomial of degree d, say, with no multiple roots. Consider a projective nonsingular model C of the curve $y^2 = f(x)$, as given by lemma 14. We can give C the complex-analytic topology. If g is the genus of the curve, then we know that

$$H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

We want to really find an isomorphism between $H_1(C, \mathbb{Z})$ and $\mathbb{Z}^{2g}$. This is equivalent with finding a $\mathbb{Z}$-basis of $H_1(C, \mathbb{Z})$. Write

$$f(x) = a(x - x_1)(x - x_2) \cdots (x - x_d),$$

where $a, x_1, \ldots, x_d \in \mathbb{C}$ and d is either $2g + 1$ or $2g + 2$. If we restrict the morphism π to $C_0 - \{(x_1, 0), \ldots, (x_d, 0)\}$, where C_0 is the affine part of C defined by (59), then we get a double unramified covering of topological spaces

$$\pi : Y := C_0 - \{(x_1, 0), \ldots, (x_d, 0)\} \to X := \mathbb{C} - \{x_1, \ldots, x_d\}.$$

It is well-known that a closed path in X lifts to a closed path in Y if and only if the total winding number around the points $x_1, \ldots, x_d$ is even. And clearly, every closed path in Y projects to a closed path in X. Also, every closed path in C is clearly homologous to a closed path in Y. So to determine a basis for $H_1(C, \mathbb{Z})$ we must look at liftings of closed paths in X that have an even total winding number.

Fix a point $z_0 \in X$ and a point $\zeta_0 \in Y$ above it. Every closed path in X is homologous to a closed path based at z_0. So from now on we will assume that our paths are based at z_0. Furthermore, we will assume that all the liftings of the paths start at ζ_0. If γ is a closed path in X, then $\widetilde{\gamma}$ will denote the lifting of γ to Y, and $[\widetilde{\gamma}]$ will denote the homology class of $\widetilde{\gamma}$ in $H_1(C, \mathbb{Z})$. Now choose for each $i \in \{1, \ldots, d\}$ an analytic closed path γ_i in X that has winding number 1 around x_i and winding number equal to 0 around the other zeroes of f.

Lemma 16 *The following homology classes, with notation as described above, are a basis for $H_1(C, \mathbb{Z})$:*

$$\mathfrak{a}_i := \left[\widetilde{\gamma_{2i-1}\gamma_{2i}}\right],$$
$$\mathfrak{b}_i := \left[\widetilde{\gamma_{2i}\cdots\gamma_{2g+1}}\right],$$

where $i = 1, \ldots, g$.

Proof: See [10]. □

Let's now consider parametrised families of hyperelliptic curves of the form

$$C_k : y^2 = f(k, x),$$

where $f(k, x)$ is polynomial in x and rational in k. Set $d = \deg_x(f)$. Let T be a subset of $\mathbb{C}$ such that for all $k \in T$, the polynomial $f(k, x)$ exists, is of degree d and has no multiple roots. We can identify $H_1(C_k, \mathbb{Z})$ with a lattice in $\mathbb{C}^g$, where $g = \lfloor \frac{d-1}{2} \rfloor$, the genus of the curves C_k, using the following map:

$$h_k : H_1(C_k, \mathbb{Z}) \to \mathbb{C}^g : [\gamma] \mapsto \int_\gamma \begin{pmatrix} \frac{dx}{y} \\ \vdots \\ \frac{x^{g-1}dx}{y} \end{pmatrix}. \tag{60}$$

We wonder if we can choose a parametrised family of bases $[\sigma_{k,1}], \ldots, [\sigma_{k,2g}]$ of $H_1(C_k, \mathbb{Z})$ such that $h_k([\sigma_{k,j}])$ depends continuously on k for each j. It turns out that this can be done locally:

Proposition 4 *Choose a point $P \in T$. Then there exists an open neighbourhood $U \subset T$ of P for which the following holds: for each $k \in U$ there is a basis $[\sigma_{k,1}], \ldots, [\sigma_{k,2g}]$ of $H_1(C_k, \mathbb{Z})$ which we can choose in such a way that the function $h_k([\sigma_{k,j}])$ is a continuous function of k from U to $\mathbb{C}^g$ for each j.*

Proof: Write $f(P,x) = a(x - x_1(P)) \cdots (x - x_d(P))$. The inverse function theorem shows that for each zero $x_i(P)$ of $f(P,x)$, there is an open set $U_i \subset T$ and an open neighborhood $V_i \ni x_i(P)$ such that for each $k \in U_i$ the polynomial $f(k,x)$ has only 1 zero $x_i(k)$ in V_i and furthermore $x_i(k)$ is an analytic (hence continuous) function of k.

Let z_0 be a point in $\mathbb{C}$ which is not a zero of $f(P,x)$, and choose closed paths γ_i as above. As $x_i(k)$ is continuous, there is an open set $U' \subset \bigcap_{i=1}^d U_i$ such that for all i, $x_i(k)$ does not meet γ_i for any $k \in U$ and such that the winding number of γ_i is 1 around $x_i(k)$ and 0 around the other roots of $f(k,x)$. Let $\zeta_0(P)$ be a lifting of z_0 to C_P, as described above. Then there is an open neighborhood V of P on which we can define $\sqrt{f(k,x_0)}$ as an analytic function of k such that $\zeta_0(P) = (z_0, \sqrt{f(P,z_0)})$. Define $\zeta_0(k) = (z_0, \sqrt{f(k,z_0)})$ as lifting of z_0 to C_k over π.

The next thing we want to prove is that for each i, there is an open neighborhood $V_i \subset U' \cap V$ of P such that the lift $\widetilde{\gamma_{k,i}}$ of γ_i with starting point $\zeta_0(k)$ is an analytic function from $V_i \times [0,1]$ to $\mathbb{C}^2$. Fix i. By the inverse function theorem, for each $t \in [0,1]$ we have an open neigborhood $W_{i,t} \subset [0,1]$ of t and an open set $V_{i,t} \subset U' \cap V$ such that $\widetilde{\gamma_i}$ is an analytic function from $V_{i,t} \times W_{i,t}$ to $\mathbb{C}^2$. As $[0,1]$ is compact, finitely many of those $W_{i,t}$ cover $[0,1]$ and it's clear that we can take V_i to be intersection of the corresponding $V_{i,t}$. Let U denote the intersection of all V_i.

Because $\widetilde{\gamma_a\gamma_b} = \widetilde{\gamma_a} \cdot \iota(\widetilde{\gamma_b})$ it follows now that any path of the form $\widetilde{\gamma_a\gamma_b}$ is a continuous function from $U \times [0,1]$ to $\mathbb{C}^2$ which is analytic on $U \times]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$. This implies that the representants of the canonical homology basis from lemma 16 are continuous functions from $U \times [0,1]$ to $\mathbb{C}^2$ such that for each $k \in U$, the paths $\mathfrak{a}_{k,i}$ and $\mathfrak{b}_{k,i}$ are piecewise smooth. With this homology basis, the proposition follows immediately because of proposition 2. $\square$

We can make this result more global:

Corollary 4 *Let $\delta : [0,1] \to T$ be a path in T. Then for each $t \in [0,1]$ there is a basis $[\sigma_{t,1}], \ldots [\sigma_{t,2g}]$ of $H_1(C_{\delta(t)}, \mathbb{Z})$ such that $h_{\delta(t)}([\sigma_{t,j}])$ is a continuous function of t for each j.*

Proof: From proposition 4 it follows that we can do this locally: for each $t_0 \in [0,1]$ there exists an open neighborhood $U = U_{t_0} \subset [0,1]$ of t_0 such that for each $t \in U$ we can choose a basis $[\sigma_{t_0,t,1}], \ldots [\sigma_{t_0,t,2g}]$ of $H_1(C, \mathbb{Z})$ such that $h_{\delta(t)}([\sigma_{t_0,t,j}])$ is continuous. As $[0,1]$ is compact, finitely many of these open sets U_{t_0} cover $[0,1]$. So let $t_1 \leq \ldots \leq t_n$ be such that $[0,1] = \bigcup_{i=1}^n U_{t_i}$.

Now we proceed with induction: suppose that a basis $[\sigma_{t,1}], \ldots [\sigma_{t,2g}]$ of $H_1(C, \mathbb{Z})$ exists for which $h_{\delta(t)}([\sigma_{t,j}])$ is a continuous function of $t \in \bigcup_{i=1}^{k-1} U_{t_i}$ for each j. We will show that there exists a basis such that $h_{\delta(t)}([\sigma_{t,j}])$ is continuous for $t \in \bigcup_{i=1}^{k} U_{t_i}$. We know that there is a basis such that $h_{\delta(t)}([\sigma_{t,j}])$ is continuous for $t \in U_{t_k}$. Pick a number $t \in U_{t_{k-1}} \cap U_{t_k}$. Then $[\sigma_{t_{k-1},t,1}], \ldots [\sigma_{t_{k-1},t,2g}]$ may be another basis than $[\sigma_{t_k,t,1}], \ldots [\sigma_{t_k,t,2g}]$. Yet we can write out the basis $([\sigma_{t_{k-1},t,i}])_{i=1}^{2g}$ in terms of the basis $([\sigma_{t_k,t,i}])_{i=1}^{2g}$. So there are integers $a_{i,j}(t)$ such that $[\sigma_{t_{k-1},t,i}] = \sum_{j=1}^{2g} a_{i,j}(t)[\sigma_{t_k,t,j}]$ for each j. It is now clear that

$$h_{\delta(t)}([\sigma_{t_{k-1},t,i}]) = \sum_{j=1}^{2g} a_{i,j}(t) h_{\delta(t)}([\sigma_{t_k,t,j}]). \tag{61}$$

Let's identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$. Let's furthermore write $v_i(t) \in \mathbb{R}^{2g}$ for $h_{\delta(t)}([\sigma_{t_k,t,j}])$ as column vector in $\mathbb{R}^{2g}$ and let's write $v_i'(t) \in \mathbb{R}^{2g}$ for $h_{\delta(t)}([\sigma_{t_{k-1},t,i}])$ as column vector in $\mathbb{R}^{2g}$. Then (61) goes over into

$$\begin{pmatrix} | & & | \\ v_1'(t) & \dots & v_{2g}'(t) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1(t) & \dots & v_{2g}(t) \\ | & & | \end{pmatrix} \begin{pmatrix} a_{1,1}(t) & \dots & a_{2g,1}(t) \\ \vdots & \ddots & \vdots \\ a_{1,2g}(t) & \dots & a_{2g,2g}(t) \end{pmatrix}.$$

So

$$\begin{pmatrix} a_{1,1}(t) & \dots & a_{2g,1}(t) \\ \vdots & \ddots & \vdots \\ a_{1,2g}(t) & \dots & a_{2g,2g}(t) \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1(t) & \dots & v_{2g}(t) \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | & & | \\ v_1'(t) & \dots & v_{2g}'(t) \\ | & & | \end{pmatrix},$$

from which it follows that each $a_{i,j}$ is a continuous function of $t \in U_{t_{k-1}} \cap U_{t_k}$. As the function values are integers, it follows that each $a_{i,j}$ is a constant function. It is now immediate that if we replace the basis $([\sigma_{t_k,t,i}])$ by $\sum_{j=1}^{2g} a_{i,j}(t)[\sigma_{t_k,t,j}]$, then we get a basis $([\sigma_{t,i}])$ for which $h_{\delta(t)}([\sigma_{t,j}])$ is continuous for $t \in \bigcup_{i=1}^{k} U_k$ for each j. With induction it now follows that we can construct such a basis for which $h_{\delta(t)}([\sigma_{t,j}])$ is continuous on all of $[0,1]$. $\square$

Now we want to apply this result to study families of (piecewise smooth) closed paths on C_k. Let γ_k be a family of piecewise smooth closed paths on C_k. This means that for each $k \in T$, the path γ_k is a function from $[0,1]$ to C_k which is continuous and piecewise smooth and furthermore is satisfies $\gamma_k(0) = \gamma_k(1)$ for each k. We can view the parametrised family C_k as an algebraic surface over $\mathbb{C}$, by interpreting k as a coordinate variable. We will call a family of paths γ_k *continuous* if the mapping $T \times [0,1] \to C_k : (k,t) \mapsto (k, \gamma_k(t))$ is continuous, where C_k is viewed as a surface this way, with first coordinate equal to the k-coordinate.

Corollary 5 *Let γ_k be a continuous family of piecewise smooth closed paths on C_k. Let δ be a path in T. Then the coefficients of $\gamma_{\delta(t)}$ with respect to a basis of $H_1(C_{\delta(t)}, \mathbb{Z})$ that satisfies the conditions from corollary 4 are constant functions of t.*

Proof: It is clear that $h_k([\gamma_k])$ is continuous as a function of k. Let's write $[\gamma_{\delta(t)}] = \sum_{i=1}^{2g} a_i(t)[\sigma_{t,i}]$ with $a_i(t) \in \mathbb{Z}$ for each t. Then it is clear that $h_{\delta_t}([\gamma_{\delta(t)}] = \sum_{i=1}^{2g} a_i(t) h_{\delta(t)}([\sigma_{t,i}])$. If we identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$ and write $v_i(t)$ for $h_{\delta(t)}([\sigma_{t,i}])$ as column vector in $\mathbb{R}^{2g}$ and $w(t)$ for $h_{\delta(t)}([\gamma_t])$ as column vector, then we see that

$$\begin{pmatrix} | & & | \\ v_1(t) & \dots & v_{2g}(t) \\ | & & | \end{pmatrix} \begin{pmatrix} a_1(t) \\ \vdots \\ a_{2g}(t) \end{pmatrix} = \begin{pmatrix} | \\ w(t) \\ | \end{pmatrix}.$$

So

$$\begin{pmatrix} a_1(t) \\ \vdots \\ a_{2g}(t) \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1(t) & \dots & v_{2g}(t) \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | \\ w(t) \\ | \end{pmatrix},$$

from which it follows that each $a_i(t)$ is continuous, hence constant as it assumes only integral values. $\square$

In fact, this proof only uses that $h_\delta(t)([\gamma_{\delta(t)}])$ is continuous so that we immediately have the following corollary as well:

Corollary 6 *Suppose that $\delta : [0,1] \to T$ is a path in T. Let γ_k be a family of paths on C_k such that $h_{\delta(t)}([\gamma_{\delta(t)}])$ is a continuous function of t. Then the coefficients of $[\gamma_{\delta(t)}]$ with respect to a basis that satisfies the conditions of corollary 4 are constant functions of t.*

9 Reciprocal genus 2 curves

Let now K be any field of characteristic 0 again. Suppose that $f(x) \in K[x]$ is a polynomial 3 with no multiple roots, and that $f(0) \neq 0$. Then $f(x^2)$ has no multiple roots either. Consider the genus 2 hyperelliptic curve

$$C : y^2 = f(x^2). \tag{62}$$

When a hyperelliptic curve admits this type of equation, we will call it *reciprocal.* This curve comes equipped with a map

$$\pi : C \to \mathbb{P}^1 : (x, y) \mapsto x.$$

Besides the hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$, there are two other interesting involutions:

$$\sigma_1 : (x, y) \mapsto (-x, y) \quad \text{and} \quad \sigma_2 : (x, y) \mapsto (-x, -y). \tag{63}$$

Is easy to see that $\{\mathrm{id}, \iota, \sigma_1, \sigma_2\}$ is a subgroup of $\mathrm{Aut}_K(C)$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

From now on, suppose that $\deg(f) = 3$ so that the curve has genus 2. Let's divide out the automorphisms σ_1 and σ_2. To do this, we consider the function field $K(C) = K(x)[y]/(y^2 - f(x^2))$ and we have to determine the subfield $K(C)^{\sigma_1}$ that is fixed by σ_1. This is clearly the subfield $K(x^2, y)$. This subfield corresponds to the elliptic curve

$$E_1 : y^2 = f(x), \tag{64}$$

together with the cover

$$\phi_1 : C \to E_1 : (x, y) \mapsto (x^2, y). \tag{65}$$

We also consider the map

$$\pi_1 : E_1 \to \mathbb{P}^1 : (x, y) \mapsto x.$$

For the other automorphism σ_2, the fixed field is clearly equal to $K(x^2, xy) = K(\frac{1}{x^2}, \frac{y}{x^3})$, so that the elliptic curve and cover are in this case equal to

$$E_2 : y^2 = x^3 f(\frac{1}{x}) \quad \text{and} \quad \phi_2 : C \to E_2 : (x, y) \mapsto \left(\frac{1}{x^2}, \frac{y}{x^3}\right) = [x : y : x^3].$$

For $i \in \{1, 2\}$, the σ_i-invariant rational functions on C correspond, by construction, to the rational functions on E_i. For example, the function $g(x^2, y) \in K(C)^{\sigma_1}$ corresponds to the function $g(x, y) \in K(E_1)$. This will be very important when we will study mahler measures of polynomials that define curves of genus 2. We will denote the pull-back on C of a function g on E_i by $\phi_i^*(g)$ and the push-forward on E_i of a σ_i-invariant function g on C by $\phi_{i*}(g)$

Let's look at how the morphisms ϕ_1 and ϕ_2 look like on the (v, w)-chart from lemma 14. An easy calculation shows that

$$\phi_1(v, w) = \left(\frac{1}{v^2}, \frac{w}{v^3}\right) = [v : w : v^3] \quad \text{and} \quad \phi_2(v, w) = (v^2, w).$$

We can also look at the following map:

$$\phi : C \to E_1 \times E_2 : P \mapsto (\phi_1(P), \phi_2(P)).$$

This map is clearly an embedding. Hence we can view C via ϕ as a closed subvariety of the abelian surface $E_1 \times E_2$, as maps between projective varieties are always closed. In fact this shows that the Jacobian of C is isogeneous to $E_1 \times E_2$.

The genus 2 curves that we will look at are defined as the zero set of a *reciprocal* polynomial of the form $P(x,y) = A(x)y^2 + B(x)y + C(x)$. Note that in that case there is an involution

$$\sigma_1' : (x,y) \mapsto (\frac{1}{x}, \frac{1}{y}). \tag{66}$$

We will show that we can always put the equation of this curve into the standard form (62). By substituting $y_1 = 2A(x)y + B(x)$ one gets an equation of the shape $y_1^2 = f(x)$. If $f(x)$ here happens to have square factors, we can move them to the left hand side into the denominator of y_1 and hence we get an equation of the form (59). As $P(x,y)$ is reciprocal, it follows that $f(x)$ is reciprocal as well. We will further subsitute

$$x = \frac{x_2+1}{x_2-1}, \quad y_2 = y_1(x_2-1)^3$$

to arrive at an equation of the form (62) for the coordinates x_2 and y_2. The transformation $T(x) = (x+1)/(x-1)$ has the interesting properties that $T(T(x)) = x$, $T(1/x) = -T(x)$ and $T(-x) = 1/T(x)$ so that the involution σ from (66) goes over into the involution

$$\sigma_1 : (x_2, y_2) \mapsto (-x_2, y_2).$$

Let's denote the curve defined by the polynomial P by C' and the birationally equivalent curve in standard form (62) by C. Also denote the isomorphism between C' and C that we obtain via the above substitutions by

$$\psi : C' \to C.$$

We can also rewrite the involutions ι and σ_2 in terms of C'. Let's write ι' and σ_2' respectively. Then $\iota : (x,y) \mapsto (x, -B(x)/A(x) - y) = C(x)/(A(x)y)$. Because of reciprocity, the fraction $C(x)/A(x)$ will be a power of x, say x^m. So that

$$\iota' : (x,y) \mapsto \left(x, \frac{x^m}{y}\right) \qquad \sigma_2' = \sigma_1'\iota' : (x,y) \mapsto \left(\frac{1}{x}, \frac{y}{x^m}\right).$$

Now it becomes clear why the elliptic curve E_1 given by (64) is important for computing the Mahler measure of P: the differential $\eta(x,y)$, where x and y are the coordinates on C', is invariant under the action of σ_1'. This allows us to rewrite a regulator $r_C(x,y)$ to a regulator on E. Let $K(C)/K(E_1)$ the quadratic field extension that corresponds to the cover ϕ_1 given by (65). The Galois group of this field extension is generated by σ_1^*. We can now use the manipulation trick (19) from section 4 to push forward $r_C(x,y)$ to E. Namely,

$$\begin{aligned} r_C\{x,y\}([\gamma]) &= \frac{1}{2} r_C\left(\{x,y\}\sigma(\{x,y\})\right)([\gamma]) = \frac{1}{2} r_C \operatorname{res}_{K(C)/K(E_1)}\left(N_{K(C)/K(E_1)}\{x,y\}\right)([\gamma]) \\ &= \frac{1}{2} r_E\left(N_{K(C)/K(E_1)}\{x,y\}\right)([\phi_1\gamma]). \end{aligned}$$

Furthermore, we will see in all our examples that $(x,y)_P$ is a root of unity for each point P of a nonsingular complete model of C', where $(-,-)_P$ is the tame symbol. From (18) and the well-known fact that a non-constant morphism of nonsingular complete curves is surjective it follows that $\partial_P(\alpha)$ is a root of unity as well. Looking at lemma 6 form section 6 we see that $\eta(\alpha)$ has all its residues equal to 0.

9.1 Homology

Suppose now that $K = \mathbb{C}$. We want to compute the induced map $H_1(\phi_1) : H_1(C,\mathbb{Z}) \to H_1(E_1,\mathbb{Z})$ on homology. To do this we will suppose that C and E are given in standard form (62) and (64). We will make a homology basis as in lemma 16 from section 8. Choose $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$f(x) = a(x-\alpha_1^2)(x-\alpha_2^2)(x-\alpha_3^2),$$

where a is the leading coefficient of f. Then

$$f(x^2) = a(x-\alpha_1)(x+\alpha_1)(x-\alpha_2)(x+\alpha_2)(x-\alpha_3)(x+\alpha_3).$$

For $i \in \{1,2,3\}$, let $\gamma_i^+ : [0,1] \to \mathbb{C} - \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ be a closed piecewise smooth path, based at 0, such that the winding number of γ_i^+ around α_i is 1 and it is 0 around the other zeroes of $f(x^2)$. Now, let $\gamma_i^- : [0,1] \to \mathbb{C} - \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ be the path defined as follows: $\gamma_i^- : t \mapsto -\gamma_i^+(t)$. So γ_i^- has winding number 1 around $-\alpha_i$ and winding number 0 around the other zeroes of $f(x^2)$. Set $z_0 = 0$ and choose a point $\zeta_0 = (0, y_0) \in \pi^{-1}(0)$. If we lift paths along π as in subsection 8.1, then lemma 16 shows that the following classes form a basis of $H_1(C, \mathbb{Z})$:

$$\mathfrak{a}_1(C) := \left[\widetilde{\gamma_1^+\gamma_1^-}\right], \quad \mathfrak{a}_2(C) := \left[\widetilde{\gamma_2^+\gamma_2^-}\right], \quad \mathfrak{b}_1(C) := \left[\widetilde{\gamma_1^-\gamma_2^+\gamma_2^-\gamma_3^+}\right], \quad \mathfrak{b}_2(C) := \left[\widetilde{\gamma_2^-\gamma_3^+}\right].$$

Now, let $\gamma_i' : [0,1] \to \mathbb{C} - \{\alpha_1^2, \alpha_2^2, \alpha_3^2\}$ be the path defined as $t \mapsto (\gamma_i^+(t))^2$. We see that the winding number of γ_i' around α_i^2 is 1 and it is 0 around the other zeroes of $f(x)$:

$$\begin{aligned}\operatorname{ind}_{P^2}(\gamma_i') &= \frac{1}{2\pi i}\int_{\gamma_i'} \frac{dz}{z-P^2} = \frac{1}{2\pi i}\int_{\gamma_i^+} \frac{2z\,dz}{z^2-P^2} = \frac{1}{2\pi i}\int_{\gamma_i^+}\left(\frac{dz}{z-P} + \frac{dz}{z+P}\right)\\ &= \operatorname{ind}_P(\gamma_i^+) + \operatorname{ind}_{-P}(\gamma_i^+).\end{aligned}$$

We can lift these paths to E_1, again as in subsection 8.1, using $z_0 = 0$ and $\zeta_0 = (0, y_0)$. Using lemma 16, we get the following basis of $H_1(E_1, \mathbb{Z})$:

$$\mathfrak{a}_1(E_1) = \left[\widetilde{\gamma_1'\gamma_2'}\right], \quad \mathfrak{b}_1(E_1) = \left[\widetilde{\gamma_2'\gamma_3'}\right].$$

Now, we want to figure out the induced map $H_1(\phi_1)$ on homology. It is not so difficult to see that

$$\pi_1 \circ \phi_1 \circ \left(\widetilde{\gamma_i^+\gamma_i^-}\right) = (\gamma_i')^2$$

for $i \in \{1,2\}$. This implies that $H_1(\phi_1)(\mathfrak{a}_i(C)) = 0$, an easy way to see this is for example by applying the map h from equation (60). Also,

$$\pi_1 \circ \phi_1 \circ \left(\widetilde{\gamma_1^-\gamma_2^+}\right) = \gamma_1'\gamma_2' \quad \text{and} \quad \pi_1 \circ \phi_1 \circ \left(\widetilde{\gamma_2^-\gamma_3^+}\right) = \gamma_2'\gamma_3'.$$

As $\phi(0, y_0) = (0, y_0)$, the map ϕ sends the base point of the loops in C to the base point of the loops in E. It follows that $H_1(\phi_1)(\mathfrak{b}_1(C)) = \mathfrak{a}_1(E_1) + \mathfrak{b}_1(E_1)$ and $H_1(\phi_1)(\mathfrak{b}_2(C)) = \mathfrak{b}_1(E_1)$. In conclusion,

$$H_1(\phi_1)\left(c_1\mathfrak{a}_1(C) + c_2\mathfrak{a}_2(C) + c_3\mathfrak{b}_1(C) + c_4\mathfrak{b}_2(C)\right) = c_3\mathfrak{a}_1(E_1) + (c_3 + c_4)\mathfrak{b}_1(E_1). \tag{67}$$

10 The family $P_k(x,y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4$

Let C_k' be the family curves defined by the polynomial $P_k(x,y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4$. If $k \in \mathbb{C}$, we can view this as a curve over the field $\mathbb{Q}(k)$. The curve C_k' has generically genus 2, as one can compute with the Riemann-Hurwitz formula. Hence it is a hyperelliptic curve. We do not even need to know the Riemann-Hurwitz formula nor the fact that curves of genus 2 are always hyperelliptic to make these conclusions as we can write the curve in the following normal form:

$$C_k : y^2 = f(x^2),$$

where

$$f(x) = (k^2+k)x^3 + (-2k^2+5k+4)x^2 + (k^2-5k+8)x - k + 4.$$

A birational map between the curves C_k' and C_k can be given by

$$\psi : C_k' \to C_k : (x,y) \mapsto \left(\frac{x+1}{x-1}, \frac{4(y^2-x^4)}{y(x-1)^3(x+1)}\right), \tag{68}$$

whose inverse is equal to

$$\psi^{-1} : C_k \to C_k' : (x,y) \mapsto \left(\frac{x+1}{x-1}, \frac{2xy - (2k+1)x^4 + (2k-6)x^2 - 1}{(x-1)^4}\right). \tag{69}$$

So the elliptic curve E that we get by dividing out the involution σ from (66) is

$$E : y^2 = (k^2+k)x^3 + (-2k^2+5k+4)x^2 + (k^2-5k+8)x - k + 4.$$

Lemma 17 *If $K = \mathbb{C}$, then C_k has genus 2 for $k \in \mathbb{C} - \{-1,0,4,8\}$. Furthermore, C_k has genus 1 if $k \in \{-1,4\}$, it has genus 0 if $k = 8$ and it is a reducible curve consisting of 2 irreducible components of genus 0 if $k = 0$.*

Proof: The discriminant of $f(x^2)$ is equal to $2^{22}k^3(k-8)^2(k-4)(k+1)$. So $f(x^2)$ has degree 6 and no multiple roots if $k \in \mathbb{C} - \{-1,0,4,8\}$, so lemma 15 shows that C_k has genus 2.

Let's look at what happens if $k \in \{-1,0,4,8\}$. If $k = -1$, then $f(x) = -3x^4 + 14x^2 + 5$, so C_k has genus 1. If $k = 0$, then $f(x) = 4(x^2+1)^2$. This means that C_k is not even irreducible, it has 2 irreducible components, each of genus 0. If $k = 4$, then $f(x) = 4x^2(5x^4 - 2x^2 + 1)$, so C_k is birationally equivalent to the curve $y^2 = 5x^4 - 2x^2 + 1$, hence it has genus 1. And if $k = 8$, then $f(x) = 4(3x^2+1)^2(2x^2+1)$, so C_k is birationally equivalent to $y^2 = 2x^2 + 1$, hence of genus 0.□

Next, we will compute the values of the tame symbols $(x,y)_P$ on C_k' (for a definition of the tame symbol, see formula (17) in section 4). Being coordinates, x and y are still functions on a complete nonsingular model $\widetilde{C_k'}$ for C_k'.

Lemma 18 *For all $P \in \widetilde{C_k'}$, the tame symbol $(x,y)_P$ is a root of unity.*

Proof: If $\mathrm{ord}_P(x) = 0$ and $\mathrm{ord}_P(y) = 0$ then it is clear that $(x,y)_P = 1$. So suppose now that $\mathrm{ord}_P(x) > 0$. If $\mathrm{ord}_P(y) < 0$ then on one hand, $\mathrm{ord}_P(y^2 + (x^4+kx^3+kx^2+kx+1)y) = 2\,\mathrm{ord}_P(y) < 0$, but on the other hand, $\mathrm{ord}_P(y^2 + (x^4+kx^3+kx^2+kx+1)y) = \mathrm{ord}_P(x^4) > 0$, so this is not possible. If $\mathrm{ord}_P(y) > 0$, then P is the point $(0,0)$, which is a nonsingular point of C_k'. We see now that $\mathrm{ord}_P(y) = \mathrm{ord}_P(y^2 + (x^4+kx^3+kx^2+kx+1)y) = \mathrm{ord}_P(x^4) = 4\,\mathrm{ord}_P(x)$. Hence

$$(x,y)_P = \left(\left.\frac{x^4}{y}\right|_{(0,0)}\right)^{\mathrm{ord}_P(x)} = \left(-(y + x^4 + kx^3 + 2kx^2 + kx + 1)|_{(0,0)}\right)^{\mathrm{ord}_P(x)} = \pm 1$$

so the claim follows. The subcase $\text{ord}_P(y) = 0$ is left. Here, $\text{ord}_P(y + x^4 + kx^3 + 2kx^2 + kx + 1) = \text{ord}_P(y^2 + (x^4 + kx^3 + kx^2 + kx + 1)y) = \text{ord}_P(x^4) > 0$. So P is the nonsingular point $(0, -1)$. We can now see that

$$(x, y)_P = \frac{1}{y(P)^{\text{ord}_P(x)}} = (-1)^{\text{ord}_P(x)},$$

which is a root of unity as well.

Let's now look at the case $\text{ord}_P(x) < 0$. Remind that σ_1' is the automorphism $(x, y) \mapsto (1/x, 1/y)$ on C_k'. From (17) it follows immediately that

$$(x, y)_P = \left(\frac{1}{x}, \frac{1}{y}\right)_P = (x \circ \sigma_1', y \circ \sigma_1')_P = (x, y)_{\sigma_1'(P)}$$

and $\text{ord}_{\sigma_1'(P)}(x) = -\text{ord}_P(x) > 0$ so we have reduced the problem to a case we have already proved.

So we are left with the case $\text{ord}_P(x) = 0$. There are two subcases to examine, namely $\text{ord}_P(y) > 0$ and $\text{ord}_P(y) < 0$. As above, the former subcase implies the latter one by applying σ_1'. Thus we assume that $\text{ord}_P(y) > 0$. But then we get a contradiction from $\text{ord}_P(x^4) = \text{ord}_P(y^2 + (x^4 + kx^3 + kx^2 + kx + 1)y) \geq \text{ord}_P(y) > 0$. $\square$

10.1 Construction of closed paths γ_k on C_k'

Let's now find out for which $k \in \mathbb{R}$ there exists a function $y_+(x)$ for the polynomial $P_k(x)$ that satisfies conditions 1 and (10) from section 2.1. The equation $P_k(x, y) = 0$ is a quadratic equation for the variable y, with discriminant equal to

$$D_k(x) = (x + 1)^2(x^2 + (k - 2)x + 1)(x^4 + kx^3 + 2(k + 1)x^2 + kx + 1). \tag{70}$$

According to the *abc*-formula, we have to take the square root of this discriminant. In this *abc*-formula we see that

$$a_k(x) = 1, \quad b_k(x) = x^4 + kx^3 + 2kx^2 + kx + 1, \quad c_k(x) = x^4.$$

It is well known (see for example chapter 10 in [11]) that the square root of a holomorphic funtion f exists on an open set U if and only if for every closed path γ in U, the total winding number of γ around the zeroes of f (counting with multiplicities) is divisible by 2. So if $D_k(x)$ has an odd number of zeroes with $|x| < 1$ we do not expect that there exists a function y satisfying both condition 1 and (10). The following lemma and corollary show that this is indeed not possible:

Lemma 19 *Let $C = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. Let f be a polynomial with real coefficients without multiple zeroes. Suppose that f has an odd number of zeroes of absolute value strictly less than 1. Suppose furthermore $f(1) \neq 0$ and $f(-1) \neq 0$. Then there does not exist a continuous function $g : C \to \mathbb{C}$ such that $g(x)^2 = f(x)$ and $\overline{g(x)} = g(\overline{x})$ for each $x \in C$.*

Proof: Suppose that such a function g does exist. Let $X = \{(x, y) \in \mathbb{C}^2 : y^2 = f(x)\}$ and define $\pi : X \to \mathbb{C} : (x, y) \mapsto x$. As f has no multiple zeroes, X has no singular points. As the number of zeroes of f is finite we can find a number c with $0 < c < 1$ such that $|x| < c$ for every zero x of f of absolute value strictly less than 1 and $|x| > 1/c$ for every zero x of f of absolute value stricly greater than 1. Define the path

$$\gamma_1 : [0, 1] \to X : t \to (\exp(\pi i t), g(\exp(\pi i t)))$$

(so γ_1 is a lift of the upper half of the circle C to C'_k). Then clearly γ_1 is path-homotopic to a path γ'_2, not passing through any points with y-coordinate equal to 0 and such that the x-coordinates of all the points it passes satisfy $c < x < 1/c$. We also have a closed path

$$\gamma_2 : [0,1] \to X : t \to (\exp(2\pi i t), g(\exp(2\pi i t)))\,.$$

Clearly, γ_2 is homotopic to the path

$$\gamma'_2 : [0,1] \to X : t \to \left\{ \begin{array}{ll} \gamma'_1(2t) & \text{if } 0 \leq t < \frac{1}{2}; \\ \overline{\gamma'_1(2-2t)} & \text{if } \frac{1}{2} < t \leq 1 \end{array} \right. ,$$

where $\overline{(x,y)} = (\overline{x}, \overline{y})$, because $g(\overline{x}) = \overline{g(x)}$ for each $x \in C$. We can now look at the path $\pi \circ \gamma'_2 : [0,1] \to \mathbb{C}$, it does not meet any zeroes of f. The total winding number around the zeroes of f is odd, since the number of zeroes x with $|x| < 1$ is odd and the zeroes of f with $|x| = 1$ that contribute to the total winding number come in complex conjugate pairs. But it is well-known that any path with odd total winding number around the zeroes of f lifts to a path which is not closed. Hence we have a contradiction. $\square$

Corollary 7 *If $D_k(x)$ has an odd number of zeroes with $|x| < 1$, with k real, then there does not exist a function y satisfying both condition 1 and (10).*

Proof: Suppose that such a function y does exist. Clearly $D_k(x)$ satisfies the conditions for f in lemma 19. Taking $g(x) = 2(y(x) + b_k(x))$ in this lemma we get a contradiction. $\square$

If the number of zeroes of $D_k(x)$ with $|x| < 1$ is even, then we will construct a set U as follows: take the entire complex plane $\mathbb{C}$, and remove from it some so-called branch cuts. We can draw these branch cuts as follows: for each zero x_0 of odd multiplicity of $D_k(x)$ with $|x_0| < 1$ we draw the ray $\{\lambda x_0 : 0 \leq \lambda \leq 1\}$. For each zero x_0 of odd multiplicity of $D_k(x)$ with $|x_0| \geq 1$ we draw the ray $\{\lambda x_0 : \lambda \geq 1\}$. If it happens that there are no zeroes with $|x| < 1$ then we still remove the point 0. As the set of zeroes is invariant under complex conjugation, these branch cuts are invariant under conjugation as well.

Lemma 20 *If $D_k(x)$ has an even number of zeroes with $|x| < 1$ then the open set U constructed above satisfies $\overline{U} = U$ (of course we mean the complex conjugation here and not the toplogical closure) and we can take a square root of $D_k(x)$ on U as a holomorphic function. Furthermore, the function $\sqrt{D_k(x)}$ is defined over $\mathbb{R}$, which means that it satisfies $\sqrt{D_k(\overline{x})} = \overline{\sqrt{D_k(x)}}$ for all $x \in U$.*

Proof: The invariance of U under complex conjugation follows immediately from the construction of U. As the number of zeroes of $D_k(x)$ with $|x| < 1$ is finite we can find a number c with $0 < c < 1$ such that $|x| < c$ for every zero x of $D_k(x)$ of absolute value strictly less than 1. Because $0 \notin U$, the circle $C = \{z \in \mathbb{C} : |z| = c\}$ is a subset of U. We can now also see that this circle is a deformation retract of U, define the retraction as follows:

$$U \times [0,1] \to U : (z,t) \mapsto (1-t)z + tc\frac{z}{|z|}.$$

It follows now that the closed path $\gamma : [0,1] \to U : t \mapsto c\exp(2\pi i t)$ defines a generator of $H_1(U, \mathbb{Z})$. Because the number of zeroes of $D_k(x)$ with $|x| < c$ is even, the total winding number of γ around the zeroes of $D_k(x)$ is even. Hence the total winding number of every closed path in U around the zeroes of $D_k(x)$ is even. Hence we can define $\sqrt{D_k(x)}$ as a holomorphic function on U. There are two choices for $\sqrt{D_k(x)}$, fix one of them.

Now, consider the function

$$f : U \to \mathbb{C} : z \mapsto \frac{\sqrt{D_k(\overline{z})}}{\left(\overline{\sqrt{D_k(z)}}\right)}.$$

This function is well-defined because $\overline{U} = U$ and U does not contain any zeroes of $D_k(x)$. As $D_k(x)$ is defined over $\mathbb{R}$, it follows that $f(z)^2$ is identically 1 on U. So f takes the values 1 and -1. The set U is connected, because it is homotopy-equivalent to C. So f is constant as f is continuous, hence f is either identically 1 or identically -1.

To examine whether f is identically 1 or -1 we distinguish three cases: $k > 0$, $k < 0$ and $k = 0$. Let's start with the case $k > 0$. Let's try to prove that $D_k(x)$ has no positive real zeroes. From the well-known arithmetic-geometric mean inequality it follows that $x^2 + 1 \geq 2x$ for positive real x. Hence the factor $x^2 + (k-2)x + 1$ of (70) is strictly positive for $x > 0$. It's clear that the other two factors have no zeroes for $x > 0$ either. This implies, by construction of U, that the set of positive real numbers is a subset of U. As $D_k(x)$ has leading coefficient 1, we know that there is $x_0 > 0$ such that $D_k(x_0) > 0$. This implies that $\sqrt{D_k(x_0)}$ is real. Hence $\sqrt{D_k(\overline{x_0})} = \sqrt{D_k(x_0)} = \overline{\sqrt{D_k(x_0)}}$ or, equivalently, $f(x_0) = 1$. Hence f is identically 1 in this case, which implies the claim that $\sqrt{D_k(x)}$ is defined over $\mathbb{R}$.

Let's now check the case $k < 0$. There is a double zero $x = -1$ of $D_k(x)$ on the negative part of the real line. We'll show that this is the only zero on $\mathbb{R}_{<0}$. It's clear that the factor $x^2 + (k-2)x + 1$ is strictly positive for $x < 0$. From the arithmetic-geometric mean inequality it follows that $k(x^3 + x) = |k|(|x|^3 + |x|) \geq 2|k||x|^2 = -2kx^2$ for $x < 0$. Hence the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ is strictly positive for $x < 0$. So $x = -1$ is the only zero of $D_k(x)$ with $x < 0$. This is a zero of even multiplicity, so the negative real line is part of U. As $D_k(x)$ has even degree and leading coefficient equal to 1, there is an $x_0 < 0$ with $D_k(x_0) > 0$ and now for the same reason as above, $\sqrt{D_k(x)}$ is defined over $\mathbb{R}$.

In the case $k = 0$, the polynomial $D_k(x)$ is the square of the polynomial $x^4 - 1$. So $\sqrt{D_k(x)}$ must be $x^4 - 1$ or $-x^4 + 1$, which are both defined over $\mathbb{R}$. □

We will now examine what kind of implications this has on the functions $y(x)$ as solution of $P_k(x, y) = 0$. In the general case we get the following picture: there is a function $y(x)$ such that $|y(x)| = 1$ for x on a left part of the unit circle (left to the roots of $D_k(x)$) and $|y(x)| > 1$ for x on the right of these roots on the unit circle.

Let U' be the union of U with the set of zeroes of $D_k(x)$. At the zeroes of $D_k(x)$ we define $\sqrt{D_k(x)}$ to be 0, so that $\sqrt{D_k(x)}$ is continuous there. By the *abc*-formula we find two functions y that satisfy condition 1:

$$y(x) = \frac{-b_k(x) \pm \sqrt{D_k(x)}}{2a_k(x)} = \frac{1}{2}\left(-(x^4 + kx^3 + 2kx^2 + kx + 1) \pm \sqrt{D_k(x)}\right).$$

It is now immediate that $y(\overline{x}) = \overline{y(x)}$, so that $y(x)$ is defined over $\mathbb{R}$. We still don't know whether one of these two functions satisfies $|y(x)| \geq 1$ for $|x| = 1$, but we will examine that later. One can compute $\sqrt{D_k(x)}$ as follows: fix a point $x_0 \in U$, together with a choice of $\sqrt{D_k(x_0)}$. Choose a path γ from x_0 to x, lying in U. Then

$$\sqrt{D_k(x)} = \sqrt{D_k(x_0)} \cdot \exp\left(\frac{1}{2}\int_\gamma d\log D_k(x)\right) = \sqrt{D_k(x_0)} \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{D_k'(x)}{D_k(x)}dx\right).$$

So we should find out for which k, the discriminant $D_k(x)$ has an even number of zeroes within the disc $D(0,1)$. As $D_k(x)$ is a reciprocal polynomial, the mapping $\alpha \mapsto 1/\alpha$ is an involution on the multiset of zeroes. This implies that the number of zeroes inside $D(0,1)$ is equal to the number of zeroes outside $\overline{D}(0,1)$, so we only have to figure out how many zeroes there are on the unit circle. We will consider a number of cases: $k > 4$, $k = 4$, $0 < k < 4$, $k = 0$, $-1 < k < 0$, $k = -1$ and $k < -1$.

Let's start with the case $k > 4$. We will show that the double zero $x = -1$ are the only zeroes on the unit circle. As the degree of $D_k(x)$ is 8, we see that there are 3 zeroes in $D(0,1)$, so in this case we do not expect that a nice function y exists. Indeed, for these values of k, the polynomial $P_k(x)$ does not seem to satisfy Boyd's conjecture (see [4]). As the factorisation (70) shows, we must look for zeroes of $x^2 + (k-2)x + 1$ and for zeroes of $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$. The discriminant of $x^2 + (k-2)x + 1$ is strictly positive if $k > 4$ so this factor has two disctinct real zeroes, whose product is 1. Hence they cannot lie on the unit circle. Now we have to study the equation $x^4 + kx^3 + 2(k+1)x^2 + kx + 1 = 0$. After dividing both sides by x^2 and substituting $w = x + 1/x$ this equation goes over into the equation

$$w^2 + kw + 2k = 0. \tag{71}$$

It is easily verified that x is on the unit circle if and only if w is a real number with $|w| \leq 2$. So we must show that there are no real roots of (71) with $|w| \leq 2$. The graph of $f(w) = w^2 + kw + 2k$, as a function on the reals, is a convex parabola. As $k \geq 4$, the top of the parabola is at $w = -k/2 \leq -2$. Because $f(-2) = 4 > 0$, it follows that $f(w) > 0$ for any $w \geq -2$, so f has no zeroes on $[-2, 2]$ hence the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ has no zeroes on the unit circle, which is what we wanted to prove.

In the case $k = 4$, the zero $x = -1$ of $D_k(x)$ has multiplicity 4: another 2 of these zeroes are coming from the factor $x^2 + (k-2)x + 1$. For the same reason as in the case $k > 4$, the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ has no zeroes on the unit circle. So the number of zeroes in $D(0,1)$ is 2 in this case. Hence by lemma 20 there exists a function y satisfying condition 1 and (10).

Now we have to figure out whether there exists such a function y which satisfies $|y(x)| \geq 1$ for all $x \in \mathbb{C}$ with $|x| = 1$. After choosing branch cuts and hence an open set U, there are exactly two holomorphic choices for $\sqrt{D_k(x)}$ on U, hence two choices for $y(x)$ on U' where U' denotes the union of U and the zero set of $D_k(x)$. We can compute the two possible values for $y(1)$: they are roots of the equation $y^2 + 18y + 1 = 0$. The discriminant of this equation is strictly positive and the product of the two roots is 1. Hence this equation has exacly one real root y with $|y| > 1$. Let $y_+(x)$ be the choice of $y(x)$ with $y(1)$ equal to this root and let $y_-(x)$ be the other choice of $y(x)$. Set the choice of $\sqrt{D_k(x)}$ in such a way that $y_+(x) = (-b(x) + \sqrt{D_k(x)})/(2a(x))$ and $y_-(x) = (-b_k(x) - \sqrt{D_k(x)})/(2a_k(x))$.

We will show that $|y_+(x)| \geq 1$ for all $x \in \mathbb{C}$ with $|x| = 1$. Let's start with finding out for which x on the unit circle the absolute value of $y_+(x)$ is exactly equal to 1. It is clear that $y_+(x)y_-(x) = x^4$ for each $x \in U'$. Hence

$$|y_+(x)| \cdot |y_-(x)| = 1 \quad \text{for all } x \text{ with } |x| = 1. \tag{72}$$

So, for x on the unit circle, $|y_+(x)| = 1$ if and only if $|y_+(x)| = |y_-(x)|$ if and only if $|-b_k(x) + \sqrt{D_k(x)}| = |-b_k(x) - \sqrt{D_k(x)}|$ if and only if the complex numbers $b_k(x)$ and $\sqrt{D_k(x)}$ are perpendicular as vectors in the plane if and only if $D_k(x) = 0$ or $b_k(x)/\sqrt{D_k(x)} \in i\mathbb{R}$ if and only if

$$D_k(x) = 0 \quad \text{or} \quad (b_k(x))^2/D_k(x) \in \mathbb{R}_{\leq 0}.$$

It may seem like a coincidence that such a complex number is negative real but in fact the number $(b_k(x)^2)/D_k(x)$ is always real if $|x| = 1$. This can be easily verified by checking that $(b_k(x)^2)/D_k(x)$

is its own complex conjugate, using that $\overline{x} = 1/x$. Let $f(x) = (b_k(x)^2)/D_k(x)$ as a function from the unit circle to $\mathbb{R}$. We want to know at which points f changes sign if we walk on the unit circle. This can only happen at a zero or pole of odd multiplicity, hence at a zero of D_k of odd multiplicity. As we are in the case $k = 4$, we have seen above that D_k has only 1 zero on the unit circle, and it is of multiplicity 4. After substituting $x = 1$, one sees that, for each x on the unit circle, $D_k(x) = 0$ or $f(x) \geq 0$. So the only points on the unit circle where $|y_+(x)| = 1$ are the zeroes of D_k and the zeroes of b_k. Let's compute the zeroes of $b_k(x)$. After dividing the equation $b_k(x) = 0$ by x^2 and substituting $w = x + 1/x$, this equation goes over into

$$w^2 + kw + 2(k-1) = 0. \tag{73}$$

In this particular case $k = 4$ and one can see that the discriminant is negative, so it has no real roots for w hence $b_k(x)$ has no zeroes on the unit circle. Hence, for x on the unit circle,

$$|y_+(x)| = 1 \quad \text{if and only if} \quad x = -1.$$

Now from the fact that $|y_+(x)|$ is a continuous function and $|y_+(1)| > 1$, it follows that $|y_+(x)| \geq 1$ for all $x \in \mathbb{C}$ with $|x| = 1$.

Let's now go to the case $0 < k < 4$. In this case the discriminant of the factor $x^2 + (k-2)x + 1$ is strictly negative so the two zeroes are conjugate to each other. The product of the two zeroes of $x^2 + (k-2)x + 1$ is 1 so these zeroes lie on the unit circle. Let's check the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ now. Again, if x is a zero with $|x| = 1$ and $w = x + 1/x$, then w should be a real root of absolute value at most 2 of (71). However, the discriminant of $w^2 + kw + 2k$ equals $k(k-8)$ so this is negative as $0 < k < 4$. So (71) cannot have any real roots at all. So on the unit circle $D_k(x)$ has the double zero $x = -1$ and two zeroes coming from $x^2 + (k-2)x + 1$. So there are 4 zeroes on the unit circle, hence $D_k(x)$ has 2 zeroes in $D(0, 1)$. So also in this case there exists by lemma 20 a function $y(x)$ satisfying condition 1 and (10).

Again, we have to figure out whether there exists a function $y(x)$ with $|y(x)| \geq 1$ for all $|x| = 1$. We will do a similar investigation as in the previous case. For $y(1)$ we get the equation $y^2 + (4k+2)y + 1 = 0$. The discriminant of this equation is stricly positive and the product of the roots is 1. So there is a root with $|y(1)| > 1$ and the other root has $|y(1)| < 1$. Again, define $y_+(x)$ to be the y-function with $|y_+(1)| > 1$ and $y_-(x)$ to be the other one. As before we choose $\sqrt{D_k(x)}$ such that $y_+(x) = -b_k(x)/2 + \sqrt{D_k(x)}/2$. Again, $|y_+(x)| = 1$ for x on the unit circle if and only $D_k(x) = 0$ or the real number $b_k(x)^2/D_k(x)$ is negative or 0. At $x = 1$ the value of this function is $(4k+2)^2/(16k(k+1)) > 0$. If we walk around the unit circle, $b_k(x)^2/D_k(x)$ switches sign at the zeroes of $D_k(x)$ of odd multiplicity, hence at the two zeroes that come from the factor $x^2 + (k-2)x + 1$. So, walking around the unit circle from 1 to 1, the function $b_k(x)^2/D_k(x)$ between these 2 zeroes of $D_k(x)$. Let's now check whether $b_k(x)$ has zeroes on the unit circle. Suppose $b_k(x)$ has a zero x_0 on the unit circle. Again this zero corresponds to a real root w_0 of (73) of absolute value at most 2. If this equation has real roots, then the discriminant $k^2 - 8k + 8$ is nonnegative and the largest real root is equal to $(-k + \sqrt{k^2 - 8k + 8})/2$. The real parts of the zeroes of $D_k(x)$ on the unit circle are $-(k-2)/2$. From the fact that $\Re x_0 = w_0/2$ it follows that

$$\Re x_0 \leq \frac{-k + \sqrt{k^2 - 8k + 8}}{4} < \frac{-k + \sqrt{k^2 - 8k + 16}}{4} = -\frac{k-2}{2},$$

where in the last equality we use that $k < 4$ so $\sqrt{k^2 - 8k + 16} = 4 - k$. We see that if $b_k(x)$ has zeroes on the unit circle, then they must lie on the left of the zeroes of $D_k(x)$ on the unit circle that are not -1, so this does not give additional points where $|y_+(x)| = 1$.

So, for x on the unit circle, $|y_+(x)| = 1$ if and only if x lies on the left of the zeroes of $D_k(x)$ on the unit circle that are not -1. From the continuity and the fact that $|y_+(1)| > 1$ it follows now that $|y_+(x)| \geq 1$ for all x on the unit circle.

The case $k = 0$ is not very interesting: we can use lemma 2 and identity (4), together with lemma 1 to compute $m(P_k(x,y))$:

$$\begin{aligned} m(P_k(x,y)) &= m(y^2 + (x^4+1)y + x^4) = m(y+1) + m(y+x^4) = \\ &= m(y+x^4) = m(y+x) = m(y+xy) = m(y) + m(x+1) = 0. \end{aligned}$$

Let's examine the case $-1 \leq k < 0$ now. The discriminant of $x^2 + (k-2)x + 1$ is stricly positive in this case so this factor does not have any zeroes on the unit circle. Again the zeroes of $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ on the unit circle are in a 2-1 correspondence to the real roots of (71) of absolute value at most 2. Set $f(w) = w^2 + kw + 2k$. One can easily verify that $f(-2) > 0$, $f(0) < 0$ and $f(2) \leq 0$ so that (71) has two real roots in the interval $[-2,2]$. Hence $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ has 4 zeroes on the unit circle. Together with the double zero $x = -1$, there are 6 zeroes of $D_k(x)$ on the unit circle. Hence there is 1 zero inside the unit circle, so there is no function $y(x)$ that satisfies condition 1.

Let's next assume that $k < -1$. The discriminant of $x^2+(k-2)x+1$ is strictly positive so this factor does not have zeroes on the unit circle. Let's again figure out how many zeroes the polynomial $f(w) = w^2 + kw + 2k$ has on the interval $[-2,2]$. As is easily seen, $f(-2) > 0$ and $f(0) < 0$ so $f(w)$ has a zero between -2 and 0. We can also see that $f(2) < 0$ so the other zero of $f(w)$ must be stricly greater than 2. Hence the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ of $D_k(x)$ has exactly 2 zeroes on the unit circle. So $D_k(x)$ has 4 zeroes on the unit circle, 2 zeroes with $|x| < 1$ and 2 zeroes with $|x| > 1$. So in this case there exists a nice function $y(x)$ that satisfies condition 1 and (10).

Again for $y(1)$ the equation $y^2 + (4k+2) + 1 = 0$ holds, which has a strictly positive discriminant. Hence there is a root with $|y(1)| > 1$ and one with $|y(1)| < 1$. Let, as in the previous cases, $y_+(x)$ be the choice of $y(x)$ with $y(1) > 1$ and $y_-(x)$ be the choice of $y(x)$ that has $y(1) < 1$. Also choose $\sqrt{D_k(x)}$ such that $y_+(x) = -b_k(x)/2 + \sqrt{D_k(x)}/2$. As above, $|y_+(x)| = 1$ for x on the unit circle if and only if $D_k(x) = 0$ or the real number $b_k(x)^2/D_k(x)$ is negative. The same arguments as before show that this function is negative or 0 for x on the unit circle where $\Re x$ is smaller than the real part of the roots of $D_k(x)$ on the unit circle, that are not equal to -1. Again we have to find out where $b_k(x)$ can vanish on the unit circle. Suppose that $b_k(x)$ has a zero x_0 on the unit circle. Then this zero corresponds to a root w_0 of (73) in $[-2,2]$ and $\Re x_0 = w_0/2$. A zero x_1 of $D_k(x)$ on the unit circle, not equal to -1, comes from the factor $x^4 + kx^3 + 2(k+1)x^2 + kx + 1$ hence corresponds to a root w_1 of (71) in $[-2,2]$ and $\Re x_1 = w_1/2$. The roots of (73) are $(-k + \sqrt{k^2 - 8k + 8})/2$ and $(-k - \sqrt{k^2 - 8k + 8})/2$. From the fact that $(-k + \sqrt{k^2 - 8k + 8})/2 > 2$ it follows that

$$w_0 = \frac{-k - \sqrt{k^2 - 8k + 8}}{2}.$$

Similarly,

$$w_1 = \frac{-k - \sqrt{k^2 - 8k}}{2}$$

and we see that $w_0 < w_1$ hence $\Re x_0 < \Re x_1$. So $b_k(x)$ does not give us additional points on the unit circle where $|y_+(x)| = 1$.

So for x on the unit circle, $|y(x)| = 1$ if and only if x lies on the left of the zeroes of $D_k(x)$ on the unit circle that are not -1 (we say that a complex number z lies on the left of a complex number w if $\Re z < \Re w$). As before we see that $|y_+(x)| \geq 1$ for all x on the unit circle.

Let's summarize all this in the following proposition:

Proposition 5 *For $k \in]-\infty,-1[\cup]0,4]$ there exists an open set $U \subset \mathbb{C}$ and a function holomorphic function $y_+(x)$ on U that satisfy the following conditions:*

- $\overline{U} = U$,
- $P_k(x, y(x)) = 0$ *for all* $x \in U$,
- *condition 1 from section 2.1,*
- $y_+(\overline{x}) = \overline{y_+(x)}$ *for all* $x \in U$,
- $|y_+(x)| \geq 1$ *for all* $x \in U$ *of absolute value 1.*

The function $y_-(x)$ satisfies the first 3 conditions from proposition 5 as well. And because of (72), which holds for any $k \in]-\infty,-1[\cup]0,4]$, the function $y_-(x)$ satisfies

$$|y_-(x)| \leq 1 \quad \text{for all } |x| = 1.$$

For any $x_0 \in U'$, the zeroes of the polynomial $P(x_0, y)$ are $y_+(x_0)$ and $y_-(x_0)$. Let γ_k be the path on $C'_k(\mathbb{C})$ defined by equation (9) from section 2.1. Then equation (11) from the same section shows that

$$m(P_k(x)) = -\frac{1}{2\pi i}\int_\gamma \eta(x,y),$$

where $\eta(x, y)$ is defined by formula (34) from section 6.

Let's figure out now whether γ_k lifts to a closed path on a nonsingular model $\widetilde{C'_k}$ of C'_k. To verify this, only the singular points on $C'_k(\mathbb{C})$ with $|x| = 1$ are interesting to look at. A standard calculation shows that the following points (x, y) with $|x| = 1$ are singular:

$$\begin{aligned} (x,y) &= (-1,-1) && \text{for all } k,\\ (x,y) &= (1,1) && \text{if } k=-1,\\ (x,y) = (-\pm i,-1) \text{ and } (x,y) &= (1,-1) && \text{if } k=0. \end{aligned}$$

Luckily, we have excluded the cases $k = -1$ and $k = 0$ above, so that we only have to concentrate on the singularity $(-1, -1)$. To study the lifting of γ_k to $\widetilde{C'_k}$ at this point, we only need to look locally at the blow-up of C'_k at $(-1, -1)$. We can do this at follows: define

$$R_k(x,y) = y^2 + (x^3 + (k-4)x^2 + (-k+6)x - 4)y - kx + k.$$

Then the following map is locally the blow-up of C'_k at $(-1, -1)$:

$$\rho_1 : Z(R_k) \to C'_k : (x,y) \mapsto (x-1, xy-1),$$

whose rational inverse is

$$\rho_1^{-1} : C'_k - \{(-1,-1)\} \to Z(R_k) : (x,y) \mapsto \left(x+1, \frac{y+1}{x+1}\right). \tag{74}$$

The geometric points in $Z(R_k)$ above $(-1, -1)$ are $(0, 2 \pm \sqrt{4-k})$, these points are defined over $\mathbb{Q}(\sqrt{4-k})$. If $\mathfrak{m}$ is the maximal ideal (x, y) in $\mathbb{Q}(\sqrt{4-k})[x, y]$, then

$$R_k(x, y+2\pm\sqrt{4-k}) \equiv \left(3(4-k) \pm (6-k)\sqrt{4-k}\right)x + 2\sqrt{4-k}\,y \bmod \mathfrak{m}^2.$$

This shows that the points $(0, 2 \pm \sqrt{4-k})$ are nonsingular points of $Z(R_k)$ when $k \neq 4$. If $k = 4$ then we can further blow up $Z(R_k)$ at the point $(0, 2)$. We get the polynomial

$$Q_4(x,y) = y^2 + (x^2+2)y + 2x,$$

and a map

$$\rho_2 : Z(Q_4) \to Z(R_4) : (x, y) \mapsto (x, xy + 2),$$

with rational inverse

$$\rho_2^{-1} : Z(R_4) - \{(0,2)\} \to Z(Q_4) : (x, y) \mapsto \left(x, \frac{y-2}{x}\right).$$

The points above $(0, 2) \in Z(R_4)$ are $(0, 0)$ and $(0, -2)$, who are clearly nonsingular. We get a composed map

$$\rho_1 \circ \rho_2 : Z(Q_4) \to C_4' : (x, y) \mapsto (x - 1, x^2 y + 2x - 1),$$

with rational inverse equal to

$$(\rho_1 \circ \rho_2)^{-1} : C_4' - \{(-1,-1)\} \to Z(Q_4) : (x, y) \mapsto \left(\frac{y - 2x - 1}{(x+1)^2}\right). \tag{75}$$

To study the lifting of the path γ_k to the blow-up, we will examine the case $k \neq 4$ first. The function $\sqrt{D_k(x)}$ constructed above is holomorphic at $x = -1$. If we compute its Taylor expansion at this point, then we see that $\sqrt{D_k(x)} = 2\sqrt{4-k}(x+1) + \mathcal{O}((x+1)^2)$ for an appropriate choice of $\sqrt{4-k}$. Also, $b_k(x) = 2 - 4(x+1) + \mathcal{O}((x+1)^2)$. From this it follows that

$$\frac{y_+(x) + 1}{x+1} = 2 + \sqrt{4-k} + \mathcal{O}(x+1).$$

Comparing this with (74), we see that the blow-up lifts γ_k at the point $(-1,-1) \in C_k'(\mathbb{C})$ to the point $(0, 2 + \sqrt{4-k}) \in Z(R_k)(\mathbb{C})$, from which it follows that the lifting of γ_k to $\widetilde{C_k'}$ is still a closed path.

It remains to study the case $k = 4$. Again, we compute the Taylor expansion of $\sqrt{D_4(x)}$ at $x = -1$, but now with an extra term. It turns out to be $\sqrt{D_4(x)} = 2\epsilon(x+1)^2 + \mathcal{O}(x+1)^3$, for an appropriate choice of $\epsilon \in \{1, -1\}$. And $b_4(x) = 2 - 4(x+1) + 2(x+1)^2 + \mathcal{O}((x+1)^3)$. It follows that

$$\frac{y_+(x) - 2x - 1}{(x+1)^2} = \epsilon - 1 + \mathcal{O}(x+1).$$

As above, comparing this with (75), we can conclude that the lifting of γ_4 to $\widetilde{C_4'}$ is still closed.

10.2 Degenerate cases

In our family of curves, there are some degenerate cases, as lemma 17 shows. It turns out that these cases are easier to study than the rest. We will be able to express the Mahler measure in terms of the L-series of an odd Dirichlet character, defined by (2). Generally for any nontrivial Dirichlet character we can say the following: let m be the conductor of χ (i.e. the smallest positive integer such that χ is periodic with period m). Let p be the parity of χ, which is defined as 0 if $\chi(-1) = 1$ and 1 if $\chi(-1) = -1$. Now define the following function:

$$\Lambda(\chi, s) := \left(\frac{m}{\pi}\right)^{s/2} \Gamma\left(\frac{s+p}{2}\right) L(\chi, s).$$

It satisfies the equation

$$\Lambda(\chi, s) = \frac{\tau(\chi)}{i^p \sqrt{m}} \Lambda(\overline{\chi}, 1 - s),$$

where $\tau(\chi)$ is the famous Gauss sum:

$$\tau(\chi) = \sum_{k=0}^{m-1} \chi(k) e^{\frac{2\pi i k}{m}}.$$

For a proof of this, see chapter 7 of [16]. This functional equation immediately implies the equation (3) if $\chi = \chi_{-m}$.

10.2.1 The case $k = 8$

Let's study the case $k = 8$. Above it is shown that y cannot be nicely expressed in terms of x. However, in this case this will turn out not to be a big problem. We can factor $D_8(x)$ as follows:

$$D_8(x) = (x+1)^2(x^2+4x+1)^2(x^2+6x+1).$$

So in order to define a useful square root of $D_8(x)$ we will have to define a useful square root of x^2+6x+1. After that we can put $\sqrt{D_8(x)} = (x+1)(x^2+4x+1)\sqrt{x^2+6x+1}$. We draw a branch cut on the negative part of the real axis. Then on $\mathbb{C}-\mathbb{R}_{<0}$ we can define a square root of x^2+6x+1 and then we can use the *abc*-formula to express y in terms of x. We choose $\sqrt{x^2+6x+1}$ to be that squareroot of x^2+6x+1 such that $|y(1)| > 1$. We see that $\sqrt{x^2+6x+1} = 2\sqrt{2}$ for $x = 1$. The corresponding y-function will be denoted y_+ again. The only zero of $D(x)$ on the unit circle is the double zero $x = -1$. So again a similar argument as given in subsection 10.1 shows that $|y_+(x)| > 1$ for x with $|x| = 1$ and $x \neq -1$.

The resulting function y_+ is continuous now, so the path of integration seems to be closed. However, if we blow up the singularity $(x, y) = (-1, -1)$, the blown up path will not be closed anymore. This has to do with the fact that $\sqrt{x^2+6x+1}$ is not continuous at $x = -1$. Let's try to figure out the behaviour of $\sqrt{x^2+6x+1}$ above and below $x = -1$. We will first show that if $\sqrt{x^2+6x+1} \in \mathbb{R}$, then $x \in \mathbb{R}$. So assume that $\sqrt{x^2+6x+1} \in \mathbb{R}$. Then $(x+3)^2 - 8 = x^2+6x+1 \in \mathbb{R}_{\geq 0}$. This implies that $(x+3)^2 \in \mathbb{R}_{\geq 0}$. This is only possible if $x \in \mathbb{R}$. The Taylor expansion of $\sqrt{x^2+6x+1}$ at $x = 1$ starts with $2\sqrt{2} + \sqrt{2}(x-1) + \mathcal{O}((x-1)^2)$. So $\Im\sqrt{x^2+6x+1} > 0$ for x with $\Im x > 0$ in a sufficiently small neighbourhood of 1. As $\Im\sqrt{x^2+6x+1} \neq 0$ for $\Im x > 0$ and $\{x \in \mathbb{C} : \Im x > 0\}$ is a connected set, we see that $\Im\sqrt{x^2+6x+1} > 0$ if $\Im x > 0$. Similarly $\Im\sqrt{x^2+6x+1} < 0$ if $\Im x < 0$. It follows now that

$$\lim_{\substack{x \to -1 \\ \Im x > 0}} \sqrt{x^2+6x+1} = 2i \quad \text{and} \quad \lim_{\substack{x \to -1 \\ \Im x < 0}} \sqrt{x^2+6x+1} = -2i \tag{76}$$

because the limits clearely exist and take values in $\{-2i, 2i\}$.

Let's consider the path

$$\gamma :]0,1[\to C_8' : t \mapsto \left(-\exp(2\pi i t), y_+(-\exp(2\pi i t))\right).$$

Blowing up the singularity $(-1, -1)$ will break up this path (that's why we consider it as a function of the open interval $]0,1[$ instead of the closed interval $[0,1]$). From (11) it follows that the following formula holds:

$$m(P_8) = -\frac{1}{2\pi}\int_\gamma \eta(x, y).$$

Now we want to know what the path γ looks like after a transformation by the birational map ψ in (68). An easy calculation using (76) shows that

$$\lim_{t \mapsto 0} \psi(\gamma(t)) = (0, 2i) \quad \text{and} \quad \lim_{t \mapsto 1} \psi(\gamma(t)) = (0, -2i).$$

So $\psi \circ \gamma$ is a path on C_k from $(0, 2i)$ to $(0, -2i)$. In view of (69), we see that

$$m(P_8) = -\frac{1}{2\pi}\int_{\psi\circ\gamma} \eta\left(\frac{x+1}{x-1}, \frac{2xy - 17x^4 + 10x^2 - 1}{(x-1)^4}\right).$$

As C_8 is a curve of genus 0, it is birationally equivalent to $\mathbb{P}^1$. We identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. We take the following birational map:

$$\phi : C_k \to \mathbb{P}^1 : (x, y) \mapsto \frac{iy + (6-6i)x^3 - 6x^2 - (2-2i)x + 2}{iy - (6+6i)x^3 + 6x^2 + (2+2i) - 2}$$

with inverse equal to

$$\phi^{-1} : \mathbb{P}^1 \to C_k : z \mapsto \left(\frac{(1-i)z}{z^2-i}, \frac{2(-iz^6+5z^4+5iz^2-1)}{(z^2-i)^3} \right).$$

We have chosen this birational map in such a way that the path $\psi \circ \gamma$ goes over into a path from 0 to ∞, as one can easily verify. One can also easily verify that

$$(\phi^{-1})^* \left(\frac{x+1}{x-1} \right) = -\frac{(z+1)(z-i)}{(z-1)(z+i)}$$

and

$$(\phi^{-1})^* \left(\frac{2xy - 17x^4 + 10x^2 - 1}{(x-1)^4} \right) = -\frac{(z+1)^4}{(z-1)^4}.$$

We have already seen in lemma 18 that the tame symbols have their absolute value equal to 1 everywhere. We can apply theorem 4 and formulas (21) and (22) now to express $m(P_8)$ in terms of dilogarithms:

$$\begin{aligned} m(P_8) &= -\frac{1}{2\pi} \int_0^\infty \eta \left(-\frac{(z+1)(z-i)}{(z-1)(z+i)}, -\frac{(z+1)^4}{(z-1)^4} \right) \\ &= -\frac{2}{\pi} \left(D(1) + D(-i) - D(-1) - D(i) - D(-1) - D(i) + D(1) + D(-i) \right) = \frac{8}{\pi} D(i). \end{aligned}$$

Define

$$\chi_{-4}(n) := \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4; \\ -1 & \text{if } n \equiv -1 \bmod 4; \\ 0 & \text{if } 2 \mid n. \end{cases}$$

We use formula (23) and (3) to express $D(i)$ in terms of $L'(\chi_{-4}, -1)$:

$$D(i) = D\left(e^{\pi i/2}\right) = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^2} = L(\chi_{-4}, 2) = \frac{\pi}{2} L'(\chi_{-4}, -1).$$

Hence we have proven the following proposition, which was conjectured in [4]:

Proposition 6 *The following identity holds:*

$$m(P_8) = 4L'(\chi_{-4}, -1).$$

10.2.2 The case $k = -1$

We will study the case $k = -1$ now. This case is more difficult than the previous case, as we will see in a moment. Again, we have to cut the unit circle to define an appropriate function y in terms of x. Observe that in this case we have

$$D_{-1}(x) = (x-1)^2(x+1)^2(x^2-3x+1)(x^2+x+1).$$

The zeroes of $x^2 - 3x + 1$ are positive and real. Their product is 1 and they're both distinct so one of the zeroes lies in $]0, 1[$ and the other one in $]1, \infty[$. We draw a branch cut now on the positive real axis, so that the unit circle is cut at the point $x = 1$. We also draw straight line segments from 0 to ζ_3 and from 0 to $\overline{\zeta_3}$ as branch cuts. On the remaining part of $\mathbb{C}$ we can define a square root of $(x^2-3x+1)(x^2+x+1)$. The value of $(x^2-3x+1)(x^2+x+1)$ at $x = 1$ is -3. We choose $\sqrt{(x^2-3x+1)(x^2+x+1)}$ in such a way that

$$\lim_{\substack{x \to 1 \\ \Im x > 0}} \sqrt{(x^2-3x+1)(x^2+x+1)} = -\sqrt{3}i.$$

As $(x^2-3x+1)(x^2+x+1)$ has exactly 1 zero inside the unit circle, which is an odd number, it follows that

$$\lim_{\substack{x\to 1\\ \Im x<0}} \sqrt{(x^2-3x+1)(x^2+x+1)} = \sqrt{3}i.$$

And we choose $y_+(x)$ accordingly. Around $x=1$, one can now easily verify that, for $\Im x>0$, the following formula holds:

$$y_+(\exp(it)) = 1+\sqrt{3}t+2it+\mathcal{O}(t^2),$$

so that $|y_+(\exp(it))|>1$ for $t>0$ sufficiently small. Similarly one can see that $|y_+(\exp(it))|>1$ for $t<0$ sufficiently small. Again from the same techniques as in subsection 10.1 it follows now immediately that $|y_+(x)|>1$ if $\Re x > -\frac{1}{2}$ and $|y_+(x)|=1$ if $\Re x \le -\frac{1}{2}$ for x on the unit circle.

Let γ be the path

$$\gamma :]0,1[\to C'_{-1} : t \mapsto (\exp(2\pi i t), y_+(\exp(2\pi i t))).$$

Again, the function $y_+(x)$ is continuous on the unit circle but the path γ will break up if we blow up C'_{-1} in the singular point $(1,1)$. In this case, blowing up in $(-1,-1)$ will not break up γ as was already shown at the end of subsection 10.1.

Let's now look at C_{-1}:

$$C_{-1} : y^2 = f_{-1}(x^2), \quad \text{where } f_{-1}(x) = -(x-5)(3x+1).$$

So the degree of f_{-1} is equal to 2 instead of 3. The polynomial $f_{-1}(x^2)$ has no multiple roots so the results of section 8 show that C_{-1} is a curve of genus 1. One can see that the points of C_{-1} above the point $(1,1)$ of C'_{-1} are at infinity so we have to look at the other affine piece (see lemma 14):

$$C_{-1,1} : w^2 = v^4 f_{-1}(1/v^2) = (5v^2-1)(v^2+3).$$

Composing the birational map ψ_{-1} with the glueing morphism $\theta : C_{-1} \to C_{-1,1} : (x,y) \mapsto (1/x, y/x^2)$ gives

$$\theta\circ\psi_{-1} : (x,y) \mapsto \left(\frac{x-1}{x+1}, \frac{4(y^2-x^4)}{y(x+1)^3(x-1)}\right).$$

Using that $y_+(x) = 1+(2-\sqrt{3}i)(x-1)+\mathcal{O}((x-1)^2)$ if $\Im(x)>0$ one can easily see that

$$\lim_{t\to 0} \theta\circ\psi_{-1}\circ\gamma(t) = (0,-\sqrt{3}i)$$

and similarly we can use $y_+(x) = 1+(2+\sqrt{3}i)(x-1)+\mathcal{O}((x-1)^2)$ to see that

$$\lim_{t\to 1} \theta\circ\psi_{-1}\circ\gamma(t) = (0,\sqrt{3}i)$$

Let $\widetilde{C_{-1}}$ be a nonsingular complete model of C_{-1}. Let P_1 be the point that corresponds with $(0,-\sqrt{3}i)$ on $C_{-1,1}$ and let P_2 be the point that corresponds with $(0,\sqrt{3}i)$ on $C_{-1,1}$. Then $\psi\circ\gamma$ is a closed, piecewise smooth path on $\widetilde{C_{-1}}$ from P_1 to P_2.

Again, we know now that

$$m(P_{-1}) = -\frac{1}{2\pi}\int_{\psi\circ\gamma} \eta(f,g),$$

where

$$f(x,y) = \frac{x+1}{x-1} \quad \text{and} \quad g(x,y) = \frac{2xy+x^4-8x^2-1}{(x-1)^4}.$$

Above we already saw that $\eta(\sigma_1^*(f),\sigma_1^*(g)) = \eta(f,g)$ so

$$m(P_{-1}) = -\frac{1}{4\pi}\int_{\psi\circ\gamma} \left(\eta(f,g)+\eta(\sigma_1^*(f),\sigma_1^*(g))\right). \tag{77}$$

As already mentioned in section 6, we can view η as a function from $\bigwedge^2 \mathbb{Q}(C_{-1})^* \bigotimes \mathbb{Q}$ to $M(\widetilde{C_{-1}})$.

So we want to do some manipulations in $\bigwedge^2 \mathbb{Q}(C_{-1})^* \bigotimes \mathbb{Q}$ now. In view of (36), we must try to rewrite $f \wedge g$ in terms of elements of the form $h \wedge (1-h)$. Set

$$a(x,y) = \frac{(x^2+6x+1)y + 2x^3 - 14x^2 - 18x - 2}{(x-1)((x+1)y + x^2 - 8x - 1)} \quad \text{and} \quad b(x,y) = \frac{-(x-1)^2}{(x+1)y + x^2 - 8x - 1},$$

then

$$a(x^2,y)f(x,y) + b(x^2,y)g(x,y) = 1.$$

And because

$$\begin{aligned} a(x^2,y)f(x,y) \wedge b(x^2,y)g(x,y) = & \left(a(x^2,y) \wedge b(x^2,y)\right) + \left(a(x^2,y) \wedge g(x,y)\right) \\ & + \left(f(x,y) \wedge b(x^2,y)\right) + \left(f(x,y) \wedge g(x,y)\right), \end{aligned}$$

we see that

$$\begin{aligned} f(x,y) \wedge g(x,y) = & \left(a(x^2,y)f(x,y) \wedge (1 - a(x^2,y)f(x,y))\right) - \left(a(x^2,y) \wedge b(x^2,y)\right) \\ & - \left(a(x^2,y) \wedge g(x,y)\right) - \left(f(x,y) \wedge b(x^2,y)\right). \end{aligned} \tag{78}$$

If we apply σ_1^* to (78), then we see that

$$\begin{aligned} \sigma_1^* f(x,y) \wedge \sigma_1^* g(x,y) = & \left(a(x^2,y)\sigma_1^* f(x,y) \wedge (1 - a(x^2,y)\sigma_1^* f(x,y))\right) - \left(a(x^2,y) \wedge b(x^2,y)\right) \\ & - \left(a(x^2,y) \wedge \sigma_1^* g(x,y)\right) - \left(\sigma_1^* f(x,y) \wedge b(x^2,y)\right). \end{aligned}$$

From the fact that $\sigma_1^*(f) = 1/f$ it follows that $(f(x,y) \wedge b(x^2,y)) + (\sigma_1^* f(x,y) \wedge b(x^2,y)) = 0$. Similarly, $(a(x^2,y) \wedge g(x,y)) + (a(x^2,y) \wedge \sigma_1^* g(x,y)) = 0$ so that

$$\begin{aligned} (f(x,y) \wedge g(x,y)) + (\sigma_1^* f(x,y) \wedge \sigma_1^* g(x,y)) = & \left(a(x^2,y)f(x,y) \wedge (1 - a(x^2,y)f(x,y))\right) \\ & + \left(a(x^2,y)\sigma_1^* f(x,y) \wedge (1 - a(x^2,y)\sigma_1^* f(x,y))\right) \\ & - 2\left(a(x^2,y) \wedge b(x^2,y)\right). \end{aligned} \tag{79}$$

Now,

$$a(x^2,y)f(x,y)|_{P_1} = \frac{5+\sqrt{3}i}{4} \quad \text{and} \quad a(x^2,y)f(x,y)|_{P_2} = \frac{5-\sqrt{3}i}{4}.$$

So from (36) it follows that

$$\begin{aligned} \int_{\psi\circ\gamma} \eta\left(a(x^2,y)f(x,y) \wedge (1 - a(x^2,y)f(x,y))\right) &= D\left(\frac{5-\sqrt{3}i}{4}\right) - D\left(\frac{5+\sqrt{3}i}{4}\right) \\ &= 2D\left(\frac{5-\sqrt{3}i}{4}\right), \end{aligned}$$

where in the last identity we make use of $D(\overline{z}) = -D(z)$. Also,

$$a(x^2,y)\sigma_1^* f(x,y)\big|_{P_1} = \frac{5+\sqrt{3}i}{4} \quad \text{and} \quad a(x^2,y)\sigma_1^* f(x,y)\big|_{P_2} = \frac{5-\sqrt{3}i}{4},$$

hence

$$\int_{\psi\circ\gamma} \eta\left(a(x^2,y)\sigma_1^* f(x,y) \wedge (1 - a(x^2,y)\sigma_1^* f(x,y))\right) = 2D\left(\frac{5-\sqrt{3}i}{4}\right). \tag{80}$$

So now we have to focus on $a(x^2, y) \wedge b(x^2, y)$. The functions $a(x^2, y)$ and $b(x^2, y)$ are invariant under the action of σ_1^*. So we can push them forward along ϕ_1 to the functions $a(x, y)$ and $b(x, y)$ on E_1, which is a rational curve in this case. We have that

$$\int_{\psi\circ\gamma} \eta(a(x^2, y) \wedge b(x^2, y)) = \int_{\phi_1\circ\psi\circ\gamma} \eta(a(x, y) \wedge b(x, y)). \tag{81}$$

And we can also see that

$$\phi_1(P_1) = [1 : -\sqrt{3}i : 0] \quad \text{and} \quad \phi_1(P_2) = [1 : \sqrt{3}i : 0].$$

We consider the following isomorphism between E_1 and $\mathbb{P}^1$:

$$\rho : E_1 \to \mathbb{P}^1 : (x, y) \mapsto \frac{y}{x-5},$$

with inverse

$$\rho^{-1} : \mathbb{P}^1 \to E_1 : z \mapsto \left(\frac{5z^2-1}{z^2+3}, \frac{-16z}{z^2+3}\right).$$

Then

$$\rho \circ \phi_1(P_1) = \rho\left([1 : -\sqrt{3}i : 0]\right) = -\sqrt{3}i \quad \text{and} \quad \rho \circ \phi_1(P_2) = \rho\left([1 : \sqrt{3}i : 0]\right) = \sqrt{3}i.$$

Now set

$$A(z) := a(\rho^{-1}(z)) = \frac{(z+1)^2(3z-1)}{(z-1)(z^2+4z-1)} \quad \text{and} \quad B(z) := b(\rho^{-1}(z)) = \frac{(z-1)^2}{z^2+4z-1}$$

Then it comes down to simplifying $A(z) \wedge B(z)$. The following way of rewriting is immediate:

$$\begin{aligned} A(z) \wedge B(z) &= 4\left((z+1) \wedge (z-1)\right) - 2\left((z+1) \wedge (z^2+4z-1)\right) + 2\left((3z-1) \wedge (z-1)\right) \\ &\quad - \left((3z-1) \wedge (z^2+4z-1)\right) + \left((z-1) \wedge (z^2+4z-1)\right) \\ &\quad - 2\left((z^2+4z-1) \wedge (z-1)\right) \\ &= 4\left((z+1) \wedge (z-1)\right) + 2\left((3z-1) \wedge (z-1)\right) \\ &\quad + \left((z^2+4z-1) \wedge \frac{(3z-1)(z+1)^2}{(z-1)^3}\right). \end{aligned} \tag{82}$$

Let's start with rewriting the most interesting term: $(z^2+4z-1) \wedge \frac{(3z-1)(z+1)^2}{(z-1)^3}$. We do not want to factor z^2+4z-1 here because that will give us very nasty dilogarithm terms in the end. Instead we can use some identities to rewrite it:

$$(z^2+4z-1) - (z-1)^2 = 2(3z-1).$$

From this it follows that

$$\frac{1}{2(3z-1)}(z^2+4z-1) + \frac{-1}{2(3z-1)}(z-1)^2 = 1.$$

As also

$$\begin{aligned} \frac{1}{2(3z-1)}(z^2+4z-1) \wedge \frac{-1}{2(3z-1)}(z-1)^2 = &-2\left(2(3z-1) \wedge (z-1)\right) \\ &+ \left((z^2+4z-1) \wedge \frac{(z-1)^2}{3z-1}\right) - \left((z^2+4z-1) \wedge 2\right), \end{aligned}$$

we see that

$$\begin{aligned} (z^2+4z-1) \wedge \frac{(z-1)^2}{3z-1} = &\left(\frac{z^2+4z-1}{2(3z-1)} \wedge \left(1 - \frac{z^2+4z-1}{2(3z-1)}\right)\right) + 2\left(2(3z-1) \wedge (z-1)\right) \\ &+ \left((z^2+4z-1) \wedge 2\right). \end{aligned}$$

And from the identity

$$(z^2+4z-1)-(z+1)^2 = 2(z-1)$$

it follows in a similar way that

$$\begin{aligned}(z^2+4z-1)\wedge\frac{(z+1)^2}{z-1} &= \left(\frac{z^2+4z-1}{2(z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(z-1)}\right)\right)+2\left(2(z-1)\wedge(z+1)\right)\\ &\quad+\left((z^2+4z-1)\wedge 2\right).\end{aligned}$$

Putting this together yields

$$\begin{aligned}(z^2+4z-1)\wedge\frac{(3z-1)(z+1)^2}{(z-1)^3} &= \left(\frac{z^2+4z-1}{2(z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(z-1)}\right)\right)\\ &\quad-\left(\frac{z^2+4z-1}{2(3z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(3z-1)}\right)\right)\\ &\quad+2\left(2(z-1)\wedge(z+1)\right)-2\left(2(3z-1)\wedge(z-1)\right).\end{aligned}$$

Using this, we can rewrite (82) to

$$\begin{aligned}A(z)\wedge B(z) &= \left(\frac{z^2+4z-1}{2(z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(z-1)}\right)\right)-\left(\frac{z^2+4z-1}{2(3z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(3z-1)}\right)\right)\\ &\quad+2\left((z+1)\wedge(z-1)\right)-2\left(2\wedge(z-1)\right)+2\left(2\wedge(z+1)\right)\end{aligned}$$

From $(z+1)/2-(z-1)/2=1$ it follows, after some rewriting, that

$$(z+1)\wedge(z-1) = \left(\frac{z+1}{2}\wedge\left(1-\frac{z+1}{2}\right)\right)+\left(2\wedge(z-1)\right)-\left(2\wedge(z+1)\right),$$

which immediately leads to

$$\begin{aligned}A(z)\wedge B(z) &= \left(\frac{z^2+4z-1}{2(z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(z-1)}\right)\right)-\left(\frac{z^2+4z-1}{2(3z-1)}\wedge\left(1-\frac{z^2+4z-1}{2(3z-1)}\right)\right)\\ &\quad+2\left(\frac{z+1}{2}\wedge\left(1-\frac{z+1}{2}\right)\right).\end{aligned}$$

The path $\gamma' := \rho\circ\phi_1\circ\psi\circ\gamma$ that we have to integrate $\eta(A(z)\wedge B(z))$ along goes from $-\sqrt{3}i$ to $\sqrt{3}i$. If a function h is defined over the reals, then $D(h(\sqrt{3}i)) - D(h(-\sqrt{3}i)) = 2D(h(\sqrt{3}i))$ because $D(\overline{z}) = -D(z)$ for all $z\in\mathbb{C}$. Using this and (36), we see that

$$\begin{aligned}\int_{\phi_1\circ\psi\circ\gamma}\eta\left(a(x,y)\wedge b(x,y)\right) &= \int_{\gamma'}\eta\left(A(z)\wedge B(z)\right)\\ &= 2D\left(\left.\frac{z^2+4z-1}{2(z-1)}\right|_{\sqrt{3}i}\right)-2D\left(\left.\frac{z^2+4z-1}{2(3z-1)}\right|_{\sqrt{3}i}\right)+4D\left(\left.\frac{z+1}{2}\right|_{\sqrt{3}i}\right)\\ &= 2D(2)-2D\left(\frac{5+\sqrt{3}i}{7}\right)+4D(\zeta_6)\end{aligned} \tag{83}$$

We can now express $m(P_{-1})$ in terms of the dilogarithm: from (77), (79), (80), (81) and (83) it follows that

$$m(P_{-1}) = -\frac{1}{\pi}\left(D\left(\frac{5-\sqrt{3}i}{4}\right)-D(2)+D\left(\frac{5+\sqrt{3}i}{7}\right)-2D(\zeta_6)\right).$$

Now, $D(2) = 0$ because $D(x) = 0$ for all $x \in \mathbb{R}$. Also, observe that $(\frac{5-\sqrt{3}i}{4})(\frac{5+\sqrt{3}i}{7}) = 1$ and because $D(1/z) = -D(z)$ for all z it follows that $D(\frac{5-\sqrt{3}i}{4}) + D(\frac{5+\sqrt{3}i}{7}) = 0$. From (24) and the fact that $D(\overline{z}) = -D(z)$ for all z it follows that $D(\zeta_3)/2 = D(\zeta_6) + D(-\zeta_6) = D(\zeta_6) - D(\zeta_3)$ so $2D(\zeta_6) = 3D(\zeta_3)$. Altogether, we see that

$$m(P_{-1}) = \frac{3}{\pi} D(\zeta_3).$$

Now define

$$\chi_{-3}(n) := \begin{cases} 0 & \text{if } n \equiv 0 \bmod 3 \\ 1 & \text{if } n \equiv 1 \bmod 3 \\ -1 & \text{if } n \equiv -1 \bmod 3 \end{cases}$$

Then we can use formula (23) and (3) to express $D(\zeta_3)$ in terms of $L'(\chi_{-3}, -1)$:

$$D(\zeta_3) = \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{3}}{n^2} = \sum_{n=1}^{\infty} \frac{\frac{\sqrt{3}}{2}\chi_{-3}(n)}{n^2} = \frac{\sqrt{3}}{2} L(\chi_{-3}, 2) = \frac{2\pi}{3} L'(\chi_{-3}, -1).$$

And again we have proven a formula which was conjectured in [4]:

Proposition 7 *The following formula holds:*

$$m(P_{-1}) = 2L'(\chi_{-3}, -1).$$

10.3 Homology of C_k

Now we want to determine a basis of $H_1(C_k', \mathbb{Z})$ and express $[\gamma_k]$ in terms of this basis. We will start with an important lemma:

Lemma 21 *The family γ_k of continuous piecewise smooth paths on C_k' is continuous, where continuity of a family of paths is defined in section 8.*

Proof: Define $y_k(x)$ as the function $y_+(x)$ that belongs to k in the construction in subsection 10.1. It is clear that is suffices to show that $y_k(x)$ a continuous function is on $(]0, 4[\cup]-\infty, -1[) \times C$, where C denotes the unit circle in the complex plane. For each k, an open set $U = U_k$ is constructed above lemma 20. Consider the set

$$V = \{(k, x) \in]0, 4[\cup] - \infty, -1[\times C : x \in U_k\}.$$

So in fact we must show that $y_k(x)$ is a continuous function on V. The analysis on the zeroes of $D_k(x)$ in subsection 10.1, together with the inverse function theorem, shows that each of the zeroes of $D_k(x)$ can be seen as a continuous function of k and that the property whether it lies on, inside, or outside the unit circle is also invariant.

We know that $y_k(x)$ is a zero of $y^2 + b_k(x)y + x^4$. The implicit function theorem tells us that the zeroes of such a polynomial are locally an analytic function of the variables k and x provided that they are distinct. We have seen that for all the points x on the unit circle that satisfy $\Re x > \Re x_0$, where $x_0 \neq -1$ is a zero of $D_k(x)$ there is exactly 1 zero of $y^2 + b_k(x)y + x^4$ that has $|y| > 1$. If we fix k and x such that this property is satisfied, then we see that in a neighborhood of (k, x) the zero y which is implied by the implicit function theorem satisfies $|y| > 1$, hence this must be $y_+(x)$.

Along a path in $\mathbb{C}$ that lies within every U_k we can do an analytic continuation of $y_k(x)$ and we still keep $y_k(x)$. So take a k_0 and an $x_0 \in U_{k_0}$. Then clearly there is a neighborhood K of k_0 and a path γ going from 1 to x_0 such that γ lies entirely within U_k for all $k \in K$. By analytic continuation of the functions $y_k(x)$ at $x = 1$ along γ it follows now that $y_k(x)$ is analytic at (k_0, x_0). So $y_k(x)$ is analytic everywhere on V. $\square$

To proceed, again we break up the parameter space in two components: $]-\infty, -1[$ and $]0, 4]$. We will fix one point in each of the two intervals to do computations with and extend all results by continuity.

We begin with studying the interval $]-\infty, -1[$. Let's focus on the point $k = -2$. We will construct a basis for $H_1(C_{-2}, \mathbb{Z})$. We will do this as in subsubsection 9.1. The polynomial $f_{-2}(x^2)$ has the following zeroes:

$$(\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_3, -\alpha_3) = \left(\sqrt{3}, -\sqrt{3}, \sqrt{2+\sqrt{5}}, -\sqrt{2+\sqrt{5}}, i\sqrt{-2+\sqrt{5}}, -i\sqrt{-2+\sqrt{5}}\right),$$

where, in this notation, we take for each square root the postive real value. For $P, Q \in \mathbb{C}$ we denote by $\delta(P,Q) : [0,1] \to \mathbb{C}$ the straight path $t \mapsto tQ + (1-t)P$. We will denote by $\epsilon(P,Q) : [0,1] \to \mathbb{C}$ the circular path $t \mapsto Q + (P-Q)\exp(2\pi i t)$. Now define the following closed piecewise smooth paths:

$$\begin{aligned}
\gamma_1^+ &= \delta\left(0, \frac{3}{2}\right) \epsilon\left(\frac{3}{2}, \sqrt{3}\right) \delta\left(\frac{3}{2}, 0\right), \\
\gamma_2^+ &= \delta\left(0, \frac{3}{2} + \frac{3}{2}i\right) \delta\left(\frac{3}{2} + \frac{3}{2}i, \sqrt{2+\sqrt{5}} + \frac{1}{10}i\right) \epsilon\left(\sqrt{2+\sqrt{5}} + \frac{1}{10}i, \sqrt{2+\sqrt{5}}\right) \\
&\quad \delta\left(\sqrt{2+\sqrt{5}} + \frac{1}{10}i, \frac{3}{2} + \frac{3}{2}i\right) \delta\left(\frac{3}{2} + \frac{3}{2}i, 0\right), \\
\gamma_3^+ &= \delta\left(0, \frac{2}{5}i\right) \epsilon\left(\frac{2}{5}i, i\sqrt{-2+\sqrt{5}}\right) \delta\left(\frac{2}{5}i, 0\right).
\end{aligned}$$

Using the notations of subsubsection 9.1, we see that the paths γ_i^+ satisfy the conditions of this same subsubsection. As in 9.1, we also have the paths γ_i^- and γ_i'. To lift these paths to C_{-2} and $E_{1,-2}$, we should fix a choice of a lifting of the point 0. As $f_{-2}(0) = 6$, we choose the point $(0, \sqrt{6})$ (for both cases), where we take the positive value of the square root here. Having done this, we get a basis $\mathfrak{a}_1(C_{-2}), \mathfrak{a}_2(C_{-2}), \mathfrak{b}_1(C_{-2}), \mathfrak{b}_2(C_{-2})$ of $H_1(C_{-2}, \mathbb{Z})$ and a basis $\mathfrak{a}_1(E_{1,-2}), \mathfrak{b}_1(E_{1,-2})$ of $H_1(E_{1,-2}, \mathbb{Z})$.

We want to compute the coefficients of $[\gamma_k]$ with respect to the basis $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2$ of $H_1(C_{-2}, \mathbb{Z})$. We do this using the map h_k defined by (60). Identify $\mathbb{C}$ with $\mathbb{R}^2$ by identifying z with $\binom{\Re z}{\Im z}$ and likewise identify $\mathbb{C}^2$ with $\mathbb{R}^4$. A numerical calculation now shows that

$$\begin{pmatrix} | & | & | & | \\ h_{-2}(\mathfrak{a}_1) & h_{-2}(\mathfrak{a}_2) & h_{-2}(\mathfrak{b}_1) & h_{-2}(\mathfrak{b}_2) \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 2.5390 & 2.5390 & -3.8085 & -1.2695 \\ 0.0000 & 1.2064 & -1.8097 & -1.2064 \\ 0.0000 & 0.0000 & 1.4122 & 1.4122 \\ 0.0000 & 0.0000 & 0.0000 & 1.1371 \end{pmatrix}.$$

Another numerical calculation shows that

$$h_{-2}([\gamma_{-2}]) = \begin{pmatrix} 0.0000 \\ 0.6032 \\ 0.0000 \\ -1.1371 \end{pmatrix}. \tag{84}$$

So for the coefficients of γ_{-2} with respect to the basis $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2$ we find

$$\begin{pmatrix} 2.5390 & 2.5390 & -3.8085 & -1.2695 \\ 0.0000 & 1.2064 & -1.8097 & -1.2064 \\ 0.0000 & 0.0000 & 1.4122 & 1.4122 \\ 0.0000 & 0.0000 & 0.0000 & 1.1371 \end{pmatrix}^{-1} \begin{pmatrix} 0.0000 \\ 0.6032 \\ 0.0000 \\ -1.1371 \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 1.0000 \\ 1.0000 \\ -1.0000 \end{pmatrix}.$$

As these coefficients must be integers, we conclude that

$$[\gamma_{-2}] = \mathfrak{a}_2(C_{-2}) + \mathfrak{b}_1(C_{-2}) - \mathfrak{b}_2(C_{-2}). \tag{85}$$

Let's prove that one can verify such a numerical calculation rigorously. If $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $|f'(t)| \leq C$ for some effective C, then

$$\left| \int_a^b f(t)dt - (b-a)f(a) \right| = \left| \int_a^b (f(t) - f(a))dt \right| \leq \int_a^b C(t-a)dt = \frac{C(b-a)^2}{2}.$$

By dividing an interval in arbitrary small pieces we one can use this estimation to obtain the numerical value of an integral of a piecewise differentiable continuous function of which the derivative is effectively bounded rigorously with an arbitrary precision. In the computation of the integrals for $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2$ all the integrands satisfy the boundedness of their derivatives. For the computation of the path γ, the integrand is singular at $t = 1/2$: we see a factor $dt/(t-1/2)$ there. We can solve this problem by locally substituting $u = 1/2 + \exp(t)$ there and then the integrand does satisfy the desired property. Also, inverting the matrix can be done rigorously with an arbitary precision, because the matrix can be obtained with an arbitrary precision and it is upper-triangular.

For the discriminant of $f_k(x^2)$, the following formula holds:

$$\operatorname{disc}\left(f_k(x^2)\right) = 2^{22}k^3(k-8)^2(k-4)(k+1). \tag{86}$$

It follows that there are no degenerate curves for $k \in]-\infty, -1[$, every curve has genus 2. Now we apply corollary 4 from section 8. From the simple connectivity of the interval $]-\infty, -1[$ it follows immediately that there is a basis $[\sigma_{k,1}], \ldots, [\sigma_{k,2}]$ of $H_1(C_k, \mathbb{Z})$ such that, after a suitable lineair transformation, $([\sigma_{-2,1}], \ldots, [\sigma_{-2,4}]) = (\mathfrak{a}_1(C_{-2}), \mathfrak{a}_2(C_{-2}), \mathfrak{b}_1(C_{-2}), \mathfrak{b}_2(C_{-2}))$ and $h_k([\sigma_{k,i}])$ is a continuous function of $k \in]-\infty, -1[$ for each $i \in \{1, \ldots, 4\}$. Combining this with lemma 21 and corollary 5 we conclude that

$$[\gamma_k] = [\sigma_{k,2}] + [\sigma_{k,3}] - [\sigma_{k,4}]$$

for $k \in]-\infty, -1[$.

Now we want to push forward γ_k to $E_{1,k}$ and figure out the corresponding homology classes in $H_1(E_{1,k}, \mathbb{Z})$. As above, it follows from the results of subsection 8.1 that there is a basis $[\sigma'_{k,1}], [\sigma'_{k,2}]$ such that $\left([\sigma'_{-2,1}], [\sigma'_{-2,2}]\right) = (\mathfrak{a}_1(E_{1,-2}), \mathfrak{b}_1(E_{1,-2}))$ and $h_k([\sigma'_{k,i}])$ is a continuous function of $k \in]-\infty, -1[$ for $i \in \{1, 2\}$. From corollary 6 it follows that the coefficients of $[\phi_1\gamma_k]$ with respect to this basis are constant. From (85) and (67) it follows that $[\phi_1\gamma_{-2}] = \mathfrak{a}_1(E_{1,-2})$ so that

$$[\phi_1\gamma_k] = [\sigma'_{k,1}]$$

for $k \in]-\infty, -1[$.

Lemma 22 *For each $k \in]-\infty, -1[$ as well as $k \in]0, 4]$, the complex number $h_k([\phi_1\gamma_k])$ is purely imaginary.*

Proof: We will make use of the properties of the function $y_+(x)$ from proposition 5. The path $\phi_1\gamma_k$ is equal to $t \mapsto (\exp(4\pi it), y_+(\exp(2\pi it)))$. So that

$$h_k([\phi_1\gamma_k]) = \int_{\phi_1\gamma_k} \frac{dx}{y} = 4\pi i \int_0^1 \frac{e^{4\pi it}dt}{y_+(e^{2\pi it})}.$$

Making use of the fact that y_+ is defined over $\mathbb{R}$, we can see that for the complex conjugate of $h_k([\phi_1\gamma_k])$ the following holds:

$$\overline{h_k([\phi_1\gamma_k])} = -4\pi i \int_0^1 \frac{e^{-4\pi it}dt}{y_+(e^{-2\pi it})}.$$

If we substitute $-t$ for t, then this becomes equal to

$$4\pi i\int_0^{-1}\frac{e^{4\pi it}dt}{y_+(e^{2\pi it})}=4\pi i\int_1^0\frac{e^{4\pi it}dt}{y_+(e^{2\pi it})}=-h_k([\phi_1\gamma_k]).$$

So $\overline{h_k([\phi_1\gamma_k])}=-h_k([\phi_1\gamma_k])$, which proves that $h_k([\phi_1\gamma_k])$ is purely imaginary $\square$

We can also compute $h_{-2}([\phi_1\gamma_{-2}])$ numerically, using (84):

$$h_{-2}([\phi_1\gamma_{-2}])=\int_{\phi_1\gamma_{-2}}\frac{dx}{y}=\int_{\gamma_{-2}}\phi^*\left(\frac{dx}{y}\right)=\int_{\gamma_{-2}}\frac{2xdx}{y}=2\cdot -1.1371i.$$

Using this, lemma 22 and the fact that $h_k(\phi_1\gamma_k)$ is a nowhere vanishing continuous function of $k\in]-\infty,-1[$, we conclude that

$$\Im h_k([\phi_1\gamma_k])<0\quad\text{for all } k\in]-\infty,-1[.$$

Now we want to examine the interval $]0,4]$. As most of the computations can be done in the same way as in the previous case, we will not present them as detailed as above. Because of (86), we see that there is a degenerate curve in this case, namely at $k=4$. So let's for the moment assume that we are examining the interval $]0,4[$ and extend all results to $k=4$ later by continuity.

We will focus on the point $k=2$. The zeroes of $f_k(x^2)$ are

$$(\alpha_1,-\alpha_1,\alpha_2,-\alpha_2,\alpha_3,-\alpha_3)=\left(i,-i,\zeta_8\sqrt[4]{3},-\zeta_8\sqrt[4]{3},\zeta_8^{-1}\sqrt[4]{3},-\zeta_8^{-1}\sqrt[4]{3}\right).$$

Take the paths

$$\gamma_i^+=\delta\left(0,\frac{99}{100}\alpha_i\right)\epsilon\left(\frac{99}{100}\alpha_i\right)\delta\left(\frac{99}{100}\alpha_i,0\right).$$

As $f_2(0)=2$, we take liftings of those paths with $z_0=0$ and $\zeta_0=(0,\sqrt{2})$, where the positive value of the square root is taken. We get corresponding classes $\mathfrak{a}_1(C_2),\mathfrak{a}_2(C_2),\mathfrak{b}_1(C_2),\mathfrak{b}_2(C_2)$ in $H_1(C_2,\mathbb{Z})$ and $\mathfrak{a}_1(E_{1,2}),\mathfrak{b}_1(E_{1,2})$ in $H_1(E_{1,2},\mathbb{Z})$. With numerical techniches one can now show that

$$[\gamma_2]=\mathfrak{a}_1(C_2)-\mathfrak{a}_2(C_2)-\mathfrak{b}_2(C_2).$$

If $([\sigma_{k,i}])_i$ is the continuation of $(\mathfrak{a}_1(C_2),\mathfrak{a}_2(C_2),\mathfrak{b}_1(C_2),\mathfrak{b}_2(C_2))$ to all of $]0,4[$ then it follows that

$$[\gamma_k]=[\sigma_{k,1}]-[\sigma_{k,2}]-[\sigma_{k,4}]$$

for $k\in]0,4[$. Hence

$$[\phi_1\gamma_k]=-[\sigma'_{k,2}], \tag{87}$$

for $k\in]0,4[$, where $([\sigma'_{k,1}],[\sigma'_{k,2}])$ is the continuation of $(\mathfrak{a}_1(E_{1,2}),\mathfrak{b}_1(E_{1,2}))$ to all of $]0,4]$ (here we can include $k=4$, as the family $E_{1,k}$ does not have a degeneracy at $k=4$). A numerical computation shows that $\Im h_2([\phi_1\gamma_2])<0$ hence, using lemma 22 and continuity it follows that

$$\Im h_k([\phi_1\gamma_k])<0\quad\text{for all } k\in]0,4]. \tag{88}$$

10.4 Pushing forward $r_C(f,g)$

We know that

$$m(P_k)=-\frac{1}{2\pi}\int_{\psi\circ\gamma_k}\eta(f,g), \tag{89}$$

where

$$f(x,y) = (\psi^{-1})^*(x) = \frac{x+1}{x-1} \quad \text{and} \quad g(x,y) = (\psi^{-1})^*(y) = \frac{2xy - (2k+1)x^4 + (2k-6)x^2 - 1}{(x-1)^4}.$$

We can use the ideas of section 6 to do further calculations now. Lemma 18 shows that the tame symbols of $\{f,g\}$ are roots of unity everywhere so that the differential $\eta(f,g)$ has all its residues equal to 0. We see that

$$\int_{\psi\circ\gamma_k} \eta(f,g) = r\{f,g\}([\psi\circ\gamma_k]).$$

Let's do some manipulations with $\{f,g\}$ now. We will work in $K_2(\mathbb{C}(C))\otimes\mathbb{Q}$. Define

$$a(x,y) = \frac{(-x^2-6x-1)y + (4k+2)x^3 + 14x^2 + (-4k+14)x + 2}{(x-1)\left((-x-1)y + (2k+1)x^2 + (-2k+6)x + 1\right)}$$

and

$$b(x,y) = \frac{(x-1)^2}{(-x-1)y + (2k+1)x^2 + (-2k+6)x + 1}.$$

Then

$$a(x^2,y)f(x,y) + b(x^2,y)g(x,y) = 1,$$

so

$$\begin{aligned} 1 &= \{a(x^2,y)f(x,y)\,,\, b(x^2,y)g(x,y)\} \\ &= \{a(x^2,y), b(x^2,y)\}\,\{a(x^2,y), g(x,y)\}\,\{f(x,y), b(x^2,y)\}\,\{f(x,y), g(x,y)\}, \end{aligned}$$

from which it follows that

$$\{f(x,y),g(x,y)\} = \{a(x^2,y),b(x^2,y)\}^{-1}\left(\{a(x^2,y),g(x,y)\}\{f(x,y),b(x^2,y)\}\right)^{-1}.$$

As $\sigma_1^*\{f(x,y),g(x,y)\} = \{f(x,y),g(x,y)\}$, we see that

$$\begin{aligned} \{f(x,y),g(x,y)\} &= \{a(x^2,y),b(x^2,y)\}^{-1}\left(\{a(x^2,y),g(-x,y)\}\{f(-x,y),b(x^2,y)\}\right)^{-1} \\ &= \{a(x^2,y),b(x^2,y)\}^{-1}\left(\{a(x^2,y),g(x,y)\}\{f(x,y),b(x^2,y)\}\right). \end{aligned}$$

By multiplying these two idenitities it follows that

$$\{f(x,y),g(x,y)\} = \{a(x^2,y),b(x^2,y)\}^{-1}. \tag{90}$$

Now, the functions $a(x^2,y)$ and $b(x^2,y)$ can be pushed forward to $E_{1,1}$ via ϕ_1 to the functions $a(x,y)$ and $b(x,y)$. The identity (90) holds in $K_2(\mathbb{C}(C))\otimes\mathbb{Q}$ so in $K_2(\mathbb{C}(C))$ it holds modulo torsion. This means that also the tame symbols $\partial_P\{a(x^2,y),b(x^2,y)\}$ are roots of unity. From (18) and the fact that ϕ is surjective it follows that $\partial_P\{a(x,y),b(x,y)\}$ is a root of unity for all $P\in E_{1,k}$, so that $\eta(a(x,y),b(x,y)$ has all its residues equal to 0. We see that

$$r_{C_k}\{f(x,y),g(x,y)\}([\psi\circ\gamma_k]) = -r_{E_{1,k}}\{a(x,y),b(x,y)\}([\phi_1\circ\psi\circ\gamma_k]).$$

In view of theorem 5 we see that we must compute $(a(x,y))\diamond(b(x,y))$ (see (33)). Let's use the equivalence relation $\equiv$ on divisors defined in subsection 5.3. Let

$$\begin{aligned} P &= (1,4), \quad T = \left(\frac{k-7}{k+1}, \frac{-4\sqrt{k^2-10k+9}}{k+1}\right), \\ U &= \left(\frac{k-6+\sqrt{k^2-16k+32}}{2(k+1)}, \frac{-3k+4-\sqrt{k^2-16k+32}}{k+1}\right). \end{aligned}$$

If k is fixed, it does not matter which values of the square roots we choose here. The divisors appearing in the calculations are invariant under the conjugations belonging to the square roots. Note that T is conjugate to $-T$ and that U is conjugate to $P-U$. Also note that

$$6P = \mathcal{O}.$$

And, as $k \notin \{0,-1\}$, we can easily see that 6 is the exact order of P for all k, because

$$2P = \left(\frac{k-3}{k+1}, \frac{4}{k+1}\right) \quad \text{and} \quad 3P = \left(\frac{k-4}{k}, 0\right).$$

One can easily compute and verify that

$$(a(x,y)) = 2[P] + [U] + [P-U] - [5P] - [2P] - [P+T] - [P-T]$$

and

$$(b(x,y)) = 2[5P] - [2P] - [P+T] - [P-T] + [\mathcal{O}].$$

Let $D = (a(x,y)) \diamond (b(x,y))$. Then it follows that

$$\begin{aligned} D \equiv\ & 5[P] + 3[2P] + [U] + [P-U] + 3[P+U] + 3[2P-U] + [P+T-U] + [P-T-U] \\ & + [T+U] - [T-U] - [P-T] - [P+T] - 3[2P+T] - 3[2P-T]. \end{aligned}$$

And thus we can use theorem 5 to express $m(P_k)$ in terms of $M_\tau(D)$.

10.5 The case $k = 2$

If $k=2$ then the curve $E_{1,k}$ is an elliptic curve with complex multiplication. In view of the results of section 7 we can try to express $M_\tau(D)$ in terms of $L(E,s)$. Note that $E_{1,2}$ has the following equation:

$$y^2 = 6x^3 + 6x^2 + 2x + 2.$$

Over $\mathbb{Q}$, the curve $E_{1,2}$ is isomorphic with

$$E : y^2 = x^3 + 1.$$

An isomorphism can be given by

$$\rho : E_{1,k} \to E : (x,y) \mapsto \left(\frac{3x+1}{2}, \frac{3y}{4}\right),$$

with inverse

$$\rho^{-1} : E \to E_{1,k} : (x,y) \mapsto \left(\frac{2x-1}{3}, \frac{4y}{3}\right).$$

The curve E is discussed in section 7.1. We see that over $\mathbb{R}$ it is isomorphic with $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau = (3+\sqrt{3}i)/6$. We know that $\mathbb{C}/\Lambda$ is isomorphic to the curve

$$E' : y^2 = 4x^3 - 140G_6(\Lambda),$$

via the isomorphism given by (13). Note that $G_6(\Lambda) < 0$. It is easy to see that E' isomorphic to E by the map

$$E' \to E : (x,y) \mapsto \left(\frac{x}{\sqrt[3]{-35G_6(\Lambda)}}, \frac{y}{\sqrt{-140G_6(\Lambda)}}\right),$$

where we take positive real roots. Note that this isomorphism is defined over $\mathbb{R}$ and it preserves the sign of the y-coordinate of points, which in turn implies that the rescaling of the lattice $\{\int_\gamma dx/y : [\gamma] \in H_1(E(\mathbb{C}), \mathbb{Z}\}$ that is induced by this isomorphism is a multiplication by a positive real number.

If we take $\sqrt{k^2-16k+32} = \sqrt{4} = 2$, then we see that

$$U = \left(-\frac{1}{3}, -\frac{4}{3}\right) = 4P.$$

So our divisor D is much simpler now:

$$D = (a(x,y)) \diamond (b(x,y)) = 2[P] - [2P] - [P-T] - [P+T] - 4[2P-T] - 4[2P+T].$$

This divisor is not fully supported on torsion points, although we would like to have that. Fortunately we have a strategy to tackle this problem. Consider the set

$$S = \{nP + mT : n \in \{0, \dots, 5\}, m \in \{-1, 0, 1\}\}.$$

Let P_1 and P_2 be two (not necessarily distinct) elements of S such that $P_1 + P_2$ is also in S. Then $[P_1] + [P_2] + [-P_1 - P_2] - 3[\mathcal{O}]$ is a principal divisor. It will belong to a function of the form $ay + bx + c$. One can see this by drawing a picture and using the geometric construction of the group law on an elliptic curve, or with a more formal method: the divisor has no poles except for $\mathcal{O}$, which is of order at most 3. Among all these functions we can look for relations of the kind $af_1 + bf_2 = cf_3$. It would then follow that $\{af_1/(cf_3), bf_2/(cf_3)\} = 1$, which is of course very useful. Having done this one finds that

$$\begin{aligned}
(3y + 6x - 2) &= [2P+T] + [2P-T] + [2P] - 3[\mathcal{O}],\\
(3y+4) &= 3[4P] - 3[\mathcal{O}],\\
(x-1) &= [P] + [5P] - 2[\mathcal{O}],
\end{aligned}$$

with relation

$$3y + 6x - 2 - (3y+4) = 6(x-1),$$

where the parentheses here of course don't indicate divisors. If we identify $E_{1,2}(\mathbb{C})$ with $\mathbb{C}/\Lambda$ in the way that is described above, we see that for

$$D' := \left(\frac{3y+6x-2}{6(x-1)}\right) \diamond \left(\frac{-(3y+4)}{6(x-1)}\right) \equiv -4[P] - 7[2P] - [P+T] - [P-T] - 4[2P+T] - 4[2P-T]$$

the identity $M_\tau(D') = 0$ holds so that

$$M_\tau(D) = M_\tau(D - D') = M_\tau\left(6[P] + 6[2P]\right).$$

Let's now figure out what point of $\mathbb{C}/\Lambda$ the point P corresponds with. It is a real-valued point of exact order six so it must correspond to $1/6$ or $-1/6$ modulo Λ. The y-coordinate of P is positive so if we can show that $\wp'_\Lambda(1/6) < 0$ then it must follow that P corresponds to $-1/6 \bmod \Lambda$. Here we go:

$$\begin{aligned}
\wp'_\Lambda\left(\frac{1}{6}\right) &= -2\sum_{\lambda\in\Lambda} \frac{1}{\left(\frac{1}{6}-\lambda\right)^3} = -432 - 2\sum_{\lambda\in\Lambda-\{0\}} \frac{1}{\left(\frac{1}{6}-\lambda\right)^3}\\
&= -432 - \sum_{\lambda\in\Lambda-\{0\}} \frac{1}{\left(\frac{1}{6}-\lambda\right)^3} + \frac{1}{\left(\frac{1}{6}+\lambda\right)^3} = -432 - \sum_{\lambda\in\Lambda-\{0\}} \frac{\lambda^2 + \frac{1}{108}}{\left(\lambda^2 - \frac{1}{36}\right)^3},
\end{aligned}$$

so if we can prove that the absolute value of this last sum is less than 432, then we are done. Note that $|\lambda| \geq \sqrt{3}/3$ for all nonzero $\lambda \in \Lambda$, so that we get the following estimation:

$$\begin{aligned}\left|\sum_{\lambda\in\Lambda-\{0\}} \frac{\lambda^2+\frac{1}{108}}{\left(\lambda^2-\frac{1}{36}\right)^3}\right| &\leq \sum_{\lambda\in\Lambda-\{0\}} \frac{\left|\lambda^2+\frac{1}{108}\right|}{\left|\lambda^2-\frac{1}{36}\right|^3} \leq \sum_{\lambda\in\Lambda-\{0\}} \frac{\left(1+\frac{1}{36}\right)|\lambda|^2}{\left(1-\frac{1}{12}\right)^3|\lambda|^6} < \frac{3}{2}\sum_{\lambda\in\Lambda-\{0\}} |\lambda|^{-4} \\ &= \frac{27}{2}\sum_{\alpha\in\mathcal{O}_K-\{0\}} |\alpha|^{-4} = 81 \sum_{a\geq 1, b\geq 0} |a+b\zeta_6|^{-4} = 81 \sum_{a\geq 1,b\geq 0} \frac{1}{\left(a^2+ab+b^2\right)^2} \\ &= 90 + 81 \sum_{\substack{a\geq 1, b\geq 0 \\ \max(a,b)\geq 2}} \frac{1}{\left(a^2+ab+b^2\right)^2} \leq 90+81\sum_{c=2}^{\infty}\sum_{\substack{a\geq 1,b\geq 0\\ \max(a,b)=c}} c^{-4} \\ &= 90+162(\zeta(3)-1) < 432.\end{aligned}$$

Using (55), (56) and (58) we see that

$$\begin{aligned}M_\tau(D) &= 6\left(M_\tau\left(-\frac{1}{6}\right)+M_\tau\left(-\frac{1}{3}\right)\right) = -6\left(M_\tau\left(\frac{1}{6}\right)+M_\tau\left(\frac{1}{3}\right)\right) \\ &= -6\left(\Im\tau\right)^2\left(K_{1,\Lambda}\left(0,\frac{1}{6},2\right)+K_{1,\Lambda}\left(0,\frac{1}{3},2\right)\right) = -\frac{1}{2}K_{1,\Lambda}\left(0,\frac{1}{6},2\right) = 9L(E,2) \\ &= \pi^2 L'(E,0).\end{aligned}$$

From (87) it follows that $[\phi_1\gamma_2]$ is an element of a basis for $H_1(E_{1,2}(\mathbb{C}),\mathbb{Z})$. From (88) and the fact that the rescaling of the lattice described above is a multiplication by a positive real number, say a, it follows that

$$h_2'([\phi_1\gamma_2]) = -\sqrt{3}i/3, \tag{91}$$

where

$$h_2' : H_1(E_{1,2}(\mathbb{C}),\mathbb{Z}) \to \Lambda : [\gamma] \mapsto ah_2([\gamma]).$$

So if δ_1 is a path with $h_2'([\delta_1]) = 1$ and if δ_2 is a path with $h_2'([\delta_2]) = \tau$, then we see that $h_2'([\delta_1]-2[\delta_2]) = -\sqrt{3}i/3$ so that

$$[\phi_1\gamma_2] = [\delta_1]-2[\delta_2].$$

As we have already shown above, the tame symbols vanish so the value of the integral we must compute only depends on the homology class of $\phi_1\gamma_2$. We can choose a paths homologous to δ_1 and δ_2 which start and end in $\mathcal{O}$. We can lift the corresponding paths on $\mathbb{C}/\Lambda$ to paths in $\mathbb{C}$ such that this lifting starts in 0. From (91) it follows that these liftings will end in 1 and τ respectively. From (42) it follows now that

$$r_{f,g}([\phi_1\gamma_2]) = \frac{\Im\left((1-2\tau)M_\tau(D)\right)}{(\Im\tau)\pi} = \frac{-\Im\left(\frac{\sqrt{3}i}{3}\pi^2L'(E,0)\right)}{\frac{\sqrt{3}i}{6}\pi} = -2\pi L'(E,0)$$

since $L'(E,0)$ is real. Using (89) we arrive at a proof of the following proposition which again was conjectured in [4]:

Proposition 8 *The following formula holds:*

$$m(P_2) = L'(E,0) \quad \textit{with } E : y^2 = x^3+1.$$

11 Conclusion

In tackling the computation of the Mahler measure of a bivariate polynomial P, one has to combine general methods with specific tricks. The structure of the algebraic curve C that is defined as the zero locus of the polynomial is of major importance for both the general methods and the specific ones.

The first step is to express $m(P)$ in terms of an integral of the closed differential $\eta(x, y) = \log|x| d\arg y - \log|y| d\arg x$ along a path on the curve C. How to proceed from this point depends highly on the type of curve we are dealing with. In any case, it is important that $\eta(x, y)$ has all its residues equal to zero.

If C is rational, then one can express $m(P)$ in terms of the dilogarithm function. We want to express the obtained dilogarithm sum to one involving only dilogarithms of roots of unity, which we can then try to transform to an L-series of a Dirichlet character, evaluated at $s = 2$. It is not clear when or how it is possible to rewrite the dilogarithm sum to one only involving roots of unity. No general method of doing is is known to me; in each special case, trial and error seems to play a role in finding dilogarithm identities that one can use to simplify the sums.

If C is an elliptic curve and if the path along which $\eta(x, y)$ is integrated is closed, then one can express $m(P)$ in terms of a sum of Eisenstein-Kronecker-Lerch series, which serve as an elliptic analogue of the dilogarithm. In this case we want to rewrite this sum of Eisenstein-Kronecker-Lerch series to one of which all the terms are evaluated at torsion points of the elliptic curve. Again no general method to do this is clear and trial and error is important.

In the special case of an elliptic curve with complex multiplication, the sum defining Eisenstein-Kronecker-Lerch series looks, by Deuring's theorem, similar to the sum defining L-series. The main difference is that the Eisenstein-Kronecker-Lerch series uses an additive character, while the L-series uses a multiplicative character. Using Gauss sums one can transform them into each other. We can use this to try to give a rigorous proof of an identity of the type $m(P) = rL'(E, 0)$, where r is rational.

Now, if P is a reciprocal polynomial, then we have an automorphism $\sigma : (x, y) \mapsto (x^{-1}, y^{-1})$ on C, of order 2. In the particular cases in this thesis, the genus of C is equal to 2 and dividing out σ gives an elliptic curve $E \cong C/\langle\sigma\rangle$. The fact that $\int_\gamma \eta(f, 1 - f) = 0$ for each closed path γ enables us to rewrite the integral $\int_\gamma \eta(x, y)$ to an integral of the shape $\int_\gamma \eta(f, g)$, where f and g are invariant under the action of σ. Hence we can push forward the integral on C to an integral on E. Again, we can express $m(P)$ in terms of Eisenstein-Kronecker-Lerch series of E. The problem here is that the divisor D in which we evaluate the Eisenstein-Kronecker-Lerch series is very complicated. It could happen that the divisors of x and y are supported on rational torsion points of the Jacobian of C, while the support of D generates a subgroup of E of higher rank and does not fully consist of rational points. It is not clear to me whether the simple shape of (x) and (y) can be used to get a simpler divisor on E. To simplify D we use the following trial and error method: look at divisors of the type $(ax + by + c)$, supported on a smartly chosen subset of E. If a triple of these functions satisfy a linear relation, then this gives a linear relation between Eisenstein-Kronecker-Lerch series. We hope to be able to use this to simplify the divisor D to one that is fully supported on torsion points. It is not clear to me whether or when this method works in general.

A special degenerate case arises when C is an elliptic curve and admits an automorphism $\sigma : (x, y) \mapsto (x^{-1}, y^{-1})$ which gives a rational curve after dividing it out. We get a dilogarithm sum on a very complicated divisor. Also here, finding relations between triples of functions seems to be a good method to find relations between dilogarithms which can be used to simplify the original sum. Still, it is not clear whether this always works.

So as usual, we see that solving a problem is one thing, but fully understanding one's own solution is much more difficult than finding it. A lot of new food for thought arises when one has solved a problem. How to put a specific solution step in a more general setting is the most difficult but also most important challenge that a mathematician can face. The inspiration has to come from examples that are simpler and more specific, therefore those examples are not necessarily less important than the general setting itself.

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